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ZARISKI-LIKE TOPOLOGY ON THE CLASSICAL PRIME SPECTRUM OF A MODULE†

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Abstract. Let R be a commutative ring with identity and let M be an R-module. A proper submodule P of M is called a classical prime submodule if $abm \in P$ for a, $b \in R$, and $m \in M$, implies that $am \in P$ or $bm \in P$. The classical prime spectrum Cl.Spec(M) is defined to be the set of all classical prime submodules of M. The aim of this paper is to introduce and study a topology on Cl.Spec(M), which generalizes the Zariski topology of R to M, called Zariski-like topology of M. In particular, we investigate this topological space from the point of view of spectral spaces. It is shown that if M is a Noetherian (or an Artinian) R-module, then Cl.Spec(M) with the Zariski-like topology is a spectral space, i.e., there exists a commutative ring S such that Cl.Spec(M) with the Zariski-like topology is homeomorphic to Spec(S) with the usual Zariski topology.

1. Introduction

All rings throughout this paper are associative commutative with identity $1 \neq 0$ and modules are unital. Let M be an R-module. If N is a submodule of M then we write $N \leq M$ and denote, the ideal $\{r \in R : rM \subseteq N\}$ by $(N : M)$.

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In the literature, there are several different generalizations of the notion of prime ideals to modules. For instance, a proper submodule P of M is called a *prime submodule* if $am \in P$ for $a \in R$, and $m \in M$, implies that $m \in P$ or $a \in (P : M)$. Prime submodules of modules were introduced by J. Dauns [9] and have been studied intensively since then (see for example, [2,5,7,14,16,17,18,22]). Also, a proper submodule P of M is called a *classical prime submodule* if $abm \in P$ for $a, b \in R$, and $m \in M$, implies that $am \in P$ or $bm \in P$. This notion of classical prime submodule has been extensively studied by the first author in [3,4] (see also, [6], in which, the notion of "weakly prime submodule" is investigated).

The classical Zariski topology on the spectrum of prime ideals of a commutative ring is one of the main tools in Algebraic Geometry. Recall that the *spectrum* Spec(R) of a ring R consists of all prime ideals of R and is non-empty. For each ideal I of R, we set $V(I) = \{\mathcal{P} \in Spec(R) : I \subseteq \mathcal{P}\}$. Then the sets $V(I)$, where I is an ideal of R, satisfy the axioms for the closed sets of a topology on Spec(R), called the *Zariski topology* (see for example, Atiyah and Macdonald [1]). Also, the prime spectrum Spec(M) is defined to be the set of all prime submodules of M. If $N \leq M$ is an R-submodule, denote by $V(N)$ the variety of N, which is the set consisting of all prime submodules of M that contain N. The R-module M is called a Top-module, if the prime spectrum of M has the property (true for the usual spectrum Spec(R)) that the set of all varieties $\xi(M) := \{V(N) \mid N \leq M\}$ is closed under finite unions, whence they constitute the closed sets in a Zariski-like topology on Spec(M). In a series of papers (see for example, [13,15,19-21]), a group of algebraists including mainly R. L. McCasland, M. E. Moore and P. F. Smith carried out an intensive and systematic study of the spectrum of prime submodules. For example, they showed in [19] that in case ${}_RM$ is finitely generated, M is a Top-module if and only if M is a multiplication module (i.e., any R-submodule $N \leq M$ is of the form $N = IM$ for a suitable ideal I of R); the paper included also discussions of when Spec$(M) = \emptyset$ and of spectra of direct sums. In [13], conditions on a finitely generated R-module M and on the ground ring R are determined under which the Spec(M) satisfies various finite generation conditions.

The classical prime spectrum Cl.Spec(M) is defined to be the set of all classical prime submodules of M. In our work, we rely on the classical prime submodules, and then introduce and study a topology on Cl.Spec(M) which generalizes the Zariski topology of rings to modules.

Let M be a nonzero R-module. For any submodule N of M we define the *classical variety* of N, denoted by $\mathbb{V}(N)$, to be the set of all classical prime submodules P of M such that $N \subseteq P$. Then,

(i) $\mathbb{V}(M) = \emptyset$ and $\mathbb{V}(0) = Cl.Spec(M)$,

(ii) $\bigcap_{i\in I} \mathbb{V}(N_i) = \mathbb{V}(\sum_{i\in I} N_i)$ for any index set I,

(iii) $\mathbb{V}(N) \cup \mathbb{V}(L) \subseteq \mathbb{V}(N \cap L)$, where N, L, $N_i \leq M$.

Now, we assume that $\mathbb{C}(M)$ denotes the collection of all subsets $\mathbb{V}(N)$ of Cl.Spec(M). Then, $\mathbb{C}(M)$ contains the empty set and Cl.Spec(M), and also $\mathbb{C}(M)$ are closed under arbitrary intersections. However, in general, $\mathbb{C}(M)$ is not closed under finite union. These inspire the following definitions.

Definition 1.1. An R-module M is called a *classical Top-module* if $\mathbb{C}(M)$ is closed under finite unions, i.e., for every submodules N and L of M there exists a submodule K of M such that $\mathbb{V}(N) \cup \mathbb{V}(L) = \mathbb{V}(K)$, for in this case, $\mathbb{C}(M)$ satisfies the axioms for the closed subsets of a topological space.

Definition 1.2. Let M be an R-module. For each submodule N of M, we put $\mathbb{U}(N) = Cl.Spec(M) \setminus \mathbb{V}(N)$ and $\mathbb{B}(M) = \{\mathbb{U}(N) : N \leq M\}$. Then, we define $\mathbb{T}(M)$ to be the collection of all unions of finite intersections of elements of $\mathbb{B}(M)$. In fact, $\mathbb{T}(M)$ is the topology on Cl.Spec(M) by the sub-basis $\mathbb{B}(M)$. We say that $\mathbb{T}(M)$ is the *Zariski-like topology* of M.

Remark 1.3. Let M be an R-module. We can see easily that the set,

$$\{\mathbb{U}(N_1) \cap \cdots \cap \mathbb{U}(N_k) \ : \ N_i \leq M, \ 1 \leq i \leq k, \ for\ some\ k \in \mathbb{N}\},$$

is a basis for the Zariski-like topology of M, and for a ring R, the Zariski-like topology of ${}_RR$ and the usual Zariski topology of the ring R coincide.

Definition 1.4. A *spectral space* is a topological space homeomorphic to the prime spectrum of a commutative ring equipped with the Zariski topology. By Hochster's characterization [12], a topology τ on a set X is spectral if and only if the following axioms hold:

(i) X is a T_0-space.

(ii) X is quasi-compact and has a basis of quasi-compact open subsets.

(iii) The family of quasi-compact open subsets of X is closed under finite intersections.
(iv) X is a sober space; i.e., every irreducible closed subset of X has a generic point.

In Section 2, we study classical Top-modules and also consider some relationships between Top-modules and classical Top-modules. In particular, it is shown that if M is a finitely generated or an Artinian R-module, then M is a classical Top-module if and only if M is a Top-module. In [13, Theorem 6.5 and Corollary 6.6], it is shown that for each finitely generated multiplication R-module M, Spec(M) with the Zariski topology is a spectral space. In Section 3, we show that for each Noetherian (or Artinian) R-module M, the aforementioned four axioms of a spectral space hold for Cl.Spec(M) with the Zariski-like topology. Thus, by applying Hochster's characterization of spectral spaces, we conclude that Cl.Spec(M) is a spectral space. Finally, in Section 4, we provide some comments on the paper entitled "The Zariski topology on the prime spectrum of a module" by Chin-Pi Lu (see [13]). In fact, it is shown that the Proposition 5.2 (3) and Proposition 6.3 of [13], are not true in general. Moreover, a correct version of [13, Propositions 5.2 (3)] and also a correct version [13, Proposition 6.3] are given.

2. **Classical top-modules**

The following two evident propositions offer several characterizations of classical prime submodules and prime submodules respectively (see also [3] and [5]).

Proposition 2.1. *Let M be an R-module. For a submodule $P \subsetneqq M$, the following statements are equivalent:*

(1) *P is classical prime.*
(2) *For every $0 \neq \bar{m} \in M/P$, $(0 : R\bar{m})$ is a prime ideal.*
(3) *$\{(0 : R\bar{m}) \mid 0 \neq \bar{m} \in M/P\}$ is a chain (linearly ordered set) of prime ideals.*
(4) *$(P : M)$ is a prime ideal, and $\{(0 : R\bar{m}) \mid 0 \neq \bar{m} \in M/P\}$ is a chain of prime ideals.*

Proposition 2.2. *Let M be an R-module. For a submodule $P \subsetneqq M$, the following statements are equivalent:*

(1) *P is prime.*

(2) *For every $0 \neq \bar{m} \in M/P$, $(0 : R\bar{m})$ is a prime ideal and $(0 : R\bar{m}) = (P : M)$.*

(3) *$(P : M)$ is a prime ideal and the set $\{(0 : R\bar{m}) : \ 0 \neq \bar{m} \in M/P\}$ is singleton.*

A submodule C of an R-module M will be called semiprime (resp. classical semiprime) if C is an intersection of prime (resp. classical prime) submodules. A prime (resp. classical prime) submodule P of M will be called *extraordinary* if whenever N and L are semiprime (resp. classical semiprime) submodules of M with $N \bigcap L \subseteq P$ then $N \subseteq P$ or $L \subseteq P$. Let N be a proper submodule of an R-module M. Then, the prime radical (resp. classical prime radical) $\sqrt[p]{N}$ (resp. $\sqrt[cl]{N}$) of N (in M) is the intersection of all prime (resp. classical prime) submodules of M containing N or, in case there are no such prime (resp. classical prime) submodules, $\sqrt[p]{N}$ (resp. $\sqrt[cl]{N}$) is M. Note that $N \subseteq \sqrt[cl]{N} \subseteq \sqrt[p]{N}$ and that $\sqrt[p]{N} = M$ (resp. $\sqrt[cl]{N} = M$) or $\sqrt[p]{N}$ (resp. $\sqrt[cl]{N}$) is a semiprime (resp. classical semiprime) submodule of M. Clearly $\mathbb{V}(N) = \mathbb{V}(\sqrt[cl]{N})$. Also, for any module M, we define $\text{rad}_R(M) = \sqrt[p]{(0)}$ (resp. $\text{cl.rad}_R(M) = \sqrt[cl]{(0)}$); this is called prime radical (resp. classical prime radical) of M. Thus, if M has a prime (resp. classical prime) submodule, then $\text{rad}_R(M)$ (resp. $\text{cl.rad}_R(M)$) is equal to the intersection of all the prime (resp. classical prime) submodules in M, but if M has no prime (resp. classical prime) submodule, then $\text{rad}_R(M) = M$ (resp. $\text{cl.rad}_R(M) = M$).

The next result should be compared with [19, Lemma 2.1].

Lemma 2.3. *For an R-module M, the following statements are equivalent:*

(1) *M is a classical Top-module.*

(2) *Every classical prime submodule of M is extraordinary.*

(3) *$\mathbb{V}(N) \bigcup \mathbb{V}(L) = \mathbb{V}(N \bigcap L)$ for every classical semiprime submodules N and L of M.*

Proof. If $\text{Cl.Spec}(M) = \emptyset$, then the result is clearly true. Suppose that $\text{Cl.Spec}(M) \neq \emptyset$.

(1) $\Rightarrow$ (2). Let P be a classical prime submodule of M and let N and L be classical semiprime submodules of M such that $N\bigcap L\subseteq P$. By hypothesis, there exists a submodule K of M such that $\mathbb{V}(N)\bigcup\mathbb{V}(L)=\mathbb{V}(K)$. Now, $N=\bigcap_{i\in I}P_i$, for some collection of classical prime submodules $P_i(i\in I)$. For each $i\in I$, $P_i\in\mathbb{V}(N)\subseteq\mathbb{V}(K)$, so that $K\subseteq P_i$. Thus, $K\subseteq\bigcap_{i\in I}P_i=N$. Similarly, $K\subseteq L$. Thus, $K\subseteq N\bigcap L$. Now, $\mathbb{V}(N)\bigcup\mathbb{V}(L)\subseteq\mathbb{V}(N\bigcap L)\subseteq\mathbb{V}(K)=\mathbb{V}(N)\bigcup\mathbb{V}(L)$. It follows that $\mathbb{V}(N)\bigcup\mathbb{V}(L)=\mathbb{V}(N\bigcap L)$. But $P\in\mathbb{V}(N\bigcap L)$ now gives $P\in\mathbb{V}(N)$ or $P\in\mathbb{V}(L)$, i.e., $N\subseteq P$ or $L\subseteq P$.
(2)$\Rightarrow$ (3). Let G and H be classical semiprime submodules of M. Clearly, $\mathbb{V}(G)\bigcup\mathbb{V}(H)\subseteq\mathbb{V}(G\bigcap H)$. Let $P\in\mathbb{V}(G\bigcap H)$. Then, $G\bigcap H\subseteq P$ and hence $G\subseteq P$ or $H\subseteq P$, i.e., $P\in\mathbb{V}(G)$ or $P\in\mathbb{V}(H)$. This proves that $\mathbb{V}(G\bigcap H)\subseteq\mathbb{V}(G)\bigcup\mathbb{V}(H)$ and hence $\mathbb{V}(G)\bigcup\mathbb{V}(H)=\mathbb{V}(G\bigcap H)$.
(3)$\Rightarrow$(1). Let S and T be two submodules of M. If $\mathbb{V}(S)$ is empty, then, $\mathbb{V}(S)\bigcup\mathbb{V}(T)=\mathbb{V}(T)$. Suppose that $\mathbb{V}(S)$ and $\mathbb{V}(T)$ are both non-empty. Then $\mathbb{V}(S)\bigcup\mathbb{V}(T)=\mathbb{V}(\sqrt[cl]{S})\bigcup\mathbb{V}(\sqrt[cl]{T})=\mathbb{V}(\sqrt[cl]{S}\bigcap\sqrt[cl]{T})$, by (3). This proves (1). $\square$

Proposition 2.4. *Every classical Top-module is a Top-module.*

Proof. Let M be a classical Top-module. By Lemma 2.3, every classical prime submodule of M is extraordinary. Since every classical prime submodule of M is a prime submodule, every prime submodule of M is also extraordinary. Now, by [19, Lemma 2.1], M is a Top-module. $\square$

We have not found any example of a Top-modules M which is not a classical Top-module. The lack of such counterexamples together with the fact that the two concepts *Top* and *classical Top* are equivalent for certain classes of modules like finitely generated modules and Artinian modules (see Theorem 2.7 and Proposition 2.8), motivates the following conjecture.

Conjecture 2.5. *Let M be an R-module. Then, M is a Top-module if and only if M is a classical Top-module.*

An R-module M is called a multiplication module if for each submodule N of M, there exists an ideal I of R such that $N=IM$. For example, cyclic modules and projective ideals are multiplication modules (for more information about multiplication modules see for example [10,19,23]). Here, we shall show that any finitely generated R-module

M is a classical Top-module if and only if M is a Top-module and if and only if M is a multiplication module. Here, we show that every multiplication module M is a classical Top-module.

Proposition 2.6. *Every multiplication module is a classical Top-module.*

Proof. Let M be a multiplication R-module. Then, by [19, Theorem 3.5], M is a Top-module. On the other hand by [6, Proposition 3.2], the classical prime submodules and the prime submodules of a multiplication module coincide. Thus, M is a classical Top-module. □

Now, we are in a position to show that Conjecture 2.5 is true for all finitely generated modules.

Theorem 2.7. *Let M be a finitely generated R-module. Then, the following statements are equivalent:*

(1) *M is a classical Top-module.*

(2) *M is a Top-module.*

(3) *M is a multiplication module.*

Proof. (1) ⇒ (2), by Proposition 2.4.
(2) ⇒ (3), by [19, Theorem 3.5].
(3) ⇒ (1), by Proposition 2.6. □

In the following proposition, we show that Conjecture 2.5 is also true for all Artinian modules.

Proposition 2.8. *Let M be an Artinian module. Then,*

$$Spec(M) = Cl.Spec(M).$$

Consequently, the two concepts Top and classical Top are equivalent for M.

Proof. Let M be an Artinian module. It is clear that Spec$(M) \subseteq$ Cl.Spec(M). Let P be a classical prime submodule of M. Then, $\bar{M} := M/P$ is an Artinian classical prime R-module and so by Proposition 2.1, $\{Ann(m) \mid 0 \neq m \in \bar{M}\}$ is a chain of prime ideals of R such that $Ann(\bar{M}) = \bigcap_{0\neq m\in\bar{M}} Ann(m)$. Clearly, for each $0 \neq m \in \bar{M}$, Rm is also an Artinian classical prime R-module. Since $Rm \cong R/Ann(m)$ and R is commutative, then Rm is an Artinian prime module. Now, by [5, Corollary 1.9], Rm is a homogenous semisimple R-module, i.e., $Ann(m) = \mathcal{P}$ is a maximal ideal. It follows that $\{Ann(m) \mid 0 \neq m \in \bar{M}\}$ is singleton. Thus, by Proposition 2.2, P is a prime submodule of M. Therefore, Cl.Spec$(M) \subseteq$ Spec(M) and so Spec(M) =Cl.Spec(M). □

Proposition 2.9. *Let M be a classical Top-module. Then, every homomorphic image of M is a classical Top-module.*

Proof. Let N be a submodule of a classical Top-module M. Let $M' = M/N$. Suppose that Cl.Spec$(M') \neq \emptyset$. Clearly, the classical prime submodules of M' are precisely the submodules P/N, where P is a classical prime submodule of M and $N \subseteq P$. Thus, any classical semiprime submodule of M' has the form C/N, where C is a classical semiprime submodule containing N. Now, apply Lemma 2.3. □

Corollary 2.10. *Let M be an R-module. Then,*

(i) M is a classical Top-module if and only if $M/cl.rad_R(M)$ is a classical Top-module.

(ii) M is a Top-module if and only if $M/rad_R(M)$ is a Top-module.

Proof. The proof is clear, using Proposition 2.9. □

Let M be an R-module. Recall that a proper submodule P of M is called virtually maximal if the factor module M/P is a homogeneous semisimple R-module, i.e., M/P is a direct sum of isomorphic simple modules.

Lemma 2.11. *Let M be an R-module. Then,*

(i) M is an Artinian R-module if and only if every prime submodule of M is virtually maximal and $M/rad_R(M)$ is a Noetherian R-module.

(ii) M is Noetherian and every prime submodule of M is virtually maximal if and only if $M/rad_R(M)$ is an Artinian R-module.

Proof. (i) Use [7, Theorem 2.7].
(ii) Use [7, Theorem 2.1]. □

Theorem 2.12. *Let M be an Artinian module such that Spec$(M) \neq \emptyset$. Then, M is a (classical) Top-module if and only if $M/rad(M)$ is a cyclic R-module.*

Proof. $(\Rightarrow)$. Let M be a (classical) Top-module. By Proposition 2.8, Spec(M) = Cl.Spec(M) and hence $rad(M) = cl.rad(M)$. By Lemma 2.11(i), $M/rad(M)$ is a Noetherian R-module. On the other hand, by Corollary 2.10(ii), $M/rad(M)$ is a finitely generated (classical) Top-module. Thus, by Theorem 2.7, $M/rad(M)$ is a multiplication R-module. Since $M/rad(M)$ is an Artinian R-module, then by [10, Corollary 2.9], $M/rad(M)$ is a cyclic R-module.
$(\Leftarrow)$. Since $M/rad(M)$ is cyclic, then $M/rad(M)$ is a homomorphic image of R and so by [19, Corollary 2.2], $M/rad(M)$ is a Top-module. Now, apply Corollary 2.10, (ii). □

Corollary 2.13. *Let M be an Artinian (classical) Top-module. Then,*

$$Max(M) = Spec(M).$$

Proof. Let M be an Artinian (classical) Top-module. Then, by Theorem 2.12, $M/rad(M)$ is a cyclic Artinian R-module. It follows that every prime submodule of $M/rad(M)$ is maximal. Clearly, for each prime submodules P of M, $P/rad(M)$ is a prime submodule of $M/rad(M)$. Thus, Max(M) =Spec(M). □

3. **Zariski-like topology of modules and spectral spaces**

A topological space X is called *irreducible* if $X \neq \emptyset$ and every finite intersection of non-empty open sets of X is non-empty. A (non-empty) subset Y of a topology space X is called an *irreducible set* if the subspace Y of X is irreducible. For this to be so, it is necessary and sufficient that, for every pair of sets Y_1, Y_2 which are closed in X and satisfy $Y \subseteq Y_1 \cup Y_2$, $Y \subseteq Y_1$ or $Y \subseteq Y_2$ (see, for example [8, page 94]).

Let Y be a closed subset of a topological space. An element $y \in Y$ is called a *generic point* of Y if $Y = \overline{\{y\}}$. Note that a generic point of the irreducible closed subset Y of a topological space is unique if the topological space is a T_0-space.

Here, we will show that for each Noetherian (or Artinian) R-module M, the four axioms (i), (ii), (iii) and (iv) of Definition 1.4 hold for Cl.Spec(M) with the Zariski-like topology, i.e., Cl.Spec(M) is a spectral space.

Lemma 3.1. *Let M be an R-module, and let Y be a nonempty subset of Cl.Spec(M). Then,*

$$\overline{Y} = \bigcup_{P \in Y} \mathbb{V}(P).$$

Proof. Clearly, $\overline{Y} \subseteq \bigcup_{P \in Y} \mathbb{V}(P)$. Suppose C is any closed subset of X such that $Y \subseteq C$. Thus, $C = \bigcap_{i \in I}(\bigcup_{j=1}^{n_i} \mathbb{V}(N_{ij}))$, for some $N_{ij} \leq M$, $i \in I$ and $n_i \in \mathbb{N}$. Let $Q \in \bigcup_{P \in Y} \mathbb{V}(P)$. Then, there exists $P_0 \in Y$ such that $Q \in \mathbb{V}(P_0)$ and so $P_0 \subseteq Q$. Since $P_0 \in C$, then for each $i \in I$ there exists $j \in \{1, 2, \cdots, n_i\}$ such that $N_{ij} \subseteq P_0$, and hence $N_{ij} \subseteq P_0 \subseteq Q$. It follows that $Q \in C$. Therefore, $\bigcup_{P \in Y} \mathbb{V}(P) \subseteq C$. □

The following theorem shows that for any R-module M, Cl.Spec(M) is always a T_0-space.

Theorem 3.2. *Let M be an R-module. Then, Cl.Spec(M) is a T_0-space.*

Proof. Let P_1, $P_2 \in$ Cl.Spec(M). Then, by Lemma 3.1, $\{\overline{P_1}\} = \{\overline{P_2}\}$ and if and only if $\mathbb{V}(P_1) = \mathbb{V}(P_2)$ and if and only if $P_1 = P_2$. Now, by the fact that a topological space is a T_0-space if and only if the closures of distinct points are distinct, we conclude that for any R-module M, Cl.Spec(M) is a T_0-space. □

Theorem 3.3. *Let M be a Noetherian R-module. Then, Cl.Spec(M) is a quasi-compact space.*

Proof. Suppose M is a Noetherian R-module. Let $\mathcal{A}$ be a family of open sets covering Cl.Spec(M), and suppose that no finite subfamily of $\mathcal{A}$ covers Cl.Spec(M). Since $\mathbb{V}(0) =$ Cl.Spec(M), then we may use the ACC on submodules to choose a submodule N maximal with respect to the property that no finite subfamily of $\mathcal{A}$ covers $\mathbb{V}(N)$. We claim that N is a classical prime submodule of M, for if not, then there exist $m \in M$ and a, $b \in R$, such that $abm \in N$, $am \notin N$ and $bm \notin N$. Thus, $N \subsetneqq N + Ram$ and $N \subsetneqq N + Rbm$. Hence, without loss of generality, there must exist a finite subfamily $\mathcal{A}'$ of $\mathcal{A}$ that covers both $\mathbb{V}(N+Ram)$ and $\mathbb{V}(N+Rbm)$. Let $P \in \mathbb{V}(N)$. Since $abm \in N$, then $abm \in P$ and since submodule P is classical prime, then $am \in P$ or $bm \in P$. Thus, either $P \in \mathbb{V}(N+Ram)$ or $P \in \mathbb{V}(N+Rbm)$, and therefore,

$$\mathbb{V}(N) \subseteq \mathbb{V}(N+Ram) \bigcup \mathbb{V}(N+Rbm).$$

Thus, $\mathbb{V}(N)$ is covered with the finite subfamily $\mathcal{A}'$, which is a contradiction. Therefore, N is a classical prime submodule of M.

Now, choose $U \in \mathcal{A}$ such that $N \in U$. Thus, N must have a neighborhood $\bigcap_{i=1}^{n} \mathbb{U}(K_i)$, for some $K_i \leq M$ and $n \in \mathbb{N}$, such that $\bigcap_{i=1}^{n} \mathbb{U}(K_i) \subseteq U$. We claim that for each i $(1 \leq i \leq n)$,

$$N \in \mathbb{U}(K_i + N) \subseteq \mathbb{U}(K_i).$$

To see this, assume that $P \in \mathbb{U}(K_i + N)$, i.e., $K_i + N \nsubseteq P$. Thus, $K_i \nsubseteq P$, i.e., $P \in \mathbb{U}(K_i)$. On the other hand, $N \in \mathbb{U}(K_i)$, i.e., $K_i \nsubseteq N$. Therefore, $K_i + N \nsubseteq P$, i.e., $P \in \mathbb{U}(K_i + N)$. Consequently,

$$N \in \bigcap_{i=1}^{n} \mathbb{U}(K_i + N) \subseteq \bigcap_{i=1}^{n} \mathbb{U}(K_i) \subseteq U.$$

Thus, $\bigcap_{i=1}^{n} \mathbb{U}(K_i')$, where $K_i' := K_i + N$, is a neighborhood of N such that $\bigcap_{i=1}^{n} \mathbb{U}(K_i') \subseteq U$. Since for each i $(1 \leq i \leq n)$, then $N \subsetneqq K_i'$, $\mathbb{V}(K_i')$ can be covered by some finite subfamily $\mathcal{A}_i'$ of $\mathcal{A}$. But,

$$\mathbb{V}(N) \setminus [\bigcup_{i=1}^{n} \mathbb{V}(K_i')] = \mathbb{V}(N) \setminus [\bigcap_{i=1}^{n} \mathbb{U}(K_i')]^c = [\bigcap_{i=1}^{n} [\mathbb{U}(K_i')] \bigcap \mathbb{V}(N) \subseteq U,$$

and so $\mathbb{V}(N)$ can be covered by $\mathcal{A}_i' \bigcup \mathcal{A}_2' \bigcup ... \bigcup \mathcal{A}_n' \bigcup \{U\}$, contrary to our choice of N. Thus, there must exist a finite subfamily of $\mathcal{A}$ which covers Cl.Spec(M). Therefore, Cl.Spec(M) is a quasi-compact space. □

We need to recall the patch topology (see for example, [11,12] for definition and more details). Let X be a topological space. By the patch topology on X, we mean the topology which has as a sub-basis for its closed sets the closed sets and compact open sets of the original space. By a patch we mean a set closed in the patch topology. The patch topology associated with a spectral space is compact and Hausdorff [12]. Also, the patch topology associated with the Zariski topology of a ring R (R is not necessarily commutative) with ACC on ideals is compact and Hausdorff (see, [11 Proposition 16.1]).

Definition 3.4. Let M be an R-module, and let $\mathbb{P}(M)$ be the family of all subsets of Cl.Spec(M) of the form $\mathbb{V}(N) \bigcap \mathbb{U}(K)$, where N, $K \leq M$. Clearly $\mathbb{P}(M)$ contains Cl.Spec(M) and the empty set, since Cl.Spec(M) equals $\mathbb{V}(0) \bigcap \mathbb{U}(M)$ and the empty set equals $\mathbb{V}(M) \bigcap \mathbb{U}(0)$. Let $\mathbb{T}_p(M)$ be the collection U of all unions of finite intersections of elements of $\mathbb{P}(M)$. Then, $\mathbb{T}_p(M)$ is a topology on Cl.Spec(M) and is called the *patch-like topology of* M (in fact, $\mathbb{P}(M)$ is a sub-basis for the patch-like topology of M).

Clearly, if $M = R$, R commutative, then the patch-like topology on R as an R-module coincides with the patch topology of R as a ring.

Proposition 3.5. *Let M be an R-module. Then, Cl.Spec(M) with the patch-like topology is a Hausdorff space.*

Proof. Suppose distinct points P, $Q \in$ Cl.Spec(M). Since $P \neq Q$, then either $P \nsubseteq Q$ or $Q \nsubseteq P$. Assume that $P \nsubseteq Q$. By Definition 3.4, $U_1 := \mathbb{U}(M) \bigcap \mathbb{V}(P)$ is a patch-like-neighborhood of P and $U_2 := \mathbb{U}(P) \bigcap \mathbb{V}(Q)$ is a patch-like-neighborhood of Q. Clearly, $\mathbb{U}(P) \bigcap \mathbb{V}(P) = \emptyset$ and hence $U_1 \cap U_2 = \emptyset$. Thus, Cl.Spec(M) is a Hausdorff space. □

The proof of the next proposition is similar to the proof of Theorem 3.3.

Proposition 3.6. *Let M be a Noetherian R-module. Then, Cl.Spec(M) with the patch-like topology is a compact space.*

We need the following evident lemma.

Lemma 3.7. *Assume τ and τ^* are two topologies on X such that $\tau \subseteq \tau^*$. If X is quasi-compact (i.e., any open cover of has a finite subcover) in τ^*, then X is also quasi-compact in τ.*

Theorem 3.8. *Let M be a Noetherian R-module. Then, for each $n \in \mathbb{N}$ and submodules N_i $(1 \leq i \leq n)$ of M, $\mathbb{U}(N_1) \cap \mathbb{U}(N_2) \cap \cdots \cap \mathbb{U}(N_n)$ is a quasi-compact subset of Cl.Spec(M) with the Zariski-like topology. Consequently, Cl.Spec(M) has a basis of quasi-compact open subsets and the family of Zariski-like quasi-compact open subsets of Cl.Spec(M) is closed under finite intersections.*

Proof. For each submodule N of M, $\mathbb{V}(N) = \mathbb{V}(N) \bigcap \mathbb{U}(M)$ is an open subset of Cl.Spec(M) with the patch-like topology (see Definition 3.4). Thus, for each submodule N of M, $\mathbb{U}(N)$ is a closed subset in Cl.Spec(M). Thus, for each $n \in \mathbb{N}$ and $N_i \leq M$ $(1 \leq i \leq n)$, $\mathbb{U}(N_1) \cap \mathbb{U}(N_2) \cap \cdots \cap \mathbb{U}(N_n)$ is also a closed subset in Cl.Spec(M) with patch-like topology. Since every closed subset of a compact space is compact, then $\mathbb{U}(N_1) \cap \mathbb{U}(N_2) \cap \cdots \cap \mathbb{U}(N_n)$ is compact in Cl.Spec(M) with the patch-like topology and so by Lemma 3.7, it is quasi-compact in Cl.Spec(M) with the Zariski-like topology. Now, by Remark 1.3,

$$\mathbb{B} = \{\mathbb{U}(N_1) \cap \mathbb{U}(N_2) \cap \cdots \cap \mathbb{U}(N_n) \,:\, N_i \leq M,\ 1 \leq i \leq n,\ for\ some\ n \in \mathbb{N}\}$$

is a basis for the Zariski-like topology of M. On the other hand, if U is a Zariski-like quasi-compact open subset of Cl.Spec(M), then $U = \bigcup_{i=1}^{m}(\bigcap_{j=1}^{n_i} \mathbb{U}(N_j))$. It follows that the family of Zariski-like quasi-compact open subsets of Cl.Spec(M) is closed under finite intersections. □

Proposition 3.9. *Let M be a Noetherian R-module. Then, every irreducible closed subset of Cl.Spec(M) (with the Zariski-like topology) has a generic point.*

Proof. Let Y be an irreducible closed subset of Cl.Spec(M) (with the Zariski-like topology). First, we show that $Y = \bigcup_{P \in Y} \mathbb{V}(P)$. Clearly, $Y \subseteq \bigcup_{P \in Y} \mathbb{V}(P)$. By Lemma 3.1, for each $P \in Y$ we have $\mathbb{V}(P) = \overline{\{P\}} \subseteq \overline{Y}$, and since $\overline{Y} = Y$, then $\bigcup_{P \in Y} \mathbb{V}(P) \subseteq Y$. Thus, $Y = \bigcup_{P \in Y} \mathbb{V}(P)$. By Definition 3.4, for each $P \in Y$, $\mathbb{V}(P)$ is an open subset of Cl.Spec(M) with the patch-like topology. On the other hand, since $Y \subseteq$ Cl.Spec(M) is closed with the Zariski-like topology, then the complement of Y is open by this topology. This yields that the complement of Y is open with the patch-like topology, i.e., $Y \subseteq$ Cl.Spec(M) is closed with the patch-like topology. By Proposition 3.6, Cl.Spec(M) is compact with the patch-like topology and since $Y \subseteq$ Cl.Spec(M) is closed, then Y is also compact. Now, since $Y = \bigcup_{P \in Y} \mathbb{V}(P)$ and each $\mathbb{V}(P)$ is patch-like-open, then there exists a finite subset Y' of Y such that $Y = \bigcup_{P \in Y'} \mathbb{V}(P)$. Now, since Y is irreducible, then $Y = \mathbb{V}(P)$ for some $P \in Y'$. Therefore, we have $Y = \mathbb{V}(P) = \overline{\{P\}}$ for some $P \in Y$, i.e., P is a generic point for Y. □

Now, we are in position to prove the main results of this section.

Theorem 3.10. *Let M be a Noetherian R-module. Then, Cl.Spec(M) (with the Zariski-like topology) is a spectral space.*

Proof. By Theorem 3.2, Cl.Spec(M) is a T_0-space. Since M is Noetherian, then by Theorem 3.3, Cl.Spec(M) is quasi-compact. By Theorem 3.8, Cl.Spec(M) has a basis of quasi-compact open subsets and the family of quasi-compact open subsets of Cl.Spec(M) are closed under finite intersections. Finally, by Proposition 3.9, each irreducible closed subset of Cl.Spec(M) has a generic point. Thus, by Hochster's characterization, Cl.Spec(M) is a spectral spaces. □

Theorem 3.11. *Let R be a ring and M be an Artinian R-module. Then, Cl.Spec(M) (with the Zariski-like topology) is a spectral space.*

Proof. By Proposition 2.8, Cl.Spec(M) =Spec(M). Also, by Lemma 2.11 (i), $M/rad_R(M)$ is a Noetherian R-module. On the other hand, one can easily see that Cl.Spec(M) is homeomorphic to the prime spectrum $M/rad_R(M)$. Now, apply Theorem 3.10. □

4. Comments on the paper entitled "The Zariski topology on the prime spectrum of a module" by Chin-Pi Lu

Let M be an R-module. For any prime ideal $\mathcal{P}$ of R, we define $Spec_{\mathcal{P}}(M) = \{P \in Spec(M) \ : \ \mathcal{P} = (P : M)\}$. As in [13], for any submodule N of M we define the *variety* of N, denoted by $\widetilde{V}(N)$, to be the set of all prime submodules P of M such that $(N : M) \subseteq (P : M)$. Assume that $\bar{\zeta}(M)$ denotes the collection of all subsets $\widetilde{V}(N)$ of Spec(M). Then, $\bar{\zeta}(M)$ contains the empty set and Spec(M), and also $\bar{\zeta}(M)$, are closed under arbitrary intersections and finite unions. Thus, it is evident that for every module M there always exists a topology, say τ, on Spec(M) having $\bar{\zeta}(M)$ as the family of all closed sets. τ is called the Zariski topology on Spec(M). This topology on Spec(M) is studied in [13]. Unfortunately, the Proposition 5.2(3) of [13] (*which says that the set $\{P\}$ is closed in Spec(M) if and only if P is a maximal submodule of M with $Spec_{\mathcal{P}}(M) = \{P\}$, where $\mathcal{P} = (P : M)$*), and Proposition 6.3 of [13] (*which says that for each R-module M, Spec(M) is a T_1-space if and only if $Max(M) = Spec(M)$, where $Max(M)$ is the set of all maximal submodules of M*), are not true in general (see the following example).

Example 4.1. Let $R = \mathbb{Z}$ and $M = \mathbb{Q}$. It is easy to check that the zero submodule is the only prime submodule of M, i.e., $\mathrm{Spec}(M) = \{(0)\}$ (see also [5]). Thus, the Zariski topology on $\mathrm{Spec}(M)$ is the trivial topology and so $\mathrm{Spec}(M)$ is trivially a T_1-space. But, $Max(M) = \emptyset$. Moreover, $\{(0)\}$ is a closed subset of $\mathrm{Spec}(M)$, but (0) is not a maximal submodule of M.

Fortunately, the Propositions 5.2(3) and 6.3 of [13] are true when the R-module M is finitely generated. Moreover, using the proofs in [13], a correct version of [13, Propositions 5.2(3)] and also a correct version [13, Proposition 6.3] are as follows, respectively, while the converses are not true in general (see Example 4.4 below). (We recall that a submodule P of M will be called *maximal prime* if P is a prime submodule of M and there is no prime submodule Q of M such that $P \subset Q$).

Proposition 4.2. *Let M be an R-module and $P \in Spec(M)$. If the set $\{P\}$ is closed in $Spec(M)$, then P is a maximal prime submodule of M, and $|Spec_{\mathcal{P}}(M)| \leq 1$, where $\mathcal{P} = (P : M)$. The converse is also true, when M is finitely generated.*

Proposition 4.3. *Let M be an R-module such that $Spec(M)$ is a T_1-space. Then, every prime submodule of M is a maximal prime submodule, and $|Spec_{\mathcal{P}}(M)| \leq 1$, for every $\mathcal{P} \in Spec(R)$. The converse is also true, when M is finitely generated.*

Example 4.4. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_2 \oplus \mathbb{Q}$. Clearly, $\{\mathbb{Z}_2 \oplus (0), (0) \oplus \mathbb{Q}\} \subseteq \mathrm{Spec}(M)$. We claim that

$$Spec(M) = \{\mathbb{Z}_2 \oplus (0), (0) \oplus \mathbb{Q}\}.$$

To see this, we assume that P is a prime submodule of M. Since $2(1, 0) = (0, 0) \in P$ and $2M \neq (0)$, then $(1, 0) \in P$, and so $\mathbb{Z}_2 \oplus (0) \subseteq P$. If $P \subseteq \mathbb{Z}_2 \oplus (0)$, then $P = \mathbb{Z}_2 \oplus (0)$. Now, let $P \nsubseteq \mathbb{Z}_2 \oplus (0)$. Then, there exist nonzero elements $m, n \in \mathbb{Z}$ such that $(k, \frac{m}{n}) \in P$, where either $k = 0$ or $k = 1$. Thus, $(0, 2m) = 2n(k, \frac{m}{n}) \in P$. It follows that for each $\frac{s}{t} \in \mathbb{Q}$, $2mt(0, \frac{s}{t}) = (0, 2ms) = s(0, 2m) \in P$. Since P is prime, then either $(0, \frac{s}{t}) \in P$ or $2mtM \subseteq P$. Clearly, $2mtM = (0) \oplus \mathbb{Q}$. It follows that $(0) \oplus \mathbb{Q} \subseteq P$ and so $P = (0) \oplus \mathbb{Q}$. Assume $P_1 = \mathbb{Z}_2 \oplus (0)$ and $P_2 = (0) \oplus \mathbb{Q}$. Clearly, both prime submodules P_1 and P_2 are maximal prime submodules of M and also $(P_1 : M) = (0)$ and $(P_2 : M) = 2\mathbb{Z}$.

Thus, $|Spec_{\mathcal{P}}(M)| \leq 1$, for every $\mathcal{P} \in Spec(R)$. But, $\{\overline{P_1}\} = \widetilde{V}(P_1) = \{P_1, P_2\}$ and hence $\{P_1\}$ is not closed in Spec(M) and so Spec(M) is not a T_1-space. Thus, the converses Propositions 4.2 and 4.3 are not true for M.

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