ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 41 (2015), No. 2, pp. 499–518

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 41 (2015), No. 2, pp. 499–518
Online ISSN: 1735-8515

ON THE NON-SPLIT EXTENSION $2^{2n\cdot}Sp(2n, 2)$

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(Communicated by Ali Reza Ashrafi)

Abstract. In this paper we give some general results on the non-split extension group $\overline{G}_n = 2^{2n\cdot}Sp(2n, 2),\ n \geq 2$. We then focus on the group $\overline{G}_4 = 2^{8\cdot}Sp(8, 2)$. We construct $\overline{G}_4$ as a permutation group acting on 512 points. The conjugacy classes are determined using the coset analysis technique. Then we determine the inertia factor groups and Fischer matrices, which are required for the computations of the character table of $\overline{G}_4$ by means of Clifford-Fischer Theory. There are two inertia factor groups namely $H_1 = Sp(8, 2)$ and $H_2 = 2^7{:}Sp(6, 2)$, the Schur multiplier and hence the character table of the corresponding covering group of H_2 were calculated. Using the information on conjugacy classes, Fischer matrices and ordinary and projective tables of H_2, we concluded that we only need to use the ordinary character table of H_2 to construct the character table of $\overline{G}_4$. The Fischer matrices of $\overline{G}_4$ are all listed in this paper. The character table of $\overline{G}_4$ is a 195×195 complex valued matrix, it has been supplied in the PhD Thesis [2] of the first author, which could be accessed online.

Keywords: Group extensions, symplectic group, character table, inertia groups, Fischer matrices.
MSC(2010): Primary: 20C15; Secondary: 20C40.

1. Introduction

In this section most of the generality comes from discussion made in the introduction section of [3]. Let $G = Sp(2n, q)$ be the symplectic group consisting of $2n \times 2n$ matrices over $\mathbb{F}_q$ that preserve a non-degenerate alternating bilinear form and let $V = q^{2n}$ be a $2n-$dimensional vector space over $\mathbb{F}_q$. Dempwolff proved in [8] that a non-split extension of the form $\overline{G}_n = 2^{2n\cdot}Sp(2n, 2)$ does exist for all $n \geq 2$, where $\overline{G}_n/2^{2n} \cong Sp(2n, 2)$ acts faithfully on 2^{2n}. Moreover, such an extension is unique up to isomorphism, since

Article electronically published on April 29, 2015.
Received: 13 December 2013, Accepted: 19 March 2014.
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$\dim_{\mathbb{F}_2} H^2(Sp(2n,2),2^{2n}) = 1$ for all $n \geq 2$, where $H^2(K,M)$ is the second cohomology group of a group K with coefficients in M. In the case $n = 2$, the non-split extension $\overline{G}_2 = 2^{4\cdot}Sp(4,2) \cong 2^{4\cdot}S_6$ is a maximal subgroup of the sporadic simple group Higman-Sims HS (see the $\mathbb{ATLAS}$ [6]). This group has been fully investigated by T. Seretlo [14], its Fischer matrices and the character table were determined. The character table of $\overline{G}_2$ is also available in a GAP library (see [9]). The group $\overline{G}_3 = 2^{6\cdot}Sp(6,2)$ was fully studied in [3] by authors, its Fischer matrices and the character table were determined. The Fischer matrices of our group $\overline{G}_4 = 2^{8\cdot}Sp(8,2)$ and its character table are not known. In this paper our main aims are to give some results on the group $\overline{G}_n$ for any $n \geq 2$ and to fully study the group $\overline{G}_4$, to determine its inertia factor groups (and their respective ordinary and projective character tables) and to compute the Fischer matrices. It will turn out that the character table of $\overline{G}_4$ is a 195×195 complex matrix and coincides with the character table of the split extension $2^8{:}Sp(8,2)$, available in GAP library. As we mentioned in [3], if one is only interested in the calculation of the character table, then it could be computed by using GAP or Magma and the generators $\overline{g}_1$ and $\overline{g}_2$ of $\overline{G}_4$. But Clifford-Fischer Theory provides many other interesting information on the group and on the character table, in particular the character table produced by Clifford-Fischer Theory, is in a special format that could not be achieved by direct computations using GAP or Magma. Also providing various examples for the applications of Clifford-Fischer Theory to both split and non-split extensions is making sense, since each group requires individual approach. The readers (particulary young researchers) will highly benefit from the theoretical background required for these computations. GAP and Magma are computational tools and would not replace good powerful and theoretical arguments.

For the notations used in this paper and the description of Clifford-Fischer theory technique, we follow [2–5].

2. The group $\overline{G}_n = 2^{2n\cdot}Sp(2n,2)$

The group $\overline{G}_n$ can be constructed in GAP in terms of permutations through the following sequence of commands.

```
gap> P:= ExtraspecialGroup(2^{2n+1}, "+");
gap> C:= CyclicGroup(4);
gap> D:= DirectProduct(P, C);
gap> NS:= MinimalNormalSubgroups(D);
gap> N:= NS[k];
gap> R:= FactorGroup(D,N);
gap> A:= AutomorphismGroup(R);
gap> MS:= MaximalNormalSubgroups(A);
gap> S:= MS[r];
gap> iso:= IsomorphismPermGroup(S);
gap> image:= Image(iso);;
gap> NrMovedPoints(image);
gap> small:= SmallerDegreePermutationRepresentation(image);;
gap> Gn:= Image(small);
```

Remark 2.1. *In the above programme, the command "MinimalNormalSubgroups(D)" always returns three copies of $\mathbb{Z}_2$ and the reader has to choose the appropriate $\mathbb{Z}_2$ in such a way that the order of the automorphism group of the quotient of D by $\mathbb{Z}_2$ is $2^{2n+1} \times |Sp(2n,2)|$. Also the command "MaximalNormalSubgroups(A)" returns a list of those normal subgroups of A that are maximal, and the group $\overline{G}_n := S$ appears as an index 2 subgroup of A. Also the last five commands of the above programme are used to convert S from a finitely presented group into a group of permutations.*

We also remark that the group $\overline{G}_n$ can be constructed in terms of permutations of a set of cardinality at least 2^{2n+1}, i.e., $\overline{G}_n \leq S_{2^{2n+1}}$, but $\overline{G}_n \nleq S_{2^{2n+1}-1}$ (in fact $\overline{G}_n \leq A_{2^{2n+1}}$). Moreover, the group $\overline{G}_n$ acts transitively on a $2^{2n+1}-2$ points, that it fixes 2 points of the set $\{1, 2, \cdots, 2^{2n+1}\}$. Hence the resulting permutation character of this action is of degree $2^{2n+1}-2$. In [3] we have seen that the generators of $\overline{G}_3 = 2^{6\cdot}Sp(6,2)$ are in terms of permutations of a set of cardinality 128 and $\overline{G}_3$ fixes 7 and 26 on its action on $\{1, 2, \cdots, 128\}$. Also we have shown how the permutation character, of degree 126, decomposes in terms of the irreducible characters of $\overline{G}_3$.

The group $\overline{G}_n/2^{2n} \cong Sp(2n,2)$ acts faithfully on 2^{2n}, it yields two orbits of lengths 1 and $2^{2n}-1$. By Brauer Theorem (Lemma 4.5.2 of [10]), it follows that the action of $\overline{G}_n$ on $\mathrm{Irr}(N)$ will also produce two orbits. These two orbits must necessarily have lengths 1 and $2^{2n}-1$ and the first orbit consists of the identity character $\mathbf{1}_N$ while the other orbit consists of the non-trivial linear characters of N. Thus, the corresponding inertia factor groups H_1 and H_2 have indices 1 and $2^{2n}-1$ in $Sp(2n,2)$, respectively. It is clear that $H_1 = G_n = Sp(2n,2)$, while a subgroup of $Sp(2n,2)$ of index $2^{2n}-1$ is the affine symplectic group $2^{2n-1}{:}Sp(2n-2,2)$, that is the stabilizer in $Sp(2n,2)$ of a non-zero vector of the vector space $\mathbb{V} = 2^{2n}$, which is maximal in $Sp(2n,2)$. From Section 5.3 of [2], it follows that the irreducible characters of $\overline{G}_n$ are distributed into two blocks of characters $\mathcal{K}_1$ and $\mathcal{K}_2$ corresponding to the ordinary characters of $H_1 = Sp(2n,2)$ and a projective character table of $H_2 = 2^{2n-1}{:}Sp(2n-2,2)$, respectively. Thus, the number of irreducible characters of $\overline{G}_n$ is given by the following formula:

(2.1)
$$|\mathrm{Irr}(2^{2n\cdot}Sp(2n,2))| = |\mathrm{Irr}(Sp(2n,2))| + |\mathrm{IrrProj}(2^{2n-1}{:}Sp(2n-2,2),\alpha^{-1})|,$$

for some factor set α of the Schur multiplier of H_2. In his PhD Thesis [14], Seretlo constructed the ordinary character table of $\overline{G}_2 = 2^{4\cdot}Sp(4,2) \cong 2^{4\cdot}S_6$ by means of Clifford-Fischer theory, where he showed that there are two inertia factor groups $H_1 = S_6$ and $H_2 = 2 \times S_4$. He proved that for the construction of the character table of $\overline{G}_2$ we only need the ordinary character tables of H_1 and H_2. In [3] we showed that we need to use the ordinary character tables of the two inertia factors of $\overline{G}_3$. For the case $n = 4$, where the group $\overline{G}_4$ is the focus of the discussion of this paper, the computations of the Schur multiplier of the second inertia factor group $2^7{:}Sp(6,2)$ reveals 2. By similar arguments used in [3] one can show that we need to use the ordinary character table of $2^7{:}Sp(6,2)$ to construct the character table of $\overline{G}_4$. By Section 5.2 of [2], it follows that the identity Fischer matrix of $\overline{G}_n$ for any $n \geq 2$ is of the form:

$\mathcal{F}_1$

$g_1 = 1_{Sp(2n,2)}$		g_{11}	g_{12}
$o(g_{1j})$		1	2
$\lvert C_{\overline{G}}(g_{1j})\rvert$		$\lvert\overline{G}_n\rvert$	$\lvert\overline{G}_n\rvert/2^{2n}-1$
(k,m)	$\lvert C_{H_k}(g_{1km})\rvert$		
$(1,1)$	$\lvert G_n\rvert$	1	1
$(2,1)$	$\lvert G_n\rvert/2^{2n}-1$	$2^{2n}-1$	-1
m_{1j}		1	$2^{2n}-1$

Corollary 2.2. *The group $\overline{G}_n$ has several irreducible characters with degrees multiple of $2^{2n}-1$.*

Proof. Let $g_{i1}, g_{i2}, \cdots, g_{ic(g_i)}$, be representatives of the conjugacy classes of $\overline{G}$ corresponding to $[g_i]_G$, obtained through the coset analysis technique. Using the notations of Section 3 of [3] the values of the irreducible characters of $\overline{G}$, contained in the block $\mathcal{K}_k$, on the classes $g_{i1}, g_{i2}, \cdots, g_{ic(g_i)}$ are given by $\mathcal{K}_{ik}\mathcal{F}_{ik}$. Here $\mathcal{K}_{ik}$ is the fragment of the projective character table of H_k with factor set α_k^{-1} consisting of columns corresponding to the $\alpha_k^{-1}-$regular classes of H_k that fuse to $[g_i]_G$. The sub-matrix $\mathcal{F}_{ik}$ of $\mathcal{F}_i$ consists of rows corresponding to the pairs $(k,1), (k,2), \cdots, (k, r_{ik})$ of the set J_i as defined in [3]. Now for $(i,k) = (1,2)$ from the above table representing $\mathcal{F}_1$ we get $\mathcal{F}_{12} = (2^{2n}-1 \quad -1)$. Thus, multiplying $\mathcal{K}_{12}$ by $\mathcal{F}_{12}$ shows that $\overline{G}_n$ has at least $\lvert\mathrm{IrrProj}(2^{2n-1}{:}Sp(2n-2,2), \alpha^{-1})\rvert$ irreducible characters with degrees multiple of $2^{2n}-1$. $\square$

In the special cases $\overline{G}_5$ (resp. $\overline{G}_6$) the degrees of the irreducible characters contained in $\mathcal{K}_2$ which correspond to $H_2 = 2^9{:}Sp(10,2)$ (resp. $H_2 = 2^{11}{:}Sp(12,2)$) are multiples of 1023 (resp. 4095). Using the programm supplied in Section 2 one can obtain the character table of $\overline{G}_5$ (resp. $\overline{G}_6$), where one can re-arrange (see Remark 2.3 below) the rows and columns in such a way that the characters of $Sp(10,2)$ (resp. $Sp(12,2)$) form the block $\mathcal{K}_1$ and characters of $2^9{:}Sp(10,2)$ (resp. $2^{11}{:}Sp(12,2)$) form the other block $\mathcal{K}_2$. After this re-arrangement of characters, one can see that there is a character of $\overline{G}_5$ (resp. $\overline{G}_6$) contained in $\mathcal{K}_2$ of degree 1023 (resp. 4095). Recall from Lemma 4.2.4 of [2] that for any finite group K, if α is any non-trivial factor set of $M(K)$, then $\deg(\chi) > 1$ for any $\chi \in \mathrm{IrrProj}(K, \alpha^{-1})$. Now since $1023 \mid \deg(\chi)$ (resp. $4095 \mid \deg(\chi)$) for any $\chi \in \mathrm{Irr}(\overline{G}_5)$ (resp. $\chi \in \mathrm{Irr}(\overline{G}_6)$) such that χ is contained in $\mathcal{K}_2$ and since there is a character in $\mathcal{K}_2$ of degree 1023 (resp. 4095), it follows that the fragment (more precisely the column of degrees) of the projective character table of $2^9{:}Sp(10,2)$ (resp. $2^{11}{:}Sp(12,2)$) that we will use to construct (by means of Clifford-Fischer theory) the character table of $\overline{G}_5$ (resp. $\overline{G}_6$) contains a character of degree 1. This shows that the associated factor set α of $M(2^9{:}Sp(10,2))$ (resp. $M(2^{11}{:}Sp(12,2))$) is trivial. Hence we will use the ordinary character table of $2^9{:}Sp(10,2)$ (resp. $2^{11}{:}Sp(12,2)$) to construct the character table of $\overline{G}_5$ (resp. $\overline{G}_6$).

From the above, Eq. (2.1) becomes

(2.2) $|\mathrm{Irr}(2^{2n\cdot}Sp(2n,2))| = |\mathrm{Irr}(Sp(2n,2))| + |\mathrm{Irr}(2^{2n-1}{:}Sp(2n-2,2))|, 2 \leq n \leq 6.$

In Table 1 we list the number of ordinary irreducible characters of $\overline{G}_n = 2^{2n\cdot}Sp(2n,2)$ for small values of n.

TABLE 1. The number of ordinary irreducible characters of $\overline{G}_n = 2^{2n\cdot}Sp(2n,2)$, $n \in \{2,3,4,5,6\}$

n	$\lvert\mathrm{Irr}(Sp(2n,2))\rvert$	$\lvert\mathrm{Irr}(2^{2n-1}{:}Sp(2n-2,2))\rvert$	$\lvert\mathrm{Irr}(2^{2n\cdot}Sp(2n,2))\rvert$
2	11	10	21
3	30	37	67
4	81	114	195
5	198	322	520
6	477	839	1316

Remark 2.3. *Note that for the group $\overline{G}_5$ (resp. $\overline{G}_6$), we have used the programm, supplied at the beginning of this section to obtain its character table, which is not necessarily to be in the format of Clifford-Fischer theory. To re-arrange the characters of $\overline{G}_5$ (resp. $\overline{G}_6$) we only look at the two columns corresponding to the identity of $\overline{G}_5$ (resp. $\overline{G}_6$) and to the class of $\overline{G}_5$ (resp. $\overline{G}_6$) consisting of the 1023 (resp. 4095) involutions contained in the normal subgroup $N = 2^{10}$ (resp. $N = 2^{12}$). If the degree of a character was repeated in the column corresponding to the class of the 1023 (resp. 4095) involutions, then this character must be in block $\mathcal{K}_1$, otherwise it is in $\mathcal{K}_2$. Also note that we have used the information that we have on the character table of $\overline{G}_5$ (resp. $\overline{G}_6$) and the identity Fischer matrix of $\overline{G}_5$ (resp. $\overline{G}_6$) to determine the type of character table (projective or ordinary) of $2^9{:}Sp(10,2)$ (resp. $2^{11}{:}Sp(12,2)$) that is required.*

Conjecture: We speculate that for any $n \in \mathbb{N}^{\geq 2}$ we only need to use the ordinary character table of $H_2 = 2^{2n-1}{:}Sp(2n-2,2)$ to construct the character table of $\overline{G}_n$ and thus the theory of projective representations is not involved in the construction of the character tables of $\overline{G}_n$. That is to say, Eq. (2.2) is true for all $n \in \mathbb{N}^{\geq 2}$.

If the above conjecture is true, then by applications of Theorem 3 of [7] we obtain all the degrees of $\mathrm{Irr}(\overline{G}_n)$, which are listed below:

- the degrees of $\mathrm{Irr}(Sp(2n,2))$
- the degrees of $\mathrm{Irr}(Sp(2n-2,2)$
- the degrees of $\mathrm{Irr}(2^{2n-3}{:}Sp(2n-4,2)$ multiplied by $2^{2n-2}-1$
- the degrees of $\mathrm{Irr}(O^+(2n-2,2)$ multiplied by $2^{n-2}(2^{n-1}+1)$ and
- the degrees of $\mathrm{Irr}(O^-(2n-2,2)$ multiplied by $2^{n-2}(2^{n-1}-1)$.

3. Generators of the group $\overline{G} = 2^{8\cdot}Sp(8,2)$

From now on, we let $\overline{G}$ be the group $\overline{G}_4 = 2^{8\cdot}Sp(8,2)$. In [2] we produced two permutations $\overline{g}_1$ and $\overline{g}_2$ of the alternating simple group A_{512}, with $o(\overline{g}_1) = 17$, $o(\overline{g}_2) = 15$, $o(\overline{g}_1\overline{g}_2) = 14$ such that $\langle \overline{g}_1, \overline{g}_2 \rangle = \overline{G}$. The generators $\overline{g}_1$ and $\overline{g}_2$ both fix the points 9 and 42 and $\overline{G}$ acts transitively on the set $\Omega = \{1, 2, \cdots, 512\} \setminus \{9, 42\}$. Hence we have a permutation character $\chi(\overline{G}|\Omega) = \chi$ of degree 510. The values of χ on $\overline{G}-$classes were listed in Table 11.1 of [2] and using Table 11.13 of [2] we can see that $\chi = \chi_1 + \chi_5 + \chi_6 + \chi_{82}$.

Now having the group $\overline{G}$ constructed in GAP, it is easy to obtain all its normal subgroups. In fact the only non-trivial proper normal subgroup that $\overline{G}$ contains is a group of order 256 and thus must be isomorphic to the elementary abelian group $N = 2^8$. In [2] we also listed 8 permutations $n_1, n_2, \cdots, n_8$ of A_{512} that generate the normal subgroup N.

In Magma or GAP one can easily check for the complements of any normal subgroup N of $\overline{G}$. In our case the set of complements of $N = \langle n_1, n_2, \cdots, n_8 \rangle$ in $\overline{G} = \langle \overline{g}_1, \overline{g}_2 \rangle$, is empty. Also one can check that $\overline{G}/N \cong Sp(8,2)$. This shows that the group $\overline{G}$ constructed using the generators $\overline{g}_1$ and $\overline{g}_2$ is indeed a non-split extension of the elementary abelian group $N = 2^8$ by $Sp(8,2)$.

4. The conjugacy classes of $\overline{G} = 2^{8\cdot}Sp(8,2)$

In this section we use the method of the coset analysis, discussed in [2], to construct the character table of $\overline{G}$. We supply the conjugacy classes of $\overline{G}$ in Table 2, where in this table:

- k_i's represent the number of fixed points on the action of $N = \langle n_1, n_2, \cdots, n_8 \rangle$ on the coset $N\overline{g}_i$,
- f_{ij}'s represent the number of orbits (of the action of N on $N\overline{g}_i$) fused together under the action of $\overline{G} = \langle \overline{g}_1, \overline{g}_2 \rangle$,
- m_{ij}'s are weights (attached to each class of $\overline{G}$) that will be used later in computing the Fischer matrices of $\overline{G}$. These weights are computed through the formula

$$m_{ij} = [N_{\overline{G}}(N\overline{g}_i) : C_{\overline{G}}(g_{ij})] = |N| \frac{|C_G(g_i)|}{|C_{\overline{G}}(g_{ij})|}. \tag{4.1}$$

Table 2: The conjugacy classes of $\overline{G} = 2^{8\cdot}Sp(8,2)$

$[g_i]_G$	k_i	f_{ij}	m_{ij}	$[g_{ij}]_{\overline{G}}$	$o(g_{ij})$	$\lvert[g_{ij}]_{\overline{G}}\rvert$	$\lvert C_{\overline{G}}(g_{ij})\rvert$
$g_1 = 1A$	$k_1 = 256$	$f_{11} = 1$	$m_{11} = 1$	g_{11}	1	1	24257337753600
		$f_{12} = 255$	$m_{12} = 255$	g_{12}	2	255	47563407360
$g_2 = 2A$	$k_2 = 128$	$f_{21} = 1$	$m_{21} = 2$	g_{21}	4	510	23781703680
		$f_{22} = 63$	$m_{22} = 126$	g_{22}	4	32130	377487360
		$f_{23} = 64$	$m_{23} = 128$	g_{23}	2	32640	371589120
		$f_{31} = 1$	$m_{31} = 4$	g_{31}	2	21420	566231040

continued on next page

Table 2 (continued from previous page)

$[g_i]_G$	k_i	f_{ij}	m_{ij}	$[g_{ij}]_{\overline{G}}$	$o(g_{ij})$	$\|[g_{ij}]_{\overline{G}}\|$	$\|C_{\overline{G}}(g_{ij})\|$
$g_3 = 2B$	$k_3 = 64$	$f_{32} = 15$	$m_{32} = 60$	g_{32}	2	321300	37748736
		$f_{33} = 48$	$m_{33} = 192$	g_{33}	4	1028160	11796480
$g_4 = 2C$	$k_4 = 64$	$f_{41} = 1$	$m_{41} = 4$	g_{41}	4	64260	188743680
		$f_{42} = 15$	$m_{42} = 60$	g_{42}	4	963900	12582912
		$f_{43} = 15$	$m_{43} = 64$	g_{43}	2	1028160	11796480
		$f_{44} = 32$	$m_{44} = 128$	g_{44}	4	2056320	5898240
$g_5 = 2D$	$k_5 = 16$	$f_{51} = 1$	$m_{51} = 16$	g_{51}	2	1028160	11796480
		$f_{52} = 15$	$m_{52} = 240$	g_{52}	4	15422400	786432
$g_6 = 2E$	$k_6 = 32$	$f_{61} = 1$	$m_{61} = 8$	g_{61}	4	2570400	4718592
		$f_{62} = 3$	$m_{62} = 24$	g_{62}	4	7711200	1572864
		$f_{63} = 4$	$m_{63} = 32$	g_{63}	2	10281600	1179648
		$f_{64} = 12$	$m_{64} = 96$	g_{64}	4	30844800	393216
		$f_{65} = 12$	$m_{65} = 96$	g_{65}	4	30844800	393216
$g_7 = 2F$	$k_7 = 16$	$f_{71} = 1$	$m_{71} = 16$	g_{71}	4	15422400	786432
		$f_{72} = 1$	$m_{72} = 16$	g_{72}	2	15422400	786432
		$f_{73} = 6$	$m_{73} = 96$	g_{73}	4	92534400	131072
		$f_{74} = 8$	$m_{74} = 128$	g_{74}	4	123379200	98304
$g_8 = 3A$	$k_8 = 64$	$f_{81} = 1$	$m_{81} = 4$	g_{81}	3	43520	278691840
		$f_{82} = 63$	$m_{82} = 252$	g_{82}	6	2741760	4423680
$g_9 = 3B$	$k_9 = 1$	$f_{91} = 1$	$m_{91} = 256$	g_{91}	3	155975680	77760
$g_{10} = 3C$	$k_{10} = 16$	$f_{10,1} = 1$	$m_{10,1} = 16$	$g_{10,1}$	3	58490880	207360
		$f_{10,2} = 15$	$m_{10,2} = 240$	$g_{10,2}$	6	877363200	13824
$g_{11} = 3D$	$k_{11} = 4$	$f_{11,1} = 1$	$m_{11,1} = 64$	$g_{11,1}$	3	779878400	15552
		$f_{11,2} = 3$	$m_{11,2} = 192$	$g_{11,2}$	6	2339635200	5184
$g_{12} = 4A$	$k_{12} = 32$	$f_{12,1} = 1$	$m_{12,1} = 8$	$g_{12,1}$	8	4112640	2949120
		$f_{12,2} = 15$	$m_{12,2} = 120$	$g_{12,2}$	8	61689600	196608
		$f_{12,3} = 16$	$m_{12,3} = 128$	$g_{12,3}$	4	65802240	184320
$g_{13} = 4B$	$k_{13} = 32$	$f_{13,1} = 1$	$m_{13,1} = 8$	$g_{13,1}$	4	4112640	2949120
		$f_{13,2} = 15$	$m_{13,2} = 120$	$g_{13,2}$	8	61689600	196608
		$f_{13,3} = 16$	$m_{13,3} = 128$	$g_{13,3}$	4	65802240	184320
$g_{14} = 4C$	$k_{14} = 16$	$f_{14,1} = 1$	$m_{14,1} = 16$	$g_{14,1}$	4	20563200	589824
		$f_{14,2} = 3$	$m_{14,2} = 48$	$g_{14,2}$	4	61689600	196608
		$f_{14,3} = 12$	$m_{14,3} = 192$	$g_{14,3}$	4	246758400	49152
$g_{15} = 4D$	$k_{15} = 16$	$f_{15,1} = 1$	$m_{15,1} = 16$	$g_{15,1}$	4	61689600	196608
		$f_{15,2} = 3$	$m_{15,2} = 48$	$g_{15,2}$	4	185068800	65536
		$f_{15,3} = 4$	$m_{15,2} = 64$	$g_{15,2}$	4	246758400	49152
		$f_{15,4} = 8$	$m_{15,2} = 128$	$g_{15,2}$	4	493516800	24576
$g_{16} = 4E$	$k_{16} = 8$	$f_{16,1} = 1$	$m_{16,1} = 32$	$g_{16,1}$	8	246758400	49152
		$f_{16,2} = 3$	$m_{16,2} = 96$	$g_{16,2}$	8	740275200	16384
		$f_{16,3} = 4$	$m_{16,3} = 128$	$g_{16,3}$	4	987033600	12288
$g_{17} = 4F$	$k_{17} = 8$	$f_{17,1} = 1$	$m_{17,1} = 32$	$g_{17,1}$	8	246758400	49152
		$f_{17,2} = 3$	$m_{17,2} = 96$	$g_{17,2}$	8	740275200	16384
		$f_{17,3} = 4$	$m_{17,3} = 128$	$g_{17,3}$	4	987033600	12288
$g_{18} = 4G$	$k_{18} = 8$	$f_{18,1} = 1$	$m_{18,1} = 32$	$g_{18,1}$	4	493516800	24576
		$f_{18,2} = 1$	$m_{18,2} = 32$	$g_{18,2}$	4	493516800	24576
		$f_{18,3} = 3$	$m_{18,3} = 96$	$g_{18,3}$	4	1480550400	8192
		$f_{18,4} = 3$	$m_{18,4} = 96$	$g_{18,4}$	4	1480550400	8192
$g_{19} = 4H$	$k_{19} = 16$	$f_{19,1} = 1$	$m_{19,1} = 16$	$g_{19,1}$	8	246758400	49152
		$f_{19,2} = 3$	$m_{19,2} = 48$	$g_{19,2}$	8	740275200	16384
		$f_{19,3} = 4$	$m_{19,3} = 64$	$g_{19,3}$	8	987033600	12288
		$f_{19,4} = 4$	$m_{19,4} = 64$	$g_{19,4}$	4	987033600	12288
		$f_{19,5} = 4$	$m_{19,5} = 64$	$g_{19,5}$	4	987033600	12288

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Table 2 (continued from previous page)

$[g_i]_G$	k_i	f_{ij}	m_{ij}	$[g_{ij}]_{\overline{G}}$	$o(g_{ij})$	$\|[g_{ij}]_{\overline{G}}\|$	$\|C_{\overline{G}}(g_{ij})\|$
$g_{20} = 4I$	$k_{20} = 8$	$f_{20,1} = 1$	$m_{20,1} = 32$	$g_{20,1}$	8	1480550400	8192
		$f_{20,2} = 1$	$m_{20,2} = 32$	$g_{20,2}$	8	1480550400	8192
		$f_{20,3} = 2$	$m_{20,3} = 64$	$g_{20,3}$	8	2961100800	4096
		$f_{20,4} = 2$	$m_{20,4} = 64$	$g_{20,4}$	4	2961100800	4096
		$f_{20,5} = 2$	$m_{20,5} = 64$	$g_{20,5}$	4	2961100800	4096
$g_{21} = 4J$	$k_{21} = 4$	$f_{21,1} = 1$	$m_{21,1} = 64$	$g_{21,1}$	4	3948134400	3072
		$f_{21,2} = 3$	$m_{21,2} = 192$	$g_{21,2}$	8	11844403200	1024
$g_{22} = 4K$	$k_{22} = 4$	$f_{22,1} = 1$	$m_{22,1} = 64$	$g_{22,1}$	8	5922201600	2084
		$f_{22,2} = 1$	$m_{22,2} = 64$	$g_{22,2}$	4	5922201600	2084
		$f_{22,3} = 2$	$m_{22,3} = 128$	$g_{22,3}$	8	11844403200	1024
$g_{23} = 4L$	$k_{23} = 4$	$f_{23,1} = 1$	$m_{23,1} = 64$	$g_{23,1}$	8	5922201600	2084
		$f_{23,2} = 1$	$m_{23,2} = 64$	$g_{23,2}$	4	5922201600	2084
		$f_{23,3} = 2$	$m_{23,3} = 128$	$g_{23,3}$	8	5922201600	2084
$g_{24} = 5A$	$k_{24} = 16$	$f_{24,1} = 1$	$m_{24,1} = 16$	$g_{24,1}$	5	210567168	57600
		$f_{24,2} = 15$	$m_{24,2} = 240$	$g_{24,2}$	10	3158507520	3840
$g_{25} = 5B$	$k_{25} = 1$	$f_{25,1} = 1$	$m_{25,1} = 256$	$g_{25,1}$	5	40428896256	300
$g_{26} = 6A$	$k_{26} = 32$	$f_{26,1} = 1$	$m_{26,1} = 8$	$g_{26,1}$	12	5483520	2211840
		$f_{26,2} = 15$	$m_{26,2} = 120$	$g_{26,2}$	12	82252800	147456
		$f_{26,3} = 16$	$m_{26,3} = 128$	$g_{26,3}$	6	87736320	138240
$g_{27} = 6B$	$k_{27} = 16$	$f_{27,1} = 1$	$m_{27,1} = 16$	$g_{27,1}$	6	54835200	221184
		$f_{27,2} = 3$	$m_{27,2} = 48$	$g_{27,2}$	6	164505600	73728
		$f_{27,3} = 12$	$m_{27,3} = 192$	$g_{27,3}$	12	658022400	18432
$g_{28} = 6C$	$k_{28} = 16$	$f_{28,1} = 1$	$m_{28,1} = 16$	$g_{28,1}$	12	164505600	73728
		$f_{28,2} = 3$	$m_{28,2} = 48$	$g_{28,2}$	12	493516800	24576
		$f_{27,3} = 4$	$m_{28,3} = 64$	$g_{28,3}$	6	658022400	18432
		$f_{28,4} = 8$	$m_{28,4} = 128$	$g_{28,4}$	12	1316044800	9216
$g_{29} = 6D$	$k_{29} = 16$	$f_{29,1} = 1$	$m_{29,1} = 16$	$g_{29,1}$	6	175472640	69120
		$f_{29,2} = 15$	$m_{29,2} = 240$	$g_{29,2}$	6	2632089600	4608
$g_{30} = 6E$	$k_{30} = 1$	$f_{30,1} = 1$	$m_{30,1} = 256$	$g_{30,1}$	6	7018905600	1728
$g_{31} = 6F$	$k_{31} = 2$	$f_{31,1} = 1$	$m_{31,1} = 128$	$g_{31,1}$	6	4679270400	2592
		$f_{31,2} = 1$	$m_{31,2} = 128$	$g_{31,2}$	12	4679270400	2592
$g_{32} = 6G$	$k_{32} = 8$	$f_{32,1} = 1$	$m_{32,1} = 32$	$g_{32,1}$	12	1316044800	9216
		$f_{32,2} = 1$	$m_{32,2} = 32$	$g_{32,2}$	6	1316044800	9216
		$f_{32,3} = 3$	$m_{32,3} = 96$	$g_{32,3}$	12	3948134400	3072
		$f_{32,4} = 3$	$m_{32,4} = 96$	$g_{32,4}$	12	3948134400	3072
$g_{33} = 6H$	$k_{33} = 8$	$f_{33,1} = 1$	$m_{33,1} = 32$	$g_{33,1}$	12	1754726400	6912
		$f_{33,2} = 3$	$m_{33,2} = 96$	$g_{33,2}$	12	5264179200	2304
		$f_{33,3} = 4$	$m_{33,3} = 128$	$g_{33,3}$	6	7018905600	1728
$g_{34} = 6I$	$k_{34} = 4$	$f_{34,1} = 1$	$m_{34,1} = 64$	$g_{34,1}$	6	3509452800	3456
		$f_{34,2} = 3$	$m_{34,2} = 192$	$g_{34,2}$	12	10528358400	1152
$g_{35} = 6J$	$k_{35} = 4$	$f_{35,1} = 1$	$m_{35,1} = 64$	$g_{35,1}$	6	7018905600	1728
		$f_{35,2} = 3$	$m_{35,2} = 192$	$g_{35,2}$	6	21056716800	576
$g_{36} = 6K$	$k_{36} = 1$	$f_{36,1} = 1$	$m_{36,1} = 256$	$g_{36,1}$	6	42113433600	288
$g_{37} = 6L$	$k_{37} = 4$	$f_{37,1} = 1$	$m_{37,1} = 64$	$g_{37,1}$	12	10528358400	1152
		$f_{37,2} = 1$	$m_{37,2} = 64$	$g_{37,2}$	6	10528358400	1152
		$f_{37,3} = 2$	$m_{37,3} = 128$	$g_{37,3}$	12	21056716800	576
$g_{38} = 6M$	$k_{38} = 4$	$f_{38,1} = 1$	$m_{38,1} = 64$	$g_{38,1}$	6	10528358400	1152
		$f_{38,2} = 3$	$m_{38,2} = 192$	$g_{38,2}$	12	31585075200	384

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Table 2 (continued from previous page)

$[g_i]_G$	k_i	f_{ij}	m_{ij}	$[g_{ij}]_{\overline{G}}$	$o(g_{ij})$	$\lvert[g_{ij}]_{\overline{G}}\rvert$	$\lvert C_{\overline{G}}(g_{ij})\rvert$
$g_{39} = 6N$	$k_{39} = 8$	$f_{39,1} = 1$	$m_{39,1} = 32$	$g_{39,1}$	12	5264179200	2304
		$f_{39,2} = 3$	$m_{39,2} = 96$	$g_{39,2}$	12	15792537600	768
		$f_{39,3} = 4$	$m_{39,3} = 128$	$g_{39,3}$	6	21056716800	576
$g_{40} = 6O$	$k_{40} = 2$	$f_{40,1} = 1$	$m_{40,1} = 128$	$g_{40,1}$	12	42113433600	288
		$f_{40,2} = 1$	$m_{40,2} = 128$	$g_{40,2}$	6	42113433600	288
$g_{41} = 6P$	$k_{41} = 4$	$f_{41,1} = 1$	$m_{41,1} = 64$	$g_{41,1}$	12	31585075200	384
		$f_{41,2} = 1$	$m_{41,2} = 64$	$g_{41,2}$	6	31585075200	384
		$f_{41,3} = 2$	$m_{41,3} = 128$	$g_{41,3}$	12	63170150400	192
$g_{42} = 7A$	$k_{42} = 4$	$f_{42,1} = 1$	$m_{42,1} = 64$	$g_{42,1}$	7	72194457600	168
		$f_{42,2} = 3$	$m_{42,2} = 192$	$g_{42,2}$	14	216583372800	56
$g_{43} = 8A$	$k_{43} = 8$	$f_{43,1} = 1$	$m_{43,1} = 32$	$g_{43,1}$	8	3948134400	3072
		$f_{43,2} = 3$	$m_{43,2} = 96$	$g_{43,2}$	8	11844403200	1024
		$f_{43,3} = 4$	$m_{43,3} = 128$	$g_{43,3}$	8	15792537600	768
$g_{44} = 8B$	$k_{44} = 8$	$f_{44,1} = 1$	$m_{44,1} = 32$	$g_{44,1}$	8	3948134400	3072
		$f_{44,2} = 3$	$m_{44,2} = 96$	$g_{44,2}$	8	11844403200	1024
		$f_{44,3} = 4$	$m_{44,3} = 128$	$g_{44,3}$	8	15792537600	768
$g_{45} = 8C$	$k_{45} = 4$	$f_{45,1} = 1$	$m_{45,1} = 64$	$g_{45,1}$	8	23688806400	512
		$f_{45,2} = 1$	$m_{45,2} = 64$	$g_{45,2}$	8	23688806400	512
		$f_{45,3} = 2$	$m_{45,3} = 128$	$g_{45,3}$	8	47377612800	256
$g_{46} = 8D$	$k_{46} = 4$	$f_{46,1} = 1$	$m_{46,1} = 64$	$g_{46,1}$	8	23688806400	512
		$f_{46,2} = 1$	$m_{46,2} = 64$	$g_{46,2}$	8	23688806400	512
		$f_{46,3} = 2$	$m_{46,3} = 128$	$g_{46,3}$	8	47377612800	256
$g_{47} = 8E$	$k_{47} = 2$	$f_{47,1} = 1$	$m_{47,1} = 128$	$g_{47,1}$	16	189510451200	64
		$f_{47,2} = 1$	$m_{47,2} = 128$	$g_{47,2}$	8	189510451200	64
$g_{48} = 8F$	$k_{48} = 2$	$f_{48,1} = 1$	$m_{48,1} = 128$	$g_{48,1}$	16	189510451200	64
		$f_{48,2} = 1$	$m_{48,2} = 128$	$g_{48,2}$	8	189510451200	64
$g_{49} = 9A$	$k_{49} = 4$	$f_{49,1} = 1$	$m_{49,1} = 64$	$g_{49,1}$	9	56151244800	216
		$f_{49,2} = 3$	$m_{49,2} = 192$	$g_{49,2}$	18	168453734400	72
$g_{50} = 9B$	$k_{50} = 1$	$f_{50,1} = 1$	$m_{50,1} = 256$	$g_{50,1}$	9	449209958400	27
$g_{51} = 10A$	$k_{51} = 8$	$f_{51,1} = 1$	$m_{51,1} = 32$	$g_{51,1}$	20	6317015040	1920
		$f_{51,2} = 3$	$m_{51,2} = 96$	$g_{51,2}$	20	18951045120	640
		$f_{51,3} = 4$	$m_{51,3} = 128$	$g_{51,3}$	10	25268060160	480
$g_{52} = 10B$	$k_{52} = 4$	$f_{52,1} = 1$	$m_{52,1} = 64$	$g_{52,1}$	10	12634030080	960
		$f_{52,2} = 3$	$m_{52,2} = 192$	$g_{52,2}$	20	37902090240	320
$g_{53} = 10C$	$k_{53} = 4$	$f_{53,1} = 1$	$m_{53,1} = 64$	$g_{53,1}$	20	37902090240	320
		$f_{53,2} = 1$	$m_{53,2} = 64$	$g_{53,2}$	10	37902090240	320
		$f_{53,3} = 2$	$m_{53,3} = 128$	$g_{53,3}$	20	75804180480	160
$g_{54} = 10D$	$k_{54} = 1$	$f_{54,1} = 1$	$m_{54,1} = 256$	$g_{54,1}$	10	606433443840	20
$g_{55} = 12A$	$k_{55} = 4$	$f_{55,1} = 1$	$m_{55,1} = 64$	$g_{55,1}$	12	2632089600	4608
		$f_{55,2} = 3$	$m_{55,2} = 192$	$g_{55,2}$	12	7896268800	1536
$g_{56} = 12B$	$k_{56} = 8$	$f_{56,1} = 1$	$m_{56,1} = 32$	$g_{56,1}$	24	2632089600	4608
		$f_{56,2} = 3$	$m_{56,2} = 96$	$g_{56,2}$	24	7896268800	1536
		$f_{56,3} = 4$	$m_{56,3} = 128$	$g_{56,3}$	12	10528358400	1152
$g_{57} = 12C$	$k_{57} = 8$	$f_{57,1} = 1$	$m_{57,1} = 32$	$g_{57,1}$	24	2632089600	4608
		$f_{57,2} = 3$	$m_{57,2} = 96$	$g_{57,2}$	24	7896268800	1536
		$f_{57,3} = 4$	$m_{57,3} = 128$	$g_{57,3}$	12	10528358400	1152
$g_{58} = 12D$	$k_{58} = 4$	$f_{58,1} = 1$	$m_{58,1} = 64$	$g_{58,1}$	12	7896268800	1536
		$f_{58,2} = 1$	$m_{58,2} = 64$	$g_{58,2}$	12	7896268800	1536
		$f_{58,3} = 2$	$m_{58,3} = 128$	$g_{58,3}$	12	15792537600	768

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Table 2 (continued from previous page)

$[g_i]_G$	k_i	f_{ij}	m_{ij}	$[g_{ij}]_{\overline{G}}$	$o(g_{ij})$	$\lvert[g_{ij}]_{\overline{G}}\rvert$	$\lvert C_{\overline{G}}(g_{ij})\rvert$
$g_{59} = 12E$	$k_{59} = 1$	$f_{59,1} = 1$	$m_{59,1} = 256$	$g_{59,1}$	12	84226867200	144
$g_{60} = 12F$	$k_{60} = 2$	$f_{60,1} = 1$	$m_{60,1} = 128$	$g_{60,1}$	24	42113433600	288
		$f_{60,2} = 1$	$m_{60,2} = 128$	$g_{60,2}$	12	42113433600	288
$g_{61} = 12G$	$k_{61} = 2$	$f_{61,1} = 1$	$m_{61,1} = 128$	$g_{61,1}$	24	42113433600	288
		$f_{61,2} = 1$	$m_{61,2} = 128$	$g_{61,2}$	12	42113433600	288
$g_{62} = 12H$	$k_{62} = 4$	$f_{62,1} = 1$	$m_{62,1} = 64$	$g_{62,1}$	24	31585075200	384
		$f_{62,2} = 1$	$m_{62,2} = 64$	$g_{62,2}$	24	31585075200	384
		$f_{62,3} = 1$	$m_{62,3} = 64$	$g_{62,3}$	12	31585075200	384
		$f_{62,4} = 1$	$m_{62,4} = 64$	$g_{62,4}$	12	31585075200	384
$g_{63} = 12I$	$k_{63} = 4$	$f_{63,1} = 1$	$m_{63,1} = 64$	$g_{63,1}$	12	42113433600	288
		$f_{63,2} = 3$	$m_{63,2} = 192$	$g_{63,2}$	12	126340300800	96
$g_{64} = 12J$	$k_{64} = 2$	$f_{64,1} = 1$	$m_{64,1} = 128$	$g_{64,1}$	24	126340300800	96
		$f_{64,2} = 1$	$m_{64,2} = 128$	$g_{64,2}$	12	126340300800	96
$g_{65} = 12K$	$k_{65} = 2$	$f_{65,1} = 1$	$m_{65,1} = 128$	$g_{65,1}$	24	126340300800	96
		$f_{65,2} = 1$	$m_{65,2} = 128$	$g_{65,2}$	12	126340300800	96
$g_{66} = 12L$	$k_{66} = 2$	$f_{66,1} = 1$	$m_{66,1} = 128$	$g_{66,1}$	12	252680601600	48
		$f_{66,2} = 1$	$m_{66,2} = 128$	$g_{66,2}$	12	252680601600	48
$g_{67} = 12M$	$k_{67} = 1$	$f_{67,1} = 1$	$m_{67,1} = 256$	$g_{67,1}$	12	505361203200	24
$g_{68} = 14A$	$k_{68} = 2$	$f_{68,1} = 1$	$m_{68,1} = 128$	$g_{68,1}$	28	433166745600	28
		$f_{68,2} = 1$	$m_{68,2} = 128$	$g_{68,2}$	14	433166745600	28
$g_{69} = 15A$	$k_{69} = 4$	$f_{69,1} = 1$	$m_{69,1} = 64$	$g_{69,1}$	15	33690746880	360
		$f_{69,2} = 3$	$m_{69,2} = 192$	$g_{69,2}$	30	101072240640	120
$g_{70} = 15B$	$k_{70} = 1$	$f_{70,1} = 1$	$m_{70,1} = 256$	$g_{70,1}$	15	134762987520	90
$g_{71} = 15C$	$k_{71} = 1$	$f_{71,1} = 1$	$m_{71,1} = 256$	$g_{71,1}$	15	808577925120	15
$g_{72} = 17A$	$k_{72} = 1$	$f_{72,1} = 1$	$m_{72,1} = 256$	$g_{72,1}$	17	713451110400	17
$g_{73} = 17B$	$k_{73} = 1$	$f_{73,1} = 1$	$m_{73,1} = 256$	$g_{73,1}$	17	713451110400	17
$g_{74} = 18A$	$k_{74} = 2$	$f_{74,1} = 1$	$m_{74,1} = 128$	$g_{74,1}$	36	336907468800	36
		$f_{74,2} = 1$	$m_{74,2} = 128$	$g_{74,2}$	18	336907468800	36
$g_{75} = 20A$	$k_{75} = 2$	$f_{75,1} = 1$	$m_{75,1} = 128$	$g_{75,1}$	40	151608360960	80
		$f_{75,2} = 1$	$m_{75,2} = 128$	$g_{75,2}$	20	151608360960	80
$g_{76} = 20B$	$k_{76} = 2$	$f_{76,1} = 1$	$m_{76,1} = 128$	$g_{76,1}$	40	151608360960	80
		$f_{76,2} = 1$	$m_{76,2} = 128$	$g_{76,2}$	20	151608360960	80
$g_{77} = 21A$	$k_{77} = 1$	$f_{77,1} = 1$	$m_{77,1} = 256$	$g_{77,1}$	21	577555660800	21
$g_{78} = 24A$	$k_{78} = 2$	$f_{78,1} = 1$	$m_{78,1} = 128$	$g_{78,1}$	24	126340300800	96
		$f_{78,2} = 1$	$m_{78,2} = 128$	$g_{78,2}$	24	126340300800	96
$g_{79} = 24B$	$k_{79} = 2$	$f_{79,1} = 1$	$m_{79,1} = 128$	$g_{79,1}$	24	126340300800	96
		$f_{79,2} = 1$	$m_{79,2} = 128$	$g_{79,2}$	24	126340300800	96
$g_{80} = 30A$	$k_{80} = 2$	$f_{80,1} = 1$	$m_{80,1} = 128$	$g_{80,1}$	600	202144481280	60
		$f_{80,2} = 1$	$m_{80,2} = 128$	$g_{80,2}$	30	202144481280	60
$g_{81} = 30B$	$k_{81} = 1$	$f_{81,1} = 1$	$m_{81,1} = 256$	$g_{81,1}$	30	404288962560	30

Remark 4.1. *Note that from Table 2, the group $\overline{G}$ contains 8 conjugacy classes of involutions, while from the character table of the split extension $2^8{:}Sp(8,2)$ (see GAP), there are 11 conjugacy classes of involutions. This confirms that the group $\overline{G}$ constructed using the generators $\overline{g}_1$ and $\overline{g}_2$ is different from the*

group $2^8{:}Sp(8,2)$ while in [2] it has been shown that the character tables of the two groups are the same.

5. The inertia factor groups of $\overline{G} = 2^{8\cdot}Sp(8,2)$

From Table 1 we can see that $|\mathrm{Irr}(\overline{G})| = |\mathrm{Irr}(\overline{G}_4)| = |\mathrm{Irr}(2^{8\cdot}Sp(8,2))| = 195$, which is the number of conjugacy classes of $\overline{G}$, listed in Table 2. Moreover, these 195 characters of $\overline{G}$ are partitioned into two blocks of characters, where the first block consists of 81 characters which correspond to $H_1 = G_4 = Sp(8,2)$, while the other block consists of 114 characters which correspond to $H_2 = 2^7{:}Sp(6,2)$. The character table of $H_1 = Sp(8,2)$ is available in the $\mathbb{ATLAS}$ or can be obtained from GAP or Magma. We will use the character table of H_2 given as Table 6.7 of Ali [1] to construct the character table of $\overline{G}$. The fusion of the conjugacy classes of $H_2 = 2^7{:}Sp(6,2)$ into classes of $Sp(8,2)$ can be found in Table 9.3 of [2]

6. Fischer matrices of $\overline{G} = 2^{8\cdot}Sp(8,2)$

In this section, we use the arithmetical properties of Fischer matrices, given by Proposition 3.6 of [5], to calculate some of the entries of the Fischer matrices and also to build an algebraic system of equations. To build these systems of equations, we first recall that we label the top and bottom of the columns of the Fischer matrix $\mathcal{F}_i$, corresponding to g_i, by the sizes of the centralizers of $g_{ij},\ 1 \le j \le c(g_i)$ in $\overline{G}$ and m_{ij}, respectively. In Table 2 we supplied $|C_{\overline{G}}(g_{ij})|$ and $m_{ij},\ 1 \le i \le 81,\ 1 \le j \le c(g_i)$. Also having obtained the fusions of the inertia factor group H_2 into $Sp(8,2)$, we are able to label the rows of the Fischer matrices as described in [2] and [5]. Since the size of the Fischer matrix $\mathcal{F}_i$ is $c(g_i)$, it follows from Table 2 that the sizes of the Fischer matrices of $\overline{G}$ range between 1 and 5 for every $i \in \{1, 2, \cdots, 81\}$.

Now with the help of the symbolic mathematical package Maxima [11], we were able to solve the systems of equations and hence we have computed all the Fischer matrices of $\overline{G}$, which we list below.

$\mathcal{F}_1$

g_1		g_{11}	g_{12}
$o(g_{1j})$		1	2
$\lvert C_{\overline{G}}(g_{1j})\rvert$		12128668876800	47563407360
(k,m)	$\lvert C_{H_k}(g_{1km})\rvert$		
$(1,1)$	47377612800	1	1
$(2,1)$	185794560	255	-1
m_{1j}		1	255

$\mathcal{F}_2$

g_2		g_{21}	g_{22}	g_{23}
$o(g_{2j})$		4	4	2
$\lvert C_{\overline{G}}(g_{2j})\rvert$		23781703680	377487360	371589120
(k,m)	$\lvert C_{H_k}(g_{2km})\rvert$			
$(1,1)$	185794560	1	1	1
$(2,1)$	185794560	1	1	-1
$(2,2)$	1474560	126	-2	0
m_{2j}		2	126	128

$\mathcal{F}_3$

g_3		g_{31}	g_{32}	g_{33}
$o(g_{3j})$		2	2	4
$\lvert C_{\overline{G}}(g_{3j})\rvert$		566231040	37748736	11796480
(k,m)	$\lvert C_{H_k}(g_{3km})\rvert$			
$(1,1)$	8847360	1	1	1
$(2,1)$	2949120	3	3	-1
$(2,2)$	147456	60	-4	0
m_{3j}		4	60	192

$\mathcal{F}_4$

g_4		g_{41}	g_{42}	g_{43}	g_{44}
$o(g_{4j})$		4	4	2	4
$\lvert C_{\overline{G}}(g_{4j})\rvert$		188743680	12582912	11796480	5898240
(k,m)	$\lvert C_{H_k}(g_{4km})\rvert$				
$(1,1)$	2949120	1	1	1	1
$(2,1)$	2949120	1	1	1	-1
$(2,2)$	1474560	2	2	-2	0
$(2,3)$	49152	60	-4	0	0
m_{4j}		4	60	64	128

$\mathcal{F}_5$

g_5		g_{51}	g_{52}
$o(g_{5j})$		2	4
$\lvert C_{\overline{G}}(g_{5j})\rvert$		11796480	786432
(k,m)	$\lvert C_{H_k}(g_{5km})\rvert$		
$(1,1)$	737280	1	1
$(2,1)$	49152	15	-1
m_{5j}		16	240

$\mathcal{F}_6$

g_6		g_{61}	g_{62}	g_{63}	g_{64}	g_{65}
$o(g_{6j})$		4	4	2	4	4
$\lvert C_{\overline{G}}(g_{6j})\rvert$		4718592	1572864	1179648	393216	393216
(k,m)	$\lvert C_{H_k}(g_{6km})\rvert$					
$(1,1)$	147456	1	1	1	1	1
$(2,1)$	49152	3	3	-3	1	-1
$(2,2)$	147456	1	1	-1	-1	1
$(2,3)$	49152	3	3	3	-1	-1
$(2,4)$	6144	24	-8	0	0	0
m_{6j}		8	24	32	96	96

$\mathcal{F}_7$

g_7		g_{71}	g_{72}	g_{73}	g_{74}
$o(g_{7j})$		4	2	4	4
$\lvert C_{\overline{G}}(g_{7j})\rvert$		786432	786432	131072	98304
(k,m)	$\lvert C_{H_k}(g_{7km})\rvert$				
$(1,1)$	49152	1	1	1	1
$(2,1)$	49152	1	1	1	-1
$(2,2)$	8192	6	6	-2	0
$(2,3)$	6144	8	-8	0	0
m_{7j}		16	16	96	128

$\mathcal{F}_8$

g_8		g_{81}	g_{82}
$o(g_{8j})$		3	6
$\lvert C_{\overline{G}}(g_{8j})\rvert$		278691840	4423680
(k,m)	$\lvert C_{H_k}(g_{8km})\rvert$		
$(1,1)$	4354560	1	1
$(2,1)$	69120	63	-1
m_{8j}		4	252

$\mathcal{F}_9$

g_9		g_{91}
$o(g_{9j})$		3
$\lvert C_{\overline{G}}(g_{9j})\rvert$		77760
(k,m)	$\lvert C_{H_k}(g_{9km})\rvert$	
$(1,1)$	77760	1
m_{9j}		256

$\mathcal{F}_{10}$

g_{10}		$g_{10,1}$	$g_{10,2}$
$o(g_{10j})$		3	6
$\lvert C_{\overline{G}}(g_{10j})\rvert$		207360	13824
(k,m)	$\lvert C_{H_k}(g_{10km})\rvert$		
$(1,1)$	12960	1	1
$(2,1)$	864	15	-1
m_{10j}		16	240

$\mathcal{F}_{11}$

g_{11}		$g_{11,1}$	$g_{11,2}$
$o(g_{11j})$		3	6
$\lvert C_{\overline{G}}(g_{11j})\rvert$		15552	5184
(k,m)	$\lvert C_{H_k}(g_{11km})\rvert$		
$(1,1)$	3888	1	1
$(2,1)$	1296	3	-1
m_{11j}		64	192

$\mathcal{F}_{12}$

g_{12}		$g_{12,1}$	$g_{12,2}$	$g_{12,3}$
$o(g_{12j})$		8	8	4
$\lvert C_{\overline{G}}(g_{12j})\rvert$		2949120	196608	184320
(k,m)	$\lvert C_{H_k}(g_{12km})\rvert$			
$(1,1)$	92160	1	1	1
$(2,1)$	92160	1	1	-1
$(2,2)$	3072	30	-2	0
m_{12j}		8	120	128

$\mathcal{F}_{13}$

g_{13}		$g_{13,1}$	$g_{13,2}$	$g_{13,3}$
$o(g_{13j})$		8	8	4
$\lvert C_{\overline{G}}(g_{13j})\rvert$		2949120	196608	184320
(k,m)	$\lvert C_{H_k}(g_{13km})\rvert$			
$(1,1)$	92160	1	1	1
$(2,1)$	92160	1	1	-1
$(2,2)$	3072	30	-2	0
m_{13j}		8	120	128

$\mathcal{F}_{14}$

g_{14}		$g_{14,1}$	$g_{14,2}$	$g_{14,3}$
$o(g_{14j})$		4	4	4
$\lvert C_{\overline{G}}(g_{14j})\rvert$		589824	196608	49152
(k,m)	$\lvert C_{H_k}(g_{14km})\rvert$			
$(1,1)$	36864	1	1	1
$(2,1)$	12288	3	3	-1
$(2,2)$	3072	12	-4	0
m_{14j}		16	48	192

$\mathcal{F}_{15}$

g_{15}		$g_{15,1}$	$g_{15,2}$	$g_{15,3}$	$g_{15,4}$
$o(g_{15j})$		4	4	4	4
$\lvert C_{\overline{G}}(g_{15j})\rvert$		196608	65536	49152	24576
(k,m)	$\lvert C_{H_k}(g_{15km})\rvert$				
$(1,1)$	12288	1	1	1	1
$(2,1)$	6144	2	2	-2	0
$(2,2)$	12288	1	1	1	-1
$(2,3)$	1024	12	-4	0	0
m_{15j}		16	48	64	128

$\mathcal{F}_{16}$

g_{16}		$g_{16,1}$	$g_{16,2}$	$g_{16,3}$
$o(g_{16j})$		8	8	4
$\lvert C_{\overline{G}}(g_{16j})\rvert$		49152	16384	12288
(k,m)	$\lvert C_{H_k}(g_{16km})\rvert$			
$(1,1)$	6144	1	1	1
$(2,1)$	6144	1	1	-1
$(2,2)$	1024	6	-2	0
m_{16j}		32	96	128

$\mathcal{F}_{17}$

g_{17}		$g_{17,1}$	$g_{17,2}$	$g_{17,3}$
$o(g_{17j})$		8	8	4
$\lvert C_{\overline{G}}(g_{17j})\rvert$		49152	16384	12288
(k,m)	$\lvert C_{H_k}(g_{17km})\rvert$			
$(1,1)$	6144	1	1	1
$(2,1)$	6144	1	1	-1
$(2,2)$	1024	6	-2	0
m_{17j}		32	96	128

$\mathcal{F}_{18}$

g_{18}		$g_{18,1}$	$g_{18,2}$	$g_{18,3}$	$g_{18,4}$
$o(g_{18j})$		4	4	4	4
$\lvert C_{\overline{G}}(g_{18j})\rvert$		24576	24576	8192	8192
(k,m)	$\lvert C_{H_k}(g_{18km})\rvert$				
$(1,1)$	3072	1	1	1	1
$(2,1)$	1024	3	3	-1	1
$(2,2)$	3072	1	-1	-1	-1
$(2,3)$	1024	3	-3	1	-1
m_{18j}		32	32	96	96

$\mathcal{F}_{19}$

g_{19}		$g_{19,1}$	$g_{19,2}$	$g_{19,3}$	$g_{19,4}$	$g_{19,5}$
$o(g_{19,j})$		8	8	8	4	4
$\|C_{\overline{G}}(g_{19j})\|$		49152	16384	12288	12288	12288
(k,m)	$\|C_{H_k}(g_{19km})\|$					
$(1,1)$	3072	1	1	1	1	1
$(2,1)$	3072	1	1	-1	1	-1
$(2,2)$	3072	1	1	-1	-1	1
$(2,3)$	3072	1	1	1	-1	-1
$(2,4)$	256	12	-4	0	0	0
m_{19j}		16	48	64	64	64

$\mathcal{F}_{20}$

g_{20}		$g_{20,1}$	$g_{20,2}$	$g_{20,3}$	$g_{20,4}$	$g_{20,5}$
$o(g_{20,j})$		8	8	8	4	4
$\|C_{\overline{G}}(g_{20j})\|$		8192	8192	4096	4096	4096
(k,m)	$\|C_{H_k}(g_{20km})\|$					
$(1,1)$	1024	1	1	1	1	1
$(2,1)$	1024	1	1	-1	1	-1
$(2,2)$	1024	1	1	-1	-1	1
$(2,3)$	1024	1	1	1	-1	-1
$(2,4)$	256	4	-4	0	0	0
m_{20j}		32	32	64	64	64

$\mathcal{F}_{21}$

g_{21}		$g_{21,1}$	$g_{21,2}$
$o(g_{21j})$		4	8
$\|C_{\overline{G}}(g_{21j})\|$		3072	1024
(k,m)	$\|C_{H_k}(g_{21km})\|$		
$(1,1)$	768	1	1
$(2,1)$	256	3	-1
m_{21j}		64	192

$\mathcal{F}_{22}$

g_{22}		$g_{22,1}$	$g_{22,2}$	$g_{22,3}$
$o(g_{22j})$		8	4	8
$\|C_{\overline{G}}(g_{22j})\|$		2048	2048	1024
(k,m)	$\|C_{H_k}(g_{22km})\|$			
$(1,1)$	512	1	1	1
$(2,1)$	512	1	1	-1
$(2,2)$	256	2	-2	0
m_{22j}		64	64	128

$\mathcal{F}_{23}$

g_{23}		$g_{23,1}$	$g_{23,2}$	$g_{23,3}$
$o(g_{23j})$		8	4	8
$\|C_{\overline{G}}(g_{23j})\|$		2048	2048	1024
(k,m)	$\|C_{H_k}(g_{23km})\|$			
$(1,1)$	512	1	1	1
$(2,1)$	512	1	1	-1
$(2,1)$	256	2	-2	0
m_{23j}		64	64	128

$\mathcal{F}_{24}$

g_{24}		$g_{24,1}$	$g_{24,2}$
$o(g_{24j})$		5	10
$\|C_{\overline{G}}(g_{24j})\|$		57600	3840
(k,m)	$\|C_{H_k}(g_{24km})\|$		
$(1,1)$	3600	1	1
$(2,1)$	240	15	-1
m_{24j}		16	240

$\mathcal{F}_{25}$

g_{25}		$g_{25,1}$
$o(g_{25j})$		5
$\|C_{\overline{G}}(g_{25j})\|$		300
(k,m)	$\|C_{H_k}(g_{25km})\|$	
$(1,1)$	300	1
m_{25j}		256

$\mathcal{F}_{26}$

g_{26}		$g_{26,1}$	$g_{26,2}$	$g_{26,3}$
$o(g_{26j})$		12	12	6
$\lvert C_{\overline{G}}(g_{26j})\rvert$		2211840	147456	138240
(k,m)	$\lvert C_{H_k}(g_{26km})\rvert$			
$(1,1)$	69120	1	1	1
$(2,1)$	69120	1	1	-1
$(2,2)$	2304	30	-2	0
m_{26j}		8	120	128

$\mathcal{F}_{27}$

g_{27}		$g_{27,1}$	$g_{27,2}$	$g_{27,3}$
$o(g_{27j})$		6	6	12
$\lvert C_{\overline{G}}(g_{27j})\rvert$		221184	73728	18432
(k,m)	$\lvert C_{H_k}(g_{27km})\rvert$			
$(1,1)$	13824	1	1	1
$(2,1)$	4608	3	3	-1
$(2,2)$	1152	12	-4	0
m_{27j}		16	48	192

$\mathcal{F}_{28}$

g_{28}		$g_{28,1}$	$g_{28,2}$	$g_{28,3}$	$g_{28,4}$
$o(g_{28j})$		12	12	6	12
$\lvert C_{\overline{G}}(g_{28j})\rvert$		73728	24576	18432	9216
(k,m)	$\lvert C_{H_k}(g_{28km})\rvert$				
$(1,1)$	4608	1	1	1	1
$(2,1)$	4608	1	1	1	-1
$(2,2)$	2304	2	2	-2	0
$(2,3)$	384	12	-4	0	0
m_{28j}		16	48	64	128

$\mathcal{F}_{29}$

g_{29}		$g_{29,1}$	$g_{29,2}$
$o(g_{29j})$		6	6
$\lvert C_{\overline{G}}(g_{29j})\rvert$		69120	4608
(k,m)	$\lvert C_{H_k}(g_{29km})\rvert$		
$(1,1)$	4320	1	1
$(2,1)$	288	15	-1
m_{29j}		16	240

$\mathcal{F}_{30}$

g_{30}		$g_{30,1}$
$o(g_{30j})$		6
$\lvert C_{\overline{G}}(g_{30j})\rvert$		1728
(k,m)	$\lvert C_{H_k}(g_{30km})\rvert$	
$(1,1)$	1728	1
m_{30j}		256

$\mathcal{F}_{31}$

g_{31}		$g_{31,1}$	$g_{31,2}$
$o(g_{31j})$		6	12
$\lvert C_{\overline{G}}(g_{31j})\rvert$		2592	2592
(k,m)	$\lvert C_{H_k}(g_{31km})\rvert$		
$(1,1)$	1296	1	1
$(2,1)$	1296	1	-1
m_{31j}		128	128

$\mathcal{F}_{32}$

g_{32}		$g_{32,1}$	$g_{32,2}$	$g_{32,3}$	$g_{32,4}$
$o(g_{32j})$		12	6	12	12
$\lvert C_{\overline{G}}(g_{32j})\rvert$		9216	9216	3072	3072
(k,m)	$\lvert C_{H_k}(g_{32km})\rvert$				
$(1,1)$	1152	1	1	1	1
$(2,1)$	384	3	3	-1	1
$(2,2)$	1152	1	-1	-1	-1
$(2,3)$	384	3	-3	1	-1
m_{32j}		32	32	96	96

$\mathcal{F}_{33}$

g_{33}		$g_{33,1}$	$g_{33,2}$	$g_{33,3}$
$o(g_{33j})$		12	12	6
$\lvert C_{\overline{G}}(g_{33j})\rvert$		6912	2304	1728
(k,m)	$\lvert C_{H_k}(g_{33km})\rvert$			
$(1,1)$	864	1	1	1
$(2,1)$	864	1	1	-1
$(2,2)$	144	6	-2	0
m_{33j}		32	96	128

$\mathcal{F}_{34}$

g_{34}		$g_{34,1}$	$g_{34,2}$
$o(g_{34j})$		6	12
$\lvert C_{\overline{G}}(g_{34j})\rvert$		3456	1152
(k,m)	$\lvert C_{H_k}(g_{34km})\rvert$		
$(1,1)$	864	1	1
$(2,1)$	288	3	-1
m_{34j}		64	192

$\mathcal{F}_{35}$

g_{35}		$g_{35,1}$	$g_{35,2}$
$o(g_{35j})$		6	6
$\lvert C_{\overline{G}}(g_{35j})\rvert$		1728	576
(k,m)	$\lvert C_{H_k}(g_{35km})\rvert$		
$(1,1)$	432	1	1
$(2,1)$	144	3	-1
m_{35j}		64	192

$\mathcal{F}_{36}$

g_{36}		$g_{36,1}$
$o(g_{36j})$		6
$\lvert C_{\overline{G}}(g_{36j})\rvert$		288
(k,m)	$\lvert C_{H_k}(g_{36km})\rvert$	
$(1,1)$	288	1
m_{36j}		256

$\mathcal{F}_{37}$

g_{37}		$g_{37,1}$	$g_{37,2}$	$g_{37,3}$
$o(g_{37j})$		12	6	12
$\lvert C_{\overline{G}}(g_{37j})\rvert$		1152	1152	576
(k,m)	$\lvert C_{H_k}(g_{37km})\rvert$			
$(1,1)$	288	1	1	1
$(2,1)$	288	1	1	-1
$(2,2)$	144	2	-2	0
m_{37j}		64	64	128

$\mathcal{F}_{38}$

g_{38}		$g_{38,1}$	$g_{38,2}$
$o(g_{38j})$		6	12
$\lvert C_{\overline{G}}(g_{38j})\rvert$		1152	384
(k,m)	$\lvert C_{H_k}(g_{38km})\rvert$		
$(1,1)$	288	1	1
$(2,1)$	96	3	-1
m_{38j}		64	192

$\mathcal{F}_{39}$

g_{39}		$g_{39,1}$	$g_{39,2}$	$g_{39,3}$
$o(g_{39j})$		12	12	6
$\lvert C_{\overline{G}}(g_{39j})\rvert$		2304	768	576
(k,m)	$\lvert C_{H_k}(g_{39km})\rvert$			
$(1,1)$	288	1	1	1
$(2,1)$	288	1	1	-1
$(2,2)$	48	6	-2	0
m_{39j}		32	96	128

$\mathcal{F}_{40}$

g_{40}		$g_{40,1}$	$g_{40,2}$
$o(g_{40j})$		12	6
$\lvert C_{\overline{G}}(g_{40j})\rvert$		288	288
(k,m)	$\lvert C_{H_k}(g_{40km})\rvert$		
$(1,1)$	144	1	1
$(2,1)$	144	1	-1
m_{40j}		128	128

$\mathcal{F}_{41}$

g_{41}		$g_{41,1}$	$g_{41,2}$	$g_{41,3}$
$o(g_{41j})$		12	6	12
$\lvert C_{\overline{G}}(g_{41j})\rvert$		384	384	192
(k,m)	$\lvert C_{H_k}(g_{41km})\rvert$			
$(1,1)$	96	1	1	1
$(2,1)$	96	1	1	-1
$(2,2)$	48	2	-2	0
m_{41j}		64	64	128

$\mathcal{F}_{42}$

g_{42}		$g_{42,1}$	$g_{42,2}$
$o(g_{42j})$		7	14
$\lvert C_{\overline{G}}(g_{42j})\rvert$		168	56
(k,m)	$\lvert C_{H_k}(g_{42km})\rvert$		
$(1,1)$	42	1	1
$(2,1)$	14	3	-1
m_{42j}		64	192

$\mathcal{F}_{43}$

g_{43}		$g_{43,1}$	$g_{43,2}$	$g_{43,3}$
$o(g_{43j})$		8	8	8
$\lvert C_{\overline{G}}(g_{43j})\rvert$		3072	1024	768
(k,m)	$\lvert C_{H_k}(g_{43km})\rvert$			
$(1,1)$	384	1	1	1
$(2,1)$	384	1	1	-1
$(2,2)$	64	6	-2	0
m_{43j}		32	96	128

$\mathcal{F}_{44}$

g_{44}		$g_{44,1}$	$g_{44,2}$	$g_{44,3}$
$o(g_{44j})$		8	8	8
$\lvert C_{\overline{G}}(g_{44j})\rvert$		3072	1024	768
(k,m)	$\lvert C_{H_k}(g_{44km})\rvert$			
$(1,1)$	384	1	1	1
$(2,1)$	384	1	1	-1
$(2,2)$	64	6	-2	0
m_{44j}		32	96	128

$\mathcal{F}_{45}$

g_{45}		$g_{45,1}$	$g_{45,2}$	$g_{45,3}$
$o(g_{45j})$		8	8	8
$\lvert C_{\overline{G}}(g_{45j})\rvert$		512	512	256
(k,m)	$\lvert C_{H_k}(g_{45km})\rvert$			
$(1,1)$	128	1	1	1
$(2,1)$	128	1	1	-1
$(2,2)$	64	2	-2	0
m_{45j}		64	64	128

$\mathcal{F}_{46}$

g_{46}		$g_{46,1}$	$g_{46,2}$	$g_{46,3}$
$o(g_{46j})$		8	8	8
$\lvert C_{\overline{G}}(g_{46j})\rvert$		512	512	256
(k,m)	$\lvert C_{H_k}(g_{46km})\rvert$			
$(1,1)$	128	1	1	1
$(2,1)$	128	1	1	-1
$(2,2)$	64	2	-2	0
m_{46j}		64	64	128

$\mathcal{F}_{47}$

g_{47}		$g_{47,1}$	$g_{47,2}$
$o(g_{47j})$		16	8
$\lvert C_{\overline{G}}(g_{47j})\rvert$		64	64
(k,m)	$\lvert C_{H_k}(g_{47km})\rvert$		
$(1,1)$	32	1	1
$(2,1)$	32	1	-1
m_{47j}		128	128

$\mathcal{F}_{48}$

g_{48}		$g_{48,1}$	$g_{48,2}$
$o(g_{48j})$		16	8
$\lvert C_{\overline{G}}(g_{48j})\rvert$		64	64
(k,m)	$\lvert C_{H_k}(g_{48km})\rvert$		
$(1,1)$	32	1	1
$(2,1)$	32	1	-1
m_{48j}		128	128

$\mathcal{F}_{49}$

g_{49}		$g_{49,1}$	$g_{49,2}$
$o(g_{49j})$		9	18
$\lvert C_{\overline{G}}(g_{49j})\rvert$		216	72
(k,m)	$\lvert C_{H_k}(g_{49km})\rvert$		
$(1,1)$	54	1	1
$(2,1)$	18	3	-1
m_{49j}		64	192

$\mathcal{F}_{50}$

g_{50}		$g_{50,1}$
$o(g_{50j})$		9
$\|C_{\overline{G}}(g_{50j})\|$		27
(k,m)	$\|C_{H_k}(g_{50km})\|$	
$(1,1)$	27	1
m_{50j}		256

$\mathcal{F}_{51}$

g_{51}		$g_{51,1}$	$g_{51,2}$	$g_{51,3}$
$o(g_{51j})$		20	20	10
$\|C_{\overline{G}}(g_{51j})\|$		1920	640	480
(k,m)	$\|C_{H_k}(g_{51km})\|$			
$(1,1)$	240	1	1	1
$(2,1)$	240	1	1	-1
$(2,2)$	40	6	-2	0
m_{51j}		32	96	128

$\mathcal{F}_{52}$

g_{52}		$g_{52,1}$	$g_{52,2}$
$o(g_{52j})$		10	20
$\|C_{\overline{G}}(g_{52j})\|$		960	320
(k,m)	$\|C_{H_k}(g_{52km})\|$		
$(1,1)$	240	1	1
$(2,1)$	80	3	-1
m_{52j}		64	192

$\mathcal{F}_{53}$

g_{53}		$g_{53,1}$	$g_{53,2}$	$g_{53,3}$
$o(g_{53j})$		20	10	20
$\|C_{\overline{G}}(g_{53j})\|$		320	320	160
(k,m)	$\|C_{H_k}(g_{53km})\|$			
$(1,1)$	80	1	1	1
$(2,1)$	80	1	1	-1
$(2,2)$	40	2	-2	0
m_{53j}		64	64	128

$\mathcal{F}_{54}$

g_{54}		$g_{54,1}$
$o(g_{54j})$		10
$\|C_{\overline{G}}(g_{54j})\|$		20
(k,m)	$\|C_{H_k}(g_{54km})\|$	
$(1,1)$	20	1
m_{54j}		256

$\mathcal{F}_{55}$

g_{55}		$g_{55,1}$	$g_{55,2}$
$o(g_{55j})$		12	12
$\|C_{\overline{G}}(g_{55j})\|$		4608	1536
(k,m)	$\|C_{H_k}(g_{55km})\|$		
$(1,1)$	1152	1	1
$(2,1)$	384	3	-1
m_{55j}		64	192

$\mathcal{F}_{56}$

g_{56}		$g_{56,1}$	$g_{56,2}$	$g_{56,3}$
$o(g_{56j})$		24	24	12
$\|C_{\overline{G}}(g_{56j})\|$		4608	1536	1152
(k,m)	$\|C_{H_k}(g_{56km})\|$			
$(1,1)$	576	1	1	1
$(2,1)$	576	1	1	-1
$(2,2)$	96	6	-2	0
m_{56j}		32	96	128

$\mathcal{F}_{57}$

g_{57}		$g_{57,1}$	$g_{57,2}$	$g_{57,3}$
$o(g_{57j})$		24	24	12
$\|C_{\overline{G}}(g_{57j})\|$		4608	1536	1152
(k,m)	$\|C_{H_k}(g_{57km})\|$			
$(1,1)$	576	1	1	1
$(2,1)$	576	1	1	-1
$(2,2)$	96	6	-2	0
m_{57j}		32	96	128

$\mathcal{F}_{58}$

g_{58}		$g_{58,1}$	$g_{58,2}$	$g_{58,3}$
$o(g_{58j})$		12	12	12
$\|C_{\overline{G}}(g_{58j})\|$		1536	1536	768
(k,m)	$\|C_{H_k}(g_{58km})\|$			
$(1,1)$	384	1	1	1
$(2,1)$	192	2	-2	0
$(2,2)$	384	1	1	-1
m_{58j}		64	64	128

$\mathcal{F}_{59}$

g_{59}		$g_{59,1}$
$o(g_{59j})$		12
$\|C_{\overline{G}}(g_{59j})\|$		144
(k,m)	$\|C_{H_k}(g_{59km})\|$	
$(1,1)$	144	1
m_{59j}		256

$\mathcal{F}_{60}$

g_{60}		$g_{60,1}$	$g_{60,2}$
$o(g_{60j})$		24	12
$\|C_{\overline{G}}(g_{60j})\|$		288	288
(k,m)	$\|C_{H_k}(g_{60km})\|$		
$(1,1)$	144	1	1
$(2,1)$	144	1	-1
m_{60j}		128	128

$\mathcal{F}_{61}$

g_{61}		$g_{61,1}$	$g_{61,2}$
$o(g_{61j})$		24	12
$\|C_{\overline{G}}(g_{61j})\|$		288	288
(k,m)	$\|C_{H_k}(g_{61km})\|$		
$(1,1)$	144	1	1
$(2,1)$	144	1	-1
m_{61j}		128	128

$\mathcal{F}_{62}$

g_{62}		$g_{62,1}$	$g_{62,2}$	$g_{62,3}$	$g_{62,4}$
$o(g_{62j})$		24	24	12	12
$\|C_{\overline{G}}(g_{62j})\|$		384	384	384	384
(k,m)	$\|C_{H_k}(g_{62km})\|$				
$(1,1)$	96	1	1	1	1
$(2,1)$	96	1	-1	1	-1
$(2,2)$	96	1	-1	-1	1
$(2,3)$	96	1	1	-1	-1
m_{62j}		64	64	64	64

$\mathcal{F}_{63}$

g_{63}		$g_{63,1}$	$g_{63,2}$
$o(g_{63j})$		12	12
$\lvert C_{\overline{G}}(g_{63j})\rvert$		288	96
(k,m)	$\lvert C_{H_k}(g_{63km})\rvert$		
$(1,1)$	72	1	1
$(2,1)$	24	3	-1
m_{63j}		64	192

$\mathcal{F}_{64}$

g_{64}		$g_{64,1}$	$g_{64,2}$
$o(g_{64j})$		24	12
$\lvert C_{\overline{G}}(g_{64j})\rvert$		96	96
(k,m)	$\lvert C_{H_k}(g_{64km})\rvert$		
$(1,1)$	48	1	1
$(2,1)$	48	1	-1
m_{64j}		128	128

$\mathcal{F}_{65}$

g_{65}		$g_{65,1}$	$g_{65,2}$
$o(g_{65j})$		24	12
$\lvert C_{\overline{G}}(g_{65j})\rvert$		96	96
(k,m)	$\lvert C_{H_k}(g_{65km})\rvert$		
$(1,1)$	48	1	1
$(2,1)$	48	1	-1
m_{65j}		128	128

$\mathcal{F}_{66}$

g_{66}		$g_{66,1}$	$g_{66,2}$
$o(g_{66j})$		12	12
$\lvert C_{\overline{G}}(g_{66j})\rvert$		48	48
(k,m)	$\lvert C_{H_k}(g_{66km})\rvert$		
$(1,1)$	24	1	1
$(2,1)$	24	1	-1
m_{66j}		128	128

$\mathcal{F}_{67}$

g_{67}		$g_{67,1}$
$o(g_{67j})$		12
$\lvert C_{\overline{G}}(g_{67j})\rvert$		24
(k,m)	$\lvert C_{H_k}(g_{67km})\rvert$	
$(1,1)$	24	1
m_{67j}		256

$\mathcal{F}_{68}$

g_{68}		$g_{68,1}$	$g_{68,2}$
$o(g_{68j})$		28	14
$\lvert C_{\overline{G}}(g_{68j})\rvert$		28	28
(k,m)	$\lvert C_{H_k}(g_{68km})\rvert$		
$(1,1)$	14	1	1
$(2,1)$	14	1	-1
m_{68j}		128	128

$\mathcal{F}_{69}$

g_{69}		$g_{69,1}$	$g_{69,2}$
$o(g_{69j})$		15	30
$\lvert C_{\overline{G}}(g_{69j})\rvert$		360	120
(k,m)	$\lvert C_{H_k}(g_{69km})\rvert$		
$(1,1)$	90	1	1
$(2,1)$	30	3	-1
m_{69j}		64	192

$\mathcal{F}_{70}$

g_{70}		$g_{70,1}$
$o(g_{70j})$		15
$\lvert C_{\overline{G}}(g_{70j})\rvert$		90
(k,m)	$\lvert C_{H_k}(g_{70km})\rvert$	
$(1,1)$	90	1
m_{70j}		256

$\mathcal{F}_{71}$

g_{71}		$g_{71,1}$
$o(g_{71j})$		15
$\lvert C_{\overline{G}}(g_{71j})\rvert$		15
(k,m)	$\lvert C_{H_k}(g_{71km})\rvert$	
$(1,1)$	15	1
m_{71j}		256

$\mathcal{F}_{72}$

g_{72}		$g_{72,1}$
$o(g_{72j})$		17
$\lvert C_{\overline{G}}(g_{72j})\rvert$		17
(k,m)	$\lvert C_{H_k}(g_{72km})\rvert$	
$(1,1)$	17	1
m_{72j}		256

$\mathcal{F}_{73}$

g_{73}		$g_{73,1}$
$o(g_{73j})$		17
$\lvert C_{\overline{G}}(g_{73j})\rvert$		17
(k,m)	$\lvert C_{H_k}(g_{73km})\rvert$	
$(1,1)$	17	1
m_{73j}		256

$\mathcal{F}_{74}$

g_{74}		$g_{74,1}$	$g_{74,2}$
$o(g_{74j})$		36	18
$\lvert C_{\overline{G}}(g_{74j})\rvert$		36	36
(k,m)	$\lvert C_{H_k}(g_{74km})\rvert$		
$(1,1)$	18	1	1
$(2,1)$	18	1	-1
m_{74j}		128	128

$\mathcal{F}_{75}$

g_{75}		$g_{75,1}$	$g_{75,2}$
$o(g_{75j})$		40	20
$\lvert C_{\overline{G}}(g_{75j})\rvert$		80	80
(k,m)	$\lvert C_{H_k}(g_{75km})\rvert$		
$(1,1)$	40	1	1
$(2,1)$	40	1	-1
m_{75j}		128	128

$\mathcal{F}_{76}$

g_{76}		$g_{76,1}$	$g_{76,2}$
$o(g_{76j})$		40	20
$\lvert C_{\overline{G}}(g_{76j})\rvert$		80	80
(k,m)	$\lvert C_{H_k}(g_{76km})\rvert$		
$(1,1)$	40	1	1
$(2,1)$	40	1	-1
m_{76j}		128	128

$\mathcal{F}_{77}$

g_{77}		$g_{77,1}$
$o(g_{77j})$		21
$\lvert C_{\overline{G}}(g_{77j})\rvert$		21
(k,m)	$\lvert C_{H_k}(g_{77km})\rvert$	
$(1,1)$	21	1
m_{77j}		256

$\mathcal{F}_{78}$

g_{78}		$g_{78,1}$	$g_{78,2}$
$o(g_{78j})$		24	24
$\lvert C_{\overline{G}}(g_{78j})\rvert$		96	96
(k,m)	$\lvert C_{H_k}(g_{78km})\rvert$		
$(1,1)$	48	1	1
$(2,1)$	48	1	-1
m_{78j}		128	128

$\mathcal{F}_{79}$

g_{79}		$g_{79,1}$	$g_{79,2}$
$o(g_{79j})$		24	24
$\lvert C_{\overline{G}}(g_{79j})\rvert$		96	96
(k,m)	$\lvert C_{H_k}(g_{79km})\rvert$		
$(1,1)$	48	1	1
$(2,1)$	48	1	-1
m_{79j}		128	128

$\mathcal{F}_{80}$

g_{80}		$g_{80,1}$	$g_{80,2}$
$o(g_{80j})$		60	30
$\lvert C_{\overline{G}}(g_{80j})\rvert$		60	60
(k,m)	$\lvert C_{H_k}(g_{80km})\rvert$		
$(1,1)$	30	1	1
$(2,1)$	30	1	-1
m_{80j}		128	128

$\mathcal{F}_{81}$

g_{81}		$g_{81,1}$
$o(g_{81j})$		30
$\lvert C_{\overline{G}}(g_{81j})\rvert$		30
(k,m)	$\lvert C_{H_k}(g_{81km})\rvert$	
$(1,1)$	30	1
m_{81j}		256

7. The character table of $\overline{G} = 2^{8\cdot}Sp(8,2)$

Having obtained

- the conjugacy classes of $\overline{G} = 2^{8\cdot}Sp(8,2)$ (Table 2),
- the fusion of classes of the inertia factor H_2 into classes of $Sp(8,2)$ (Table 9.3 of [2]),
- the Fischer matrices (see Section 6) and
- the character table of the inertia factor H_2 (Table 6.7 of [1]),

the character table of $\overline{G} = 2^{8\cdot}Sp(8,2)$ can be constructed easily by following the description of Subsection 3.1 of [5]. The character table of $\overline{G}$ is a 195×195 complex valued matrix and it coincides with the character table of the Split extension $2^8{:}Sp(8,2)$. The character table of $\overline{G}$ is not given in this paper but interested readers can refer to the Appendix of [2], which could be accessed online. This character table is not yet incorporated into the GAP library but our aim is to do so.

Acknowledgments

The first author would like to thank his supervisor (the second author) for his advice and support. The financial support from the North-West University is also acknowledged.

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