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ON THE BANDWIDTH OF MOBIUS GRAPHS

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Abstract. Bandwidth labelling is a well known research area in graph theory. We provide a new proof that the bandwidth of Mobius ladder is 4, if it is not a K_4, and investigate the bandwidth of a wider class of Mobius graphs of even strips.

Keywords: Mobius graphs, Cartesian product of graphs, labelling of graphs, bandwidth of a graph.
MSC(2010): Primary: 05C78; Secondary: 97K30.

1. **Introduction**

Graph labelling provides useful mathematical models for a wide range of applications, such as data security, mobile telecommunication systems, cryptography, various coding theory problems, communication networks, bioinformatics and x-ray crystallography [3]. Among all graph labelling problems, bandwidth numbering of graphs has perhaps attracted the most attention in the literature. The bandwidth numbering problem was proposed independently by Harper [10] and Harary [9]. Suppose that G is a finite simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For undefined terminology we refer the readers to [7]. A *labelling* f is a bijection $f : V \to X_n$ where $|V| = n$ and $X_n = \{1, 2, ..., n\}$. Let $F = \{f : V \to X_n, f \text{ a bijection}\}$. We define the *bandwidth of a labelling* f of G as $BW_f(G) = \max_{uv\in E} |f(u) - f(v)|$. The *bandwidth of* G is given by $BW(G) = \min_{f\in F} \{\max_{uv\in E} |f(u) - f(v)|\}$. We say that f is a *bandwidth labelling* of G if $BW_f(G) = BW(G)$. It is known that the bandwidth of a complete graph K_n is $n - 1$, and that the bandwidth of a non-planar graph is at least 4 [2, 3]. Let P_m, C_m denote, respectively, a path and a cycle on m vertices. The Cartesian product of two graphs G_1 and G_2, written as $G_1 \times G_2$, is defined to be the graph whose vertex set is $V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \times G_2$

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if either $u_1 = v_1$ and u_2 is adjacent to v_2 or vice versa. It is known that $BW(P_m \times P_n) = \min\{m, n\}$ [1,2,5,8], $BW(P_m \times C_n) = \min\{2m, n\}$ [1,4].

2. Bandwidth calculations for Mobius graphs

Let $2 \leq m, n$ and consider $P_m \times P_n$ with $V(P_m \times P_n) = \{(i, j) : 1 \leq i \leq m, 1 \leq j \leq n\}$. Form a new graph $M_{m,n} = M$ by adjoining the edges $(i, 1) \leftrightarrow (m - i + 1, n)$ $(1 \leq i \leq m)$. In this way we are 'identifying' the vertical sides of the 'rectangle' with a half twist so the array corresponds to a Mobius strip. We call M a *Mobius graph*. We give an alternative proof of the bandwidth of the *Mobius ladder* (the $m = 2$ case) [6] and proceed further to the hard case when $m = 2k$, i.e., Mobius graphs of even strips. However, at present we are unable to investigate the bandwidth of Mobius graphs of odd strips and this is therefore suggested as a future work.

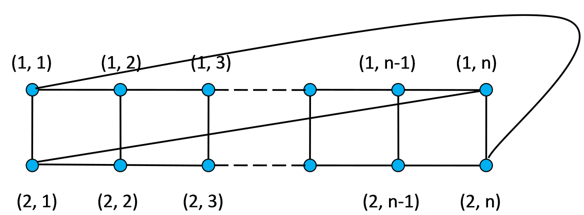

FIGURE 2.1. A Mobius Ladder $M_{2,n}$

Theorem 2.1. *The bandwidth of the Mobius ladder $M_{2,n}$ for $n > 2$ is 4. $BW(M_{2,2}) = 3$, and the bandwidth of Mobius graphs $M_{m,n}$ satisfies $\min\{m, 2n\} \leq BW(M_{2k,n}) \leq 2\min\{m, n\}$, where $m = 2k$ and $n \geq 3$.*

Proof. First we consider the case when $m = 2$, as in Figure 2.1.

There is a Hamilton cycle: $C : (1, 1), (1, 2), \ldots, (1, n), (2, 1), (2, 2), \ldots, (2, n), (1, 1)$, together with additional edges: $(1, 1) \to (2, 1), (1, 2) \to (2, 2), \ldots, (1, n) \to (2, n)$.

e.g. $n = 4$:

N.B. The central crossing point in the Figure 2.2 is not a vertex. In general $BW(M_{2,n}), n \geq 3$, is at least 4 as $M_{2,n}$ is not planar: by deleting the edges $(1, 2) \to (2, 2), (1, 3) \to (2, 3), \ldots, (1, n-2) \to (2, n-2)$ and removing degree 2 vertices $(1, 2), \ldots, (1, n-2)$ and the vertices $(2, 2), \ldots, (2, n-2)$ we find a copy of $K_{3,3} = M_{2,3}$. Since $BW(K_{3,3}) = 4$, we conclude that $BW(M_{2,n}) \geq 4$ for $n \geq 3$. We note that $M_{2,2} = K_4$, so $BW(M_{2,2}) = BW(K_4) = 4 - 1 = 3$. We conclude that for $n \geq 3$, $BW(M_{2,n}) \leq 4$ by finding a labelling of bandwidth 4.

In general, $M_{2,n}$ consists of a cycle of order $2n$ with opposite pairs of vertices joined by a single edge.

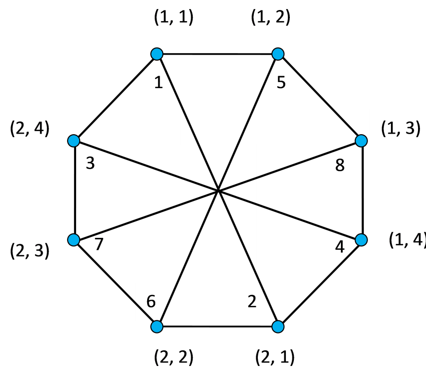

FIGURE 2.2. Labelling of $M_{2,4}$

Case 1: Suppose n is even, so $2n \equiv 0 \pmod 4$. Put $2n = 4k$, say. We label the cycle as follows, from an arbitrary point;

Numbers $\equiv 1 \pmod 4$ $1 \to 5 \to 9 \to \ldots \to 2n-3 \to$

Numbers $\equiv 0 \pmod 4$ $\to 2n \to 2n-4 \to \ldots \to 4 \to$

Numbers $\equiv 3 \pmod 4$ $\to 3 \to 7 \to \ldots \to 2n-1 \to$

Numbers $\equiv 2 \pmod 4$ $\to 2n-2 \to 2n-6 \to \ldots \to 2 \to 1$, where each congruence class contains k vertices.

This defines a labelling f of $M_{2,n}$ in which adjacent labels in the Hamilton cycle differ by at most 4. The opposite pairs in the cycle are then $(1,3), (5,7), (9,11), \ldots, (2n-3, 2n-1)$ with difference of 2 in labels, and $(2n, 2n-2), (2n-4, 2n-6), \ldots, (4,2)$ with the same difference of 2. Hence $BW(M_{2,n}) = 4$ in the case where n is even.

Case 2: Suppose n is odd, so that $2n \equiv 2 \pmod 4$; put $2n \equiv 2\,(2k+1) = 4k+2$

Numbers $\equiv 1 \pmod 4$ $1 \to 5 \to 9 \to \ldots \to 2n-1 \to$ ($k+1$ vertices)

Numbers $\equiv 0 \pmod 4$ $\to 2n-2 \to 2n-6 \to \ldots \to 4 \to$ (k vertices)

Numbers $\equiv 3 \pmod 4$ $\to 3 \to 7 \to \ldots \to 2n-3 \to$, ($k$ vertices)

Numbers $\equiv 2 \pmod 4$ $\to 2n \to 2n-4 \to \ldots \to 2 \to 1$ ($k+1$ vertices)

The opposite pairs in the cycle are then $(1,3), (5,7), \ldots, (2n-5, 2n-3), (2n-1, 2n), (2n-2, 2n-4), (2n-6, 2n-8), \ldots, (4,2)$; with all differences in labels of 2. Hence in both cases $BW_f(M_{2,n}) = 4$. Therefore $BW(M_{2,n}) = 4$, as claimed.

Next we consider the general case $m = 2k$, $k \geq 2$. The vertex set is partitioned into k disjoint cycles, each of $2n$ vertices, these being:

C_i : $(i,1), (i,2), \ldots, (i,n), (m-i+1,1), (m-i+1,2), \ldots, (m-i+1,n)$; $i = 1, 2, \ldots, k$. The other edges of the graph are: $(i,j) \to (i+1,j)$, $1 \leq i \leq k, 1 \leq j \leq n$.

Edge Count: We count the edges in two ways;

(i) As the graph $M_{2k,n}$ contains $m = 2k$ rows and n columns of the graph $P_m \times P_n$ thus we have $n-1$ edges in each of the m rows and $m-1$ edges in each of the $n-1$ columns in addition to the m edges that arises with the Mobius condition while connecting one end of each of the m rows to another row. Hence we have

$$\begin{aligned} E(M_{2k,n}) &= m(n-1) + n(m-1) + m \\ &= 2mn - m - n + m = 2mn - n. \end{aligned}$$

(ii) Since the graph $M_{2k,n}$ contains k cycles of length $2n$, it thus has $k(2n)$ edges, furthermore each vertex of C_i is adjacent to a vertex of C_{i+1} for $(1 \le i \le k-1)$, this gives $(k-1)2n$ edges. In addition C_k contains n more internal edges. Hence

$$\begin{aligned} E(M_{2k,n}) &= k(2n) + (k-1)2n + n \\ &= 2kn + 2kn - 2n + n = 4kn - n = 2mn - n. \end{aligned}$$

So $M_{2k,n}$ contains a copy of the cylinder graph $P_k \times C_{2n}$; one end cycle arises from C_1. However the end cycle C_k contains more edges and is a copy of $M_{2,n}$. Thus

$$\begin{aligned} BW(M_{2k,n}) &\ge BW(P_k \times C_{2n}) \\ &= \min\{2k, 2n\} \\ &= \min\{m, 2n\}. \end{aligned}$$

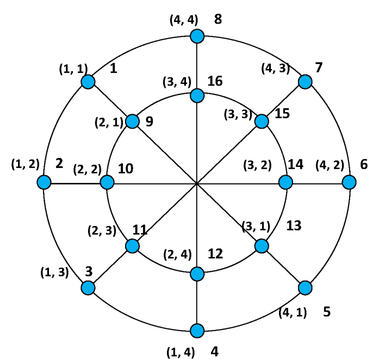

FIGURE 2.3. Bandwidth labelling of $M_{2k,n}$; $k = 2, n = 4$

Labelling: We label the graph $M_{2k,n}$ to show under what condition $BW(M_{2k,n}) \le \min\{2m, 2n\}$. First, as in Figure 2.3 if we label C_i as $(i, j) \to 2n(i-1) + j$

, for $(1 \leq i \leq k)$,$(m - i + 1, j) \mapsto 2n(i - 1) + 2k + j$, for $(1 \leq j \leq n)$ the adjacent vertices in $M_{2k,n}$ differ by at most $2n$. As $(i, j) \leftrightarrow (i + 1, j)$ has label difference $2n(i + 1) + j - (2ni + j) = 2ni + 2n + j - 2ni - j = 2n$, the edges $(i, j) \leftrightarrow (i, j + 1)$ have label differences $2n(i - 1) + j - (2n(i - 1) + j + 1) = 1$, and so on. Hence $BW(M_{2k,n}) \leq 2n$.

On the other hand we can label $M_{2k,n}$ row wise as follows; number one row (as starting point) of the cylinder from left to right as $1, 2, \ldots, 2k$; number the remaining adjacent rows clockwise as $2k + 1, 2k + 2, \ldots, 4k$; $6k + 1, 6k + 2, \ldots, 8k; \ldots$, $2(n - 2)k + 1, 2(n - 2)k + 2, \ldots, 2(n - 1)k$ if n is odd, otherwise label up to the $(\frac{n}{2} + 1)$th row as $2(n - 1)k + 1, 2(n - 1)k + 2, \ldots, 2nk = mn$ if n is even as in the following figure.

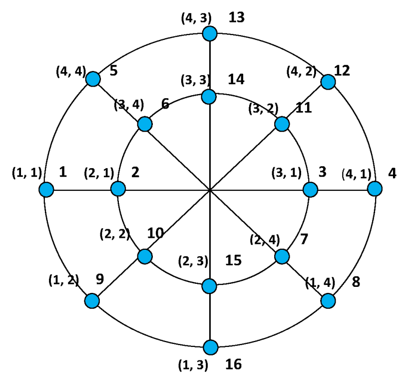

FIGURE 2.4. Alternative bandwidth labelling of $M_{2k,n}$; $k = 2, n = 4$

The remaining rows are numbered clockwise starting from the adjacent row to the first numbered row as $4k + 1, 4k + 2, \ldots, 6k$; $8k + 1, 8k + 2, \ldots, 10k$; $\ldots$; until all the rows are numbered. The order in which the rows are chosen according to the 4-labelling of $M_{2,n}$ as described earlier. This defines a labelling f of $M_{2k,n}$ in which the adjacent cells in the cycles have at most difference $4k = 2m$ and the adjacent cells in rows have at most difference 1. Hence $BW(M_{2k,n}) \leq \min\{2m, 2n\} = 2\min\{m, n\}$. Therefore

$$\min\{m, 2n\} \leq BW(M_{2k,n}) \leq 2\min\{m, n\}, \text{ where } m = 2k. \tag{2.1}$$

Note that all terms in (2.1) are equal unless $m < 2n$, i.e $2k < 2n \Leftrightarrow k < n$. $\square$

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