ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 41 (2015), No. 2, pp. 519–527

Title:

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Author(s):

A. Alotaibi, M. Mursaleen and S. A. Mohiuddine

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 41 (2015), No. 2, pp. 519–527
Online ISSN: 1735-8515

APPLICATION OF MEASURES OF NONCOMPACTNESS TO INFINITE SYSTEM OF LINEAR EQUATIONS IN SEQUENCE SPACES

A. ALOTAIBI, M. MURSALEEN* AND S. A. MOHIUDDINE

(Communicated by Hamid Reza Ebrahimi Vishki)

Abstract. G. Darbo [Rend. Sem. Math. Univ. Padova, 24 (1955) 84–92] used the measure of noncompactness to investigate operators whose properties can be characterized as being intermediate between those of contraction and compact operators. In this paper, we apply the Darbo's fixed point theorem for solving infinite system of linear equations in some sequence spaces.

Keywords: Sequence spaces; matrix transformations; Hausdorff measure of noncompactness; Darbo's fixed point theorem; infinite system of linear equations.
MSC(2010): 47H08, 47H09, 47H10, 46B45, 46B50.

1. Introduction

Measure of noncompactness are very useful tools widely used in fixed point theory, differential equations, functional equations, integral and integro-differential equations, and optimization etc. In recent years measures of noncompactness have also been used in defining geometric properties of Banach spaces as well as in characterizing compact operators between sequence spaces. Darbo formulated his celebrated fixed point theorem in 1955 which involves the notion of measure of noncompactness. Darbo's fixed point theorem is useful in establishing the existence of solutions of various classes of differential equations, especially for implicit differential equations, integral equations and integro-differential equations. In this paper, we use measures of noncompactness for solving infinite system of linear equations in some classical sequence spaces by applying the Darbo's fixed point theorem. That is, we obtain the conditions

Article electronically published on April 29, 2015.
Received: 2 January 2014, Accepted: 21 March 2014.
*Corresponding author.

for the existence of the solution of the infinite system of linear equations

$$y_n = \sum_{k=1}^{\infty} a_{nk} x_k + b_n \ (n = 1, 2, ...)$$

in sequence spaces ℓ_1 and ℓ_∞.

2. Preliminaries

We shall write w for the set of all complex sequences $x = (x_k)_{k=0}^{\infty}$. Let φ, ℓ_∞, c and c_0 denote the sets of all finite, bounded, convergent and null sequences, respectively. We write $\ell_p = \{x \in \omega : \sum_{k=0}^{\infty} |x_k|^p < \infty\}$ for $1 \leq p < \infty$. By e and $e^{(n)}$ $(n \in \mathbb{N})$, we denote the sequences such that $e_k = 1$ for $k = 0, 1, ...,$ and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ $(k \neq n)$. For any sequence $x = (x_k)_{k=0}^{\infty}$, let $x^{[n]} = \sum_{k=0}^{n} x_k e^{(k)}$ be its n–section.

A sequence $(b^{(n)})_{n=0}^{\infty}$ in a linear metric space X is called *Schauder basis* if for every $x \in X$, there is a unique sequence $(\lambda_n)_{n=0}^{\infty}$ of scalars such that $x = \sum_{n=0}^{\infty} \lambda_n b^{(n)}$. A sequence space X with a linear topology is called a K–*space* if each of the maps $p_i : X \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A K-space is called an FK–*space* if X is a complete linear metric space; a BK–*space* is a normed FK–space. An FK–space $X \supset \varphi$ is said to have AK if every sequence $x = (x_k)_{k=0}^{\infty} \in X$ has a unique representation $x = \sum_{k=0}^{\infty} x_k e^{(k)}$, that is, $x^{[n]} \to x$ as $n \to \infty$ (cf. [28]).

The classical sequence spaces c_0, c and ℓ_p $(1 \leq p < \infty)$ all have Schauder bases but ℓ_∞ has no Schauder basis; the spaces c_0 and ℓ_p $(1 \leq p < \infty)$ have AK.

Let $(X, \| \cdot \|)$ be a normed space. Then the unit sphere and closed unit ball in X are denoted by $S_X := \{x \in X : \| x \| = 1\}$ and $\bar{B}_X := \{x \in X : \| x \| \leq 1\}$.If $X \supset \varphi$ is a BK-space and $a = (a_k) \in w$, then we define

$$\|a\|_X^* = \sup_{x \in S_X} \left| \sum_{k=0}^{\infty} a_k x_k \right| \tag{2.1}$$

provided the expression on the right hand side exists and is finite.

Let $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix with complex entries a_{nk} $(n, k \in \mathbb{N})$. We write A_n for the sequence in the n^{th} row of A, i.e., $A_n = (a_{nk})_{k=0}^{\infty}$ for every $n \in \mathbb{N}$. The A-transform of the sequence $x = (x_k)_{k=0}^{\infty}$ is defined as the sequence $Ax = (A_n(x))_{n=0}^{\infty}$, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k; \ (n \in \mathbb{N}) \tag{2.2}$$

provided the series on the right converges for each $n \in \mathbb{N}$.

Let X and Y be subsets of w and $A = (a_{nk})$ an infinite matrix. Then, we say that A defines a matrix mapping from X into Y, and we denote it by writing

$A : X \to Y$, if Ax exists and is in Y for all $x \in X$. By (X, Y), we denote the class of all infinite matrices that map X into Y. If X and Y are normed spaces then we write $\mathcal{B}(X, Y)$ for the space of all bounded linear operators $L : X \to Y$ normed by $\|L\| = \sup\{\|L(x)\| : x \in S_X\}$.

Lemma 2.1 ([18,19]). (a) We have $(\ell_1, \ell_1) = \mathcal{B}(\ell_1, \ell_1)$ and $A \in (\ell_1, \ell_1)$ if and only if

$$\| A \|_{(\ell_1,\ell_1)} = \sup_k \sum_{n=0}^{\infty} | a_{nk} | < \infty. \tag{2.3}$$

If A is in any of the classes above then $\| L_A \| = \| A \|_{(\ell_1,\ell_1)}$.

(b) We have $A \in (\ell_\infty, \ell_\infty) = \mathcal{B}(\ell_\infty, \ell_\infty)$ if and only if

$$\| A \|_{(\ell_\infty,\ell_\infty)} = \sup_n \sum_{k=0}^{\infty} | a_{nk} | < \infty. \tag{2.4}$$

If A is in any of the above classes, then $\| L_A \| = \| A \|_{(\ell_\infty,\ell_\infty)}$.

3. Hausdorff measures of noncompactness

The concept of measure of noncompactness has played a basic role in nonlinear functional analysis, especially in metric and topological fixed point theory. Up to now, several papers have been published on the existence and behavior of solutions of nonlinear differential and integral equations, using the technique of measure of noncompactness. The first measure of noncompactness, the function α, was defined and studied by Kuratowski [15] in 1930. Darbo [11] used this measure to generalize both the classical Schauder fixed point principle and (a special variant of) Banach's contraction mapping principle for so called condensing operators. The Hausdorff MNC χ was introduced by Goldenstein, Gohberg and Markus [12] in 1957 (and later studied by Goldenstein and Markus [13]).

By $\mathcal{M}_X$, we denote the collection of all bounded subsets of a metric space (X, d) and by $\mathcal{N}_X$ the subfamily consisting of all relatively compact subsets of X.

Definition 3.1. Let (X, d) be a metric space and $Q \in \mathcal{M}_X$. Then the Kuratowski measure of noncompactness (α-measure or set measure of noncompactness) of Q, denoted by $\alpha(Q)$, is the infimum of the set of all numbers $\epsilon > 0$ such that Q can be covered by a finite number of sets with diameters ϵ that is

$$\alpha(Q) := \inf \left\{ \epsilon > 0 : Q \subset \bigcup_{i=1}^{n},\ S_i \subset X,\ \operatorname{diam}(S_i) < \epsilon\ (i = 1, ..., n);\ n \in \mathbb{N} \right\}.$$

Definition 3.2. If $Q \in \mathcal{M}_X$, then the Hausdorff measure of noncompactness of the set Q, denoted by $\chi(Q)$, is defined by

$$\chi(Q) := \inf \{\epsilon > 0 : Q \subset \cup_{i=1}^{n} B(x_i, r_i), x_i \in X, r_i < \epsilon (i = 1, 2, ...), n \in \mathbb{N}\}.$$

The function $\chi : \mathcal{M}_X \to [0, \infty)$ is called the Hausdorff measure of noncompactness.
The basic properties of the Hausdorff measure of noncompactness can be found in [1, 4, 18, 27] and [19].

Lemma 3.1. Let Q, Q_1 and Q_2 be bounded subsets of the metric space (X, d). Then
(i) $\chi(Q) = 0$ if and only if Q is totally bounded,
(ii) $\chi(Q) = \chi(\bar{Q})$,
(iii) $Q_1 \subset Q_2$ implies $\chi(Q_1) \leq \chi(Q_2)$,
(iv) $\chi(Q_1 \cup Q_2) = \max\{\chi(Q_1), \chi(Q_2)\}$,
Lemma 3.2. Let Q, Q_1 and Q_2 be bounded subsets of the normed space $(X, \| \, . \, \|)$. Then
(i) $\chi(Q_1 + Q_2) \leq \chi(Q_1) + \chi(Q_2)$,
(ii) $\chi(Q + x) = \chi(Q)$ for all $x \in X$,
(iii) $\chi(\lambda Q) = |\lambda| \chi(Q)$ for all $\lambda \in \mathbb{C}$,
(iv) $\chi(Q) = \chi(co(Q))$.

Let X and Y be Banach spaces and χ_1 and χ_2 be the Hausdorff measures of noncompactness on X and Y , respectively. An operator $L : X \to Y$ is said to be (χ_1, χ_2)-*bounded* if $L(Q) \in M_Y$ for all $Q \in M_X$ and there exist a constant $C \geq 0$ such that $\chi_2(L(Q)) \leq C\chi_1(Q)$ for all $Q \in M_X$. If an operator L is (χ_1,χ_2)-bounded then the number $\| L \|_{(\chi_1,\chi_2)} := \inf\{C \geq 0 : \chi_2(L(Q)) \leq C\chi_1(Q)$ for all $Q \in M_X\}$ is called the (χ_1,χ_2)–*measure of noncompactness of* L. If $\chi_1 = \chi_2 = \chi$, then we write $\| L \|_{(\chi_1,\chi_2)} = \| L \|_\chi$.
The most effective way in the characterization of compact operators between the Banach spaces is by applying the Hausdorff measure of noncompactness. This can be achieved as follows: Let X and Y be Banach spaces and $L \in B(X, Y)$. Then, the Hausdorff measure of noncompactness of L, denoted by $\| L \|_\chi$, can be determined by

$$\| L \|_\chi = \chi(L(S_X)), \tag{3.1}$$

and we have that L is compact if and only if

$$\| L \|_\chi = 0. \tag{3.2}$$

This technique has recently been used by several authors in many research papers (see for instance [8–10, 14, 16, 17, 20–26]).

The following result shows how to compute the Hausdorff measure of noncompactness in the spaces c_0 and ℓ_p $(1 \leq p < \infty)$.

Theorem 3.3 [19]. Let Q be a bounded subset of the normed space X, where X is ℓ_p for $1 \leq p < \infty$ or c_0. If $P_n : X \to X$ $(n \in N)$ is the operator defined by $P_n(x) = x^{[n]} = (x_0, x_1, \ldots, x_n, 0, 0, \ldots)$ for all $x = (x_k)_{k=0}^{\infty} \in X$, then we have

$$\chi(Q) = \lim_{n\to\infty} \Big(\sup_{x \in Q} \|(I - P_n)(x)\| \Big).$$

G. Darbo [11] used the measure of noncompactness to investigate operators whose properties can be characterized as being intermediate between those of contraction and compact operators. Darbo's fixed point theorem is a generalization of the well-known Schauder fixed point theorem.

Theorem 3.4 [11]. If C is a non-empty bounded closed convex subset of a Banach space X and $T : C \to C$ is a continuous mapping such that for any set $E \subset C$,

$$\alpha(T(E)) \leq k\alpha(E),$$

where k is a constant, $0 \leq k < 1$, then T has a fixed point.
The theorem is also true for the Hausdorff measure χ.

Darbo's fixed-point theorem is useful in establishing the existence of solutions of various classes of differential equations, especially for implicit differential equations, integral equations and integro-differential equations, see [4]. Recently, measures of noncompactness have been used in solving infinite system of differential equations and integral equations in sequence spaces (cf. [2,3,5,6], and [23]). In this paper, we use measures of noncompactness for solving infinite system of linear equations by applying the Darbo's fixed point theorem.

4. Main results

Consider the infinite system of linear equations

$$y_n = \sum_{k=1}^{\infty} a_{nk} x_k + b_n \ (n = 1, 2, ...). \tag{4.1}$$

Let us define $T : \ell_1 \to \ell_1$ by

$$y = Tx = Ax + B, \tag{4.2}$$

which is an equivalent form of (3.1), where $B = (b_n)_{n=1}^{\infty}$ is a sequence of scalars. We have to find a condition so that system (3.2) will have a unique solution.

Theorem 4.1. System (3.2) has a unique solution $x = (x_k) \in \ell_1$ if

(i) $A \in (\ell_1, \ell_1)$, and (ii) $B = (b_n) \in \ell_\infty$, and (iii) there exists a constant $k < 1$ such that

$$\sum_{n=1}^{\infty} | a_{nj} | \leq k, \ j = 1, 2, ...$$

Proof. We know by Lemma 2.1 (a) that the necessary and the sufficient condition for $A \in (\ell_1, \ell_1)$ is

$$\sup_j \sum_{n=1}^{\infty} | a_{nj} | < \infty.$$

Now, let $S = S_{\ell_1}$. Then, $T(S) = AS \in \mathcal{M}_{\ell_1}$. Thus, we have by Theorem 3.3 that

$$\begin{aligned} \chi(T(S)) &= \lim_{r\to\infty} \Big(\sup_{x\in S} \|(I - P_r)(Tx)\|_{\ell_1} \Big) \\ &= \lim_{r\to\infty} \Big(\sup_{x\in S} \|(I - P_r)(Ax + B)\|_{\ell_1} \Big) \\ &= \lim_{r\to\infty} \Big(\sup_{x\in S} \|(I - P_r)(Ax)\|_{\ell_1} \Big) \end{aligned}$$

where $P_r : \ell_1 \to \ell_1$ $(r \in \mathbb{N})$ is defined by $P_r(x) = (x_0, x_1, \ldots, x_r, 0, 0, \ldots)$ for all $x = (x_k) \in \ell_1$. This yields that

$$\chi(T(S)) = \lim_{r\to\infty} \Big(\sup_{x\in S} \Big(\sum_{n=r+1}^{\infty} |A_n(x)| \Big) \Big)$$

$$= \lim_{r\to\infty} \sup_{x\in S} \left(\sum_{n=r+1}^{\infty} | \sum_{j=0}^{\infty} a_{nj} x_j | \right)$$

$$\leq \lim_{r\to\infty} \sup_{x\in S} \sum_{n=r+1}^{\infty} \sum_{j=0}^{\infty} | a_{nj} || x_j | . \tag{4.3}$$

and hence we obtain

$$\chi(T(S)) \leq \left(\lim_{r\to\infty} \sup_j \sum_{n=r+1}^{\infty} | a_{nj} | \right) \chi(S). \tag{4.4}$$

Therefore, if $\sum_{n=1}^{\infty} | a_{nj} | \leq k < 1$ for every j, then

$$\chi(T(S)) \leq k\chi(S). \tag{4.5}$$

Now applying Theorem 3.4, we have that T has a fixed point.
Thus, if $B = (b_n) \in \ell_\infty$ and if there exists a constant $k < 1$

$$\sum_{n=1}^{\infty} | a_{nj} | \leq k, \ j = 1, 2, ...$$

then there exists a unique sequence $x = (x_k) \in \ell_1$ which is a solution of system (4.2).
This completes the proof. $\square$

Similarly, we have the following result when in (4.2) $T \in \mathcal{B}(\ell_\infty, \ell_\infty)$.
Theorem 4.2. The system (4.2) has a unique solution $x = (x_k) \in \ell_\infty$ if
(i) $A \in (\ell_\infty, \ell_\infty)$, and (ii) $B = (b_n) \in \ell_\infty$, and (iii) there exists a constant $\alpha < 1$ such that

$$\sum_{j=1}^{\infty} | a_{nj} | \leq \alpha, \ n = 1, 2, ...$$

Proof. Using Lemma 2.1 (b) then condition (2.4) for $A \in (\ell_\infty, \ell_\infty)$, and the associated operator $T \in \mathcal{B}(\ell_\infty, \ell_\infty)$ together yield

$$\chi(T(S)) \leq \left(\lim_{r\to\infty} \sup_n \sum_{j=r+1}^{\infty} | a_{nj} | \right) \chi(S).$$

Therefore, if $\sum_{j=1}^{\infty} | a_{nj} | \leq k < 1$ for every n, then

$$\chi(T(S)) \leq k\chi(S). \tag{4.5}$$

Now applying Darbo's fixed point theorem, we have that T has a fixed point.
Thus, if $B = (b_n) \in \ell_\infty$ and if there exists a constant $k < 1$

$$\sum_{n=1}^{\infty} | a_{nj} | \leq k, \ j = 1, 2, ...$$

then there exists a unique sequence $x = (x_k) \in \ell_\infty$ which is a solution of system (4.2).
This completes the proof. $\square$

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(A. Alotaibi) Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
Operator Theory and its Applications Research Group, King Abdulaziz University, Jeddah, Saudi Arabia
E-mail address: mathker11@hotmail.com

(M. Mursaleen) Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
Operator Theory and its Applications Research Group, King Abdulaziz University, Jeddah, Saudi Arabia
E-mail address: mursaleenm@gmail.com

(S. A. Mohiuddine) Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
Operator Theory and its Applications Research Group, King Abdulaziz University, Jeddah, Saudi Arabia
E-mail address: mohiuddine@gmail.com