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ON DIFFERENCE SEQUENCE SPACES DEFINED BY ORLICZ FUNCTIONS WITHOUT CONVEXITY

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Abstract. In this paper, we first define spaces of single difference sequences defined by a sequence of Orlicz functions without convexity and investigate their properties. Then we extend this idea to spaces of double sequences and present a new matrix theoretic approach for construction of such double sequence spaces.
Keywords: Difference sequence space, K-function, F-space, AK-space, Fréchet space.
MSC(2010): Primary: 46A45; Secondary: 46A99, 46E99.

1. Introduction and preliminaries

In [22], Orlicz introduced functions nowadays called Orlicz functions and constructed the sequence space (L^M). Krasnosel'skij and Rutickij further investigated the Orlicz space in [13]. For finding Banach spaces with symmetric Schauder bases having complementary subspaces isomorphic to c_0 or ℓ^p $(1 \leq p < \infty)$, Lindberg [14] initiated the study of Orlicz sequence spaces. Subsequently, Lindenstrauss and Tzafriri [15–17] studied the Orlicz sequence spaces in more detail with an aim to solve many important and interesting structural problems in Banach spaces.

Throughout the paper we use the standard notation w, ℓ_∞, c and c_0 to denote the set of all, bounded, convergent and null sequences of real numbers, respectively. By $\mathbb{N}$ we denote the set of natural numbers, and by $\mathbb{R}$ the set of real numbers. A sequence $\mathbf{x}$ will be denoted by $\mathbf{x} = (x_k)$.

A function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$ is called an *Orlicz function* (see [13, 22]).

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Lindenstrauss and Tzafriri [15] used the Orlicz function and introduced the sequence space ℓ_M,

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

and proved that this space is a Banach space with the norm

$$||(x_k)||_M = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

Every space ℓ_M contains a subspace isomorphic to the classical sequence space ℓ_p for some $p \geq 1$. The space ℓ_p, $p \geq 1$, is itself an Orlicz sequence space for $M(x) = x^p$.

In [16,17], Lindenstrauss and Tzafriri pointed out a possible generalization of the space ℓ_M to the case when M is an Orlicz function that does not satisfy the convexity condition. Later, Kalton [10] picked up the problem and succeeded in finding many interesting features distinguishing these two theories of sequence spaces. For more details, one can refer to Kamthan and Gupta [12].

A K-*function* is an Orlicz function M which is not convex.

A K-function M is said to satisfy Δ_2-*condition* if for each $\alpha > 0$, we have

$$K_{M,\alpha} = \sup_{0<x<\infty} \frac{M(\alpha x)}{M(x)} < \infty.$$

(This condition is usually called the Δ_2-*condition on* $\mathbb{R}$ satisfied by M.)

The notion of difference sequence spaces was introduced by Kizmaz [11], who studied the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [7] by introducing the spaces $\ell_\infty(\Delta^s)$, $c(\Delta^s)$ and $c_0(\Delta^s)$, $(s \in \mathbb{N})$. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [26], who studied the spaces $\ell_\infty(\Delta_m)$, $c(\Delta_m)$ and $c_0(\Delta_m)$, $(m \in \mathbb{N})$.

Let m and n be non-negative integers. Then for Z a given sequence space Dutta [3] introduced

$$Z(\Delta^n_{(m)}) = \left\{ x = (x_k) \in w : (\Delta^n_{(m)} x_k) \in Z \right\},$$

where

$$\Delta^0_{(m)} x_k = (x_k),\ \Delta^n_{(m)} x = (\Delta^n_{(m)} x_k) = (\Delta^{n-1}_{(m)} x_k - \Delta^{n-1}_{(m)} x_{k-m})\ (k \in \mathbb{N}),$$

and which is equivalent to the binomial representation

$$\Delta^n_{(m)} x_k = \sum_{i=0}^{n} (-1)^i \binom{n}{i} x_{k-mi};$$

we take here $x_{k-mi} = 0$ whenever $k - mi \leq 0$.

recently, several authors combined the concepts of difference sequences and Orlicz functions to define new classes of sequences and investigated different relevant algebraic and topological properties (see for instance [2, 4–6, 18, 19]).

Now we recall some basic definitions and results which will be useful in understanding the results of the next section. We consider only real vector spaces.

A vector space X equipped with a topology τ is called a *topological vector space* (TVS) if the operations $(x, y) \mapsto x+y$ from $X \times X \to X$ and $(\alpha, x) \mapsto \alpha x$ from $\mathbb{R} \times X \to X$ are continuous, where $X \times X$ and $\mathbb{R} \times X$ are equipped with their usual product topologies, and $\mathbb{R}$ with the usual metric topology. A topology τ on X such that (X, τ) becomes a TVS is referred to as a linear or vector topology on X. For more information about TVS see [25].

Recall that a subset U of a vector space X is *absorbing* if for each $x \in X$ there is $\lambda > 0$ such that $x \in \alpha U$ for all $\alpha \in \mathbb{R}$ with $|\alpha| > \lambda$. U is *balanced* if $\alpha U \subset U$ for each α with $|\alpha| \leq 1$.

Lemma 1.1. [25] *A vector space X equipped with a topology τ is a TVS if and only if there exists a local base β at the zero element $\mathbf{0}$ of X consisting of subsets of X such that:*

(a) *Each U in β is absorbing and balanced;*
(b) *For each $U \in \beta$ there is a $V \in \beta$ with $V + V \subset U$.*

A TVS X is Hausdorff if and only if $\cap\{U : U \in \beta\} = \{\mathbf{0}\}$; a Hausdorff TVS is metrizable if and only if it is first countable, or equivalently, if and only if there is a countable local base at $\mathbf{0}$.

A TVS (X, τ) with $\tau \equiv \tau_q$, the topology generated by a norm q on X, is called an *F^*-space*, and if in addition (X, τ_q) is complete, X is called an *F-space*.

A sequence space X with a linear topology is called a *K-space* provided each of the maps $\pi_i : X \to \mathbb{R}$, $\pi_i(x) = x_i$ is continuous, $i \geq 1$. It is known that a sequence space X equipped with a linear topology is a K-space if and only if the identity map $I : X \to w$ is continuous, where w is endowed with the topology of pointwise convergence.

A K-space X is called a *Fréchet K-space* provided X is an F-space.

For every absorbing and balanced set U of a vector space X, the function $p \equiv p_U : X \to \mathbb{R}^+$ defined by $p_U(x) = \inf\{\alpha : \alpha > 0, x \in \alpha U\}$, is called a *Minkowski functional* or the *gauge* associated with U. The function p_U associated with an absorbing and a balanced set U is also called a *pseudonorm* on X.

Lemma 1.2. *Every pseudonorm function p on X gives rise to a unique linear topology τ_p on X. Conversely, to every linear topology τ on X there corresponds a pseudonorm function p on X such that τ is equivalent to τ_p.*

2. **Spaces of single difference sequences**

In this section, we define the space $\ell^{\mathbf{M}}(\Delta^n_{(m)})$ and investigate its structural properties.

Let $\mathbf{M} = (M_k)$ be a sequence of K-functions and m, n be non-negative integers. Then we introduce the following sequence spaces:

$$\ell^{\mathbf{M}}(\Delta^n_{(m)}) = \left\{(x_k) \in w : \sum_{k=1}^{\infty} M_k \left(\frac{|\Delta^n_{(m)} x_k|}{\rho}\right) < \infty, \text{ for some} \rho > 0\right\}.$$

Taking $n = 0$ and $M_k = M$ for all $k \geq 1$, we get the famous space ℓ^M [10].

Proposition 2.1. *$\ell^{\mathbf{M}}(\Delta^n_{(m)})$ is a linear space.*

Proof. Let $\mathbf{x} = (x_k)$ and $\mathbf{y} = (y_k)$ be arbitrary sequences in $\ell^M(\Delta^n_{(m)})$. Then for some ρ_1, $\rho_2 > 0$, we have

$$\sum_{k=1}^{\infty} M_k \left(\frac{|\Delta^n_{(m)} x_k|}{\rho_1}\right) < \infty \text{ and } \sum_{k=1}^{\infty} M_k \left(\frac{|\Delta^n_{(m)} y_k|}{\rho_2}\right) < \infty.$$

Let $\rho = 2\max\{\rho_1, \rho_2\}$. One can suppose that there is a partition of $\mathbf{N}$ into two (disjoint) sets N_1 and N_2, at least one of which is infinite, such that

$$|\Delta^n_{(m)} x_k| \leq |\Delta^n_{(m)} y_k| \ \forall k \in N_1 \text{ and } |\Delta^n_{(m)} x_k| \geq |\Delta^n_{(m)} y_k| \ \forall k \in N_2.$$

Since the operator $\Delta^n_{(m)}$ is linear, and each M_k is non-decreasing, we have

$$\begin{aligned} \sum_{k\in N_1} M_k \left(\frac{|\Delta^n_{(m)}(x_k + y_k)|}{\rho}\right) &\leq \sum_{k\in N_1} M_k \left(\frac{2|\Delta^n_{(m)} y_k|}{\rho}\right) \\ &\leq \sum_{k=1}^{\infty} M_k \left(\frac{2|\Delta^n_{(m)} y_k|}{\rho}\right) \end{aligned}$$

and

$$\begin{aligned} \sum_{k\in N_2} M_k \left(\frac{|\Delta^n_{(m)}(x_k + y_k)|}{\rho}\right) &\leq \sum_{k\in N_2} M_k \left(\frac{2|\Delta^n_{(m)} x_k)|}{\rho}\right) \\ &\leq \sum_{k=1}^{\infty} M_k \left(\frac{2|\Delta^n_{(m)} x_k|}{\rho}\right). \end{aligned}$$

Therefore, we have from here

$$\begin{aligned} \sum_{k=1}^{\infty} M_k \left(\frac{|\Delta^n_{(m)}(x_k + y_k)|}{\rho}\right) &\leq \sum_{k=1}^{\infty} M_k \left(\frac{2|\Delta^n_{(m)} (_k|}{\rho}\right) \\ &+ \sum_{k=1}^{\infty} M_k \left(\frac{2|\Delta^n_{(m)} y_k|}{\rho}\right) < \infty, \end{aligned}$$

which gives $\mathbf{x}+\mathbf{y} \in \ell^{\mathbf{M}}(\Delta^n_{(m)})$.

Next, let α be any scalar and $\mathbf{x}$ as above. Then we can find $j \in \mathbb{N}$ so that $\frac{|\alpha|}{2^j} < \frac{1}{\rho_1}$. Since M_k, $k \in \mathbb{N}$, are non-decreasing functions, we have

$$\begin{aligned}\sum_{k=1}^{\infty} M_k\left(\frac{|\Delta^n_{(m)}(\alpha x_k)|}{2^j}\right) &= \sum_{k=1}^{\infty} M_k\left(\frac{|\alpha||\Delta^n_{(m)}(x_k)|}{2^j}\right)\\ &\leq \sum_{k=1}^{\infty} M_k\left(\frac{|\alpha||\Delta^n_{(m)}(x_k)|}{\rho_1}\right) < \infty,\end{aligned}$$

which means that $\alpha\mathbf{x} \in \ell^{\mathbf{M}}(\Delta^n_{(m)})$. This completes the proof. □

Proposition 2.2. $\ell^{\mathbf{M}}(\Delta^i_{(m)}) \subset \ell^{\mathbf{M}}(\Delta^n_{(m)})$, $i = 0, 1, 2, \cdots, n-1$.

Proof. Proof is easy and is omitted. □

Our next aim is to define a linear topology on $\ell^{\mathbf{M}}(\Delta^n_{(m)})$. Before defining it we prove some other results.

Let $\mathbf{M} = (M_k)$ be a sequence of K-functions and $\epsilon > 0$. Define

$$B_{\mathbf{M}}(\epsilon) := \left\{(x_k) \in w : \sum_{k=1}^{\infty} M_k\left(|\Delta^n_{(m)} x_k|\right) \leq \epsilon\right\},$$

and

$$\beta_{\mathbf{M}} := \{\rho B_{\mathbf{M}}(\epsilon) : \rho, \epsilon > 0\}.$$

Clearly, each element in $\beta_{\mathbf{M}}$ contains the zero sequence $\mathbf{0}$ – the origin of $\ell^{\mathbf{M}}(\Delta^n_{(m)})$.

Proposition 2.3. *The family $\beta_{\mathbf{M}}$ satisfies the following properties:*

(1) *If $\mathbf{x} \in \ell^{\mathbf{M}}(\Delta^n_{(m)})$, then for each member $\rho B_{\mathbf{M}}(\epsilon)$ of β we have $\mathbf{x} \in \lambda_0 \rho B_{\mathbf{M}}(\epsilon)$, for some $\lambda_0 > 0$, and thus for all $\lambda \in \mathbb{R}$ with $\lambda \geq \lambda_0$;*
(2) *For each element $U = \rho B_{\mathbf{M}}(\epsilon)$ in β and each $\lambda \in (0, 1]$, $\lambda U \subset U$;*
(3) *$\frac{\rho}{2} B_{\mathbf{M}}(\frac{\epsilon}{2}) + \frac{\rho}{2} B_{\mathbf{M}}(\frac{\epsilon}{2}) \subset \rho B_{\mathbf{M}}(\epsilon)$;*
(4) *$\cap\{U : U \in \beta_{\mathbf{M}}\} = \{\mathbf{0}\}$.*

Proof. (1) Let $\mathbf{x} \in \ell^{\mathbf{M}}(\Delta^n_{(m)})$. Then we can find $\gamma > 0$ with

$$\sum_{k=1}^{\infty} M_k\left(\frac{|\Delta^n_{(m)} x_k|}{\gamma\rho}\right) < \infty.$$

Hence there is $j \in \mathbb{N}$ such that

$$\sum_{k=j+1}^{\infty} M_k\left(\frac{|\Delta^n_{(m)} x_k|}{\gamma\rho}\right) < \frac{\epsilon}{2}.$$

There are also positive numbers $\gamma_1, \gamma_2, \cdots, \gamma_j$ such that

$$\begin{aligned} M_1\left(\frac{|\Delta^n_{(m)}x_1|}{\gamma_1\gamma\rho}\right) &< \frac{\epsilon}{2^2}, M_2\left(\frac{|\Delta^n_{(m)}x_2|}{\gamma_2\gamma\rho}\right) \\ &< \frac{\epsilon}{2^3}, \ldots, M_j\left(\frac{|\Delta^n_{(m)}x_j|}{\gamma_j\gamma\rho}\right) < \frac{\epsilon}{2^{j+1}}. \end{aligned}$$

If $\lambda_0 = \max\{\gamma, \gamma_1\gamma, \cdots, \gamma_j\gamma\}$, then for all λ with $\lambda \geq \lambda_0$ we have

$$\begin{aligned} \sum_{k=1}^{\infty} M_k\left(\frac{|\Delta^n_{(m)}x_k|}{\lambda\rho}\right) &\leq \sum_{k=1}^{j} M_k\left(\frac{|\Delta^n_{(m)}x_k|}{\gamma_k\gamma\rho}\right) \\ &+ \sum_{k=j+1}^{\infty} M_k\left(\frac{|\Delta^n_{(m)}x_k|}{\gamma\rho}\right) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence $\mathbf{x} \in \lambda\rho B_{\mathbf{M}}(\epsilon)$.

(2) Let $\lambda \in (0, 1]$ and $(x_k) \in \lambda U$, i.e. let $\sum_{k=1}^{\infty} M_k\left(\frac{|\Delta^n_{(m)}x_k|}{\lambda\rho}\right) < \epsilon$ be satisfied. Then because of $|\lambda|\rho \leq \rho$ we have

$$\sum_{k=1}^{\infty} M_k\left(\frac{|\Delta^n_{(m)}x_k|}{\rho}\right) \leq \sum_{k=1}^{\infty} M_k\left(\frac{|\Delta^n_{(m)}x_k|}{\lambda\rho}\right) < \epsilon,$$

i.e., $(x_k) \in \rho B_{\mathbf{M}}(\epsilon) = U$.

(3) Let $\mathbf{x}, \mathbf{y} \in \frac{\rho}{2} B_{\mathbf{M}}(\frac{\epsilon}{2})$. Then

$$\begin{aligned} \sum_{k=1}^{\infty} M_k\left(\frac{|\Delta^n_{(m)}(x_k + y_k)|}{\rho}\right) &\leq \sum_{k=1}^{\infty} M_k\left(\frac{2|\Delta^n_{(m)}x_k|}{\rho}\right) \\ &+ \sum_{k=1}^{\infty} M_k\left(\frac{2|\Delta^n_{(m)}y_k|}{\rho}\right) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Therefore, $\mathbf{x} + \mathbf{y} \in \rho B_{\mathbf{M}}(\epsilon)$.

(4) It is evident. $\square$

From the preceding proposition and Lemma 1.1 one obtains the following

Corollary 2.4. $(\ell^{\mathbf{M}}(\Delta^n_{(m)}), \tau_{\mathbf{M}})$ *is a Hausdroff topological vector space, where the linear topology* $\tau_{\mathbf{M}}$ *on* $\ell^{\mathbf{M}}(\Delta^n_{(m)})$ *is generated by* $\beta_{\mathbf{M}}$.

In fact, we have

Proposition 2.5. $(\ell^{\mathbf{M}}(\Delta^n_{(m)}), \tau_{\mathbf{M}})$ *is a metrizable topological vector space.*

Proof. Consider the family

$$\beta' = \{\rho B(\epsilon) : \rho, \epsilon > 0 \text{ and } \rho, \epsilon \text{ are rational numbers } \} \subset \beta.$$

This family of neighbourhoods of $\mathbf{0}$ is countable and generates the same topology $\tau_{\mathbf{M}}$ on $\ell^{\mathbf{M}}(\Delta^n_{(m)})$. Therefore, $\tau_{\mathbf{M}}$ is a metrizable topology. □

From the monotonicity and continuity of K-functions M_k, $k \in \mathbb{N}$, it directly follows the K-character of $\tau_{\mathbf{M}}$, that is $(\ell^{\mathbf{M}}(\Delta^n_{(m)}), \tau_{\mathbf{M}})$ is a K-space which in turn yields the completeness of $(\ell^{\mathbf{M}}(\Delta^n_{(m)}), \tau_{\mathbf{M}})$. Hence we get the following proposition.

Proposition 2.6. *$(\ell^{\mathbf{M}}(\Delta^n_{(m)}), \tau_{\mathbf{M}})$ is a Fréchet K-space.*

Now by imposing on each K-function M_k, the Δ_2-condition (on $\mathbb{R}$), we show that the Fréchet space $\ell^{\mathbf{M}}(\Delta^n_{(m)})$ becomes an AK-space (see, for instance, [19]).

In this connection we define

$$h^E(\Delta^n_{(m)}) = \{\mathbf{x} \in w : \sum_{k=1}^{\infty} M_k\left(\frac{|\Delta^n_{(m)} x_k|}{\rho}\right) < \infty,\ \text{for all } \rho > 0\}.$$

Clearly $h^{\mathbf{M}}(\Delta^n_{(m)})$ is a subspace of $\ell^{\mathbf{M}}(\Delta^n_{(m)})$.

Proposition 2.7. *$h^{\mathbf{M}}(\Delta^n_{(m)})$ is an AK-space.*

Proof. Let $\mathbf{x} = (x_k) \in h^{\mathbf{M}}(\Delta^n_{(m)})$ and $\epsilon > 0$ be arbitrarily chosen. Then

$$\sum_{k=1}^{\infty} M_k\left(\frac{|\Delta^n_{(m)} x_k|}{\rho}\right) < \infty,\ \text{for every } \rho > 0.$$

Hence we can find an integer s_0 such that

$$\sum_{k=s+1}^{\infty} M_k\left(\frac{|\Delta^n_{(m)} x_k|}{\rho}\right) \le \epsilon,\ \text{for all } s \ge s_0.$$

It implies that $x^{[s]} - \mathbf{x} \in \rho B_{\mathbf{M}}(\epsilon)$ for all $s \ge s_0$. (Here $x^{[s]}$ denotes the s-section of $\mathbf{x}$, i.e. $x^{[s]} = \sum_{k=1}^{s} x_k e^{(k)}$, $e^{(k)}_k = 1$, $e^{(k)}_t = 0$ for $t \neq k$.) Since ρ and $\epsilon > 0$ were arbitrary, it follows that $x^{(s)} \to \mathbf{x}$ in the topology $\tau_{\mathbf{M}}$. □

Proposition 2.8. *If each K-function M_k of the sequence $\mathbf{M} = (M_k)$ satisfies the Δ_2-condition (on $\mathbb{R}$), then $h^{\mathbf{M}}(\Delta^n_{(m)}) = \ell^{\mathbf{M}}(\Delta^n_{(m)})$.*

Proof. Let $\mathbf{x} = (x_k) \in \ell^{\mathbf{M}}(\Delta^n_{(m)})$. Then $\sum\limits_{k=1}^{\infty} M_k\left(\frac{|\Delta^n_{(m)} x_k|}{\rho}\right) < \infty$, for some $\rho > 0$. Let us choose an arbitrary $r > 0$. Then

$$\sum_{k=1}^{\infty} M_k\left(\frac{|\Delta^n_{(m)} x_k|}{r}\right) = \sum_{k=1}^{\infty} M_k\left(\frac{|\Delta^n_{(m)} x_k|}{\rho}\right) \frac{M_k\left(\frac{\rho y_k}{r}\right)}{M_k(y_k)},$$

where
$$y_k = \frac{|\Delta^n_{(m)} x_k|}{\rho}.$$
Since each M_k satisfies the Δ_2-condition we have
$$\sum_{k=1}^{\infty} M_k \left(\frac{|\Delta^n_{(m)} x_k|}{r} \right) \leq \sum_{k=1}^{\infty} K_{M_k, \frac{\rho}{r}} \; M_k \left(\frac{|\Delta^n_{(m)} x_k|}{\rho} \right).$$
Let $L = \sup\limits_k \left\{ K_{M_k, \frac{\rho}{r}} \right\}$. Hence we have
$$\sum_{k=1}^{\infty} M_k \left(\frac{|\Delta^n_{(m)} x_k|}{r} \right) \leq L \sum_{k=1}^{\infty} M_k \left(\frac{|\Delta^n_{(m)} x_k|}{\rho} \right) < \infty, \text{ for every } \; r > 0.$$
Thus $\mathbf{x} \in h^{\mathbf{M}}(\Delta^n_{(m)})$ and so $\ell^{\mathbf{M}}(\Delta^n_{(m)}) = h^{\mathbf{M}}(\Delta^n_{(m)})$. $\square$

Combining Propositions 2.6, 2.7 and 2.8 we get the most expected result in the following proposition.

Proposition 2.9. *If each K-function M_k of the sequence $\mathbf{M} = (M_k)$ satisfies the Δ_2-condition on $\mathbb{R}$, then $\ell^{\mathbf{M}}(\Delta^n_{(m)})$ is an AK-space.*

Definition of pseudonorm and Lemma 1.2 of previous section encourage us to talk about $\tau_{\mathbf{M}}$ in terms of pseudonorms which generate this topology. For each ρ and $\epsilon > 0$, let us define
$$p_{\rho,\epsilon}(\mathbf{x}) = \inf\{\alpha > 0, \mathbf{x} \in \alpha\rho B_{\mathbf{M}}(\epsilon)\}.$$
Clearly, $p_{\rho,\epsilon}(\lambda\mathbf{x}) = |\lambda| p_{\rho,\epsilon}(\mathbf{x})$ and $p_{\rho,\epsilon}(\mathbf{x} + \mathbf{y}) \leq p_{\frac{\rho}{2},\frac{\epsilon}{2}}(\mathbf{x}) + p_{\frac{\rho}{2},\frac{\epsilon}{2}}(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \ell^{\mathbf{M}}(\Delta^n_{(m)})$ and $\lambda \in \mathbb{R}$.

Hence we have the following proposition.

Proposition 2.10. *The family $\{p_{\rho,\epsilon}(\mathbf{x}) : \rho, \epsilon > 0\}$ of pseudonorms on $\ell^{\mathbf{M}}(\Delta^n_{(m)})$ generates the topology $\tau_{\mathbf{M}}$.*

Next suppose each K-function M_k of the sequence $\mathbf{M} = (M_k)$ satisfies the Δ_2-condition and let us define the function p_ϵ on $\ell^{\mathbf{M}}(\Delta^n_{(m)})$ as follows:
$$p_\epsilon(\mathbf{x}) = \inf \left\{ \alpha > 0 : \sum_{k=1}^{\infty} M_k \left(\frac{|\Delta^n_{(m)} x_k|}{\alpha} \right) \leq \epsilon \right\}.$$
Then p_ϵ is a pseudonorm on $\ell^{\mathbf{M}}(\Delta^n_{(m)})$.

For the next results we shall assume that each K-function M_k of the sequence $\mathbf{M} = (M_k)$ satisfies the Δ_2-condition.

Proposition 2.11. *The family $\{p_\epsilon : \epsilon > 0\}$ of pseudonorms on $\ell^{\mathbf{M}}(\Delta^n_{(m)})$ generates a topology $\sigma_{\mathbf{M}}$ on $\ell^{\mathbf{M}}(\Delta^n_{(m)})$.*

Proposition 2.12. *For each* $\mathbf{x} \in \ell^{\mathbf{M}}(\Delta^n_{(m)})$, $p_{\rho,\epsilon}(\mathbf{x}) = \frac{1}{\rho} p_\epsilon(\mathbf{x})$, *for each* $\rho > 0$ *and* $\epsilon > 0$.

Proof. Let $\mathbf{x} = (x_k) \in \ell^{\mathbf{M}}(\Delta^n_{(m)})$. Then

$$\begin{aligned} p_{\rho,\epsilon}(\mathbf{x}) &= \inf\{\alpha > 0 : \mathbf{x} \in \alpha\rho B_{\mathbf{M}}(\epsilon)\} \\ &= \frac{1}{\rho} \inf\left\{\alpha\rho > 0 : \sum_{k=1}^{\infty} M_k\left(\frac{|\Delta^n_{(m)} x_k|}{\alpha\rho}\right) \le \epsilon\right\} \\ &= \frac{1}{\rho} \inf\left\{r > 0 : \sum_{k=1}^{\infty} M_k\left(\frac{|\Delta^n_{(m)} x_k|}{r}\right) \le \epsilon\right\} = \frac{1}{\rho} p_\epsilon(\mathbf{x}). \end{aligned}$$

Thus $p_{\rho,\epsilon}(\mathbf{x}) = \frac{1}{\rho} p_\epsilon(\mathbf{x})$ for each $\mathbf{x} \in \ell^{\mathbf{M}}(\Delta^n_{(m)})$. □

Hence we have the following proposition.

Proposition 2.13. *The topologies* $\tau_{\mathbf{M}}$ *and* $\sigma_{\mathbf{M}}$ *are equivalent.*

3. Spaces of double difference sequences

A double real sequence $\mathbf{x} : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ is usually denoted by $\mathbf{x} = (x_{mn})$ and expressed as an infinite matrix.

In 1900, Pringsheim [23] introduced the concept of convergence of real double sequences: a double sequence $\mathbf{x} = (x_{mn})$ converges to $L \in \mathbb{R}$, denoted by $\mathsf{P}\text{-}\lim \mathbf{x} = L$ or $\mathsf{P}\text{-}\lim x_{mn} = L$, if for every $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $|x_{mn} - a| < \epsilon$ for all $m, n > n_0$. The limit L is called the *Pringsheim limit* of $\mathbf{x}$. Some initial results on double sequences can be found in the monumental Hobson's book [9] and the papers [8,24], as well as in [1]. For other useful results on double sequences, one may refer to Moricz [20] and Moricz and Rhoades [21].

The notion of regular convergence of double sequence was introduced by Hardy [8] as follows. A double sequence $\mathbf{x} = (x_{mn})$ is said to *converge regularly* if it converges in the Pringsheim's sense and the following limits exist:

$\lim_{m\to\infty} x_{mn} = L_n$, for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_{mn} = T_m$, for each $m \in \mathbb{N}$.

We denote by ${}_2w$, the set of all real double sequences. Let M be a K-function. Then we introduce the notion of *OK-space* of double sequences as follows:

$${}_2\ell^M = \left\{(x_{mn}) \in {}_2w : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} M\left(\frac{|x_{mn}|}{\rho}\right) < \infty, \text{ for some } \rho > 0\right\}.$$

It is easy to see that ${}_2\ell^M$ is a linear space. Now we present an idea how to use the difference operator to double sequences in order to introduce the spaces of double difference sequences extended by K-functions.

The first order difference operator Δ can be expressed as an infinite triangular matrix

$$\Delta=\left(\begin{array}{llllll} 1 & -1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & -1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & -1 & 0 & \ldots \\ \multicolumn{6}{l}{\dotfill} \\ \multicolumn{6}{l}{\dotfill} \end{array}\right);$$

let $\Delta_{(1)}$ denote the additive inverse of Δ, i.e. $\Delta+\Delta_{(1)}=\mathbf{0}$, the zero infinite matrix.

Define inductively

$$\Delta^2=\Delta\cdot\Delta,\ \ \Delta^2_{(1)}=-\Delta^2;\ \ \cdots\ \ ;\ \ \Delta^n=\Delta\cdot\Delta^{n-1},\ \ \Delta^n_{(1)}=-\Delta^n.$$

Next, Δ_2 can be considered as

$$\Delta_2=\left(\begin{array}{llllll} 1 & 0 & -1 & 0 & 0 & \ldots \\ 0 & 1 & 0 & -1 & 0 & \ldots \\ 0 & 0 & 1 & 0 & -1 & \ldots \\ \multicolumn{6}{l}{\dotfill} \\ \multicolumn{6}{l}{\dotfill} \end{array}\right),$$

and $\Delta_{(2)}$ as the additive inverse of Δ_2. similarly, we can have Δ_r and $\Delta_{(r)}$ for each $r\geq 2$. Hence we can define $\Delta^s_{(\,r)}$ as

$$\Delta^s_{(\,r)}=\Delta_{(r)}\cdot\Delta^{s-1}_{(\,r)}.$$

Now we can give an alternative definition of the spaces $Z\left(\Delta^s_{(\,r)}\right)$ of difference sequences as follows:

$$Z\left(\Delta^s_{(\,r)}\right)=\left\{(x_k):(A_iX)\in Z\right\},$$

where

$$X=\left[x_1\ \ x_2\ \ \cdots\ \ x_n\ \ \cdots\right]^T,\ \ \Delta^s_{(\,r)}=A=(a_{ik}),$$

and

$$A_iX=\sum_{k=1}^{\infty}a_{ik}x_k,\ \text{ for each } i\geq 1.$$

This approach to construction of difference sequence spaces is useful to study structural properties of such spaces. In particular, this approach is very useful for construction of difference double sequences.

Let a double sequence $\mathbf{a}=(a_{mn})$ be expressed as an infinite matrix

$$(a_{mn})=\left(\begin{array}{llllll} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} & \ldots \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} & \ldots \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} & \ldots \\ \multicolumn{6}{l}{\dotfill} \\ \multicolumn{6}{l}{\dotfill} \end{array}\right).$$

Now we define the set ${}_2Z\left(\Delta\right)$ of double difference sequences as follows:

$$ {}_2Z\left(\Delta\right) = \left\{\mathbf{a} = (a_{mn}) : (\Delta\mathbf{a}) \in {}_2Z\right\}, $$

where

$$
\begin{aligned}
\Delta\mathbf{a} &= \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & -1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & -1 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} & \ldots \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} & \ldots \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} \\
&= \begin{pmatrix} a_{11}-a_{21} & a_{12}-a_{22} & \ldots & a_{1n}-a_{2n} & \ldots \\ a_{21}-a_{31} & a_{22}-a_{32} & \ldots & a_{2n}-a_{3n} & \ldots \\ a_{31}-a_{41} & a_{32}-a_{42} & \ldots & a_{3n}-a_{4n} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}.
\end{aligned}
$$

We can define

$$ {}_2Z\left(\Delta^s_{(r)}\right) = \left\{\mathbf{a} = (a_{mn}) : B\mathbf{a} \in {}_2Z\right\} = \left\{\mathbf{a} = (a_{mn}) : (c_{kn}) \in {}_2Z\right\}, $$

where

$$ B = (b_{nk}) = \Delta^s_{(r)} \text{ and } B\mathbf{a} = C = (c_{kn}) $$

with

$$ c_{kn} = \sum_{m=1}^{\infty} b_{km} a_{mn}, \text{ for each } k, n \in \mathbb{N}. $$

In view of the above observations, for a K-function M we define the *OK-spaces* of double difference sequences as follows:

$$
\begin{aligned}
&{}_2\ell^M\left(\Delta^s_{(r)}\right) = \left\{\mathbf{a} = (a_{mn}) \in {}_2w : (\Delta^s_{(r)} a_{mn}) \in {}_2\ell^M\right\} = \\
&\left\{(x_{mn}) \in {}_2w : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} M\left(\frac{|\Delta^s_{(r)} a_{mn}|}{\rho}\right) < \infty, \text{ for some } \rho > 0\right\} = \\
&\left\{(c_{kn}) \in {}_2w : \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} M\left(\frac{|c_{kn}|}{\rho}\right) < \infty, \text{ for some } \rho > 0\right\}.
\end{aligned}
$$

These observations indicate that topologies of ${}_2\ell^M$ and ${}_2\ell^M\left(\Delta^s_{(r)}\right)$ are equivalent, for each $r, s \in N$. In a next paper we shall investigate a linear topology on ${}_2\ell^M$ and establish that the spaces ${}_2\ell^M$ and ${}_2\ell^M\left(\Delta^s_{(r)}\right)$ are topologically equivalent.

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