ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 41 (2015), No. 3, pp. 581–590

Title:

Volume difference inequalities for the projection and intersection bodies

Author(s):

C. J. Zhao and W. S. Cheung

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 41 (2015), No. 3, pp. 581–590
Online ISSN: 1735-8515

VOLUME DIFFERENCE INEQUALITIES FOR THE PROJECTION AND INTERSECTION BODIES

C. J. ZHAO* AND W. S. CHEUNG

(Communicated by Mohammad Bagher Kashani)

Abstract. In this paper, we introduce a new concept of volumes difference function of the projection and intersection bodies. Following this, we establish the Minkowski and Brunn-Minkowski inequalities for volumes difference function of the projection and intersection bodies.
Keywords: Projection body, intersection body, volume difference, Minkowski inequality, Brunn-Minkowski inequality.
MSC(2010): Primary: 52A40; Secondary: 52A30.

1. Introduction

The well-known classical Brunn-Minkowski inequality can be stated as follows:

If K and L are convex bodies in $\mathbb{R}^n$, then

$$V(K+L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n}, \tag{1.1}$$

with equality if and only if K and L are homothetic.

The Brunn-Minkowski inequality has in recent decades dramatically extended its influence in many areas of mathematics. Various applications have surfaced, for example, to probability and multivariate statistics, shape of crystals, geometric tomography, elliptic partial differential equations, and combinatorics (see [1, 5, 8–10, 19]). Several remarkable analogs have been established in other areas, such as potential theory and algebraic geometry (see [3,4,6,7,12,14,18,20–22]). Reverse forms of the inequality are important in the local theory of Banach spaces (see [19]). An elegant survey on this inequality is provided by Gardner (see [11]). In fact, a general version of the Brunn-Minkowski's inequality holds ([11]): If K and L are convex bodies in $\mathbb{R}^n$ and

Article electronically published on June 15, 2015.
Received: 15 March 2013, Accepted: 2 April 2014.
*Corresponding author.

$0 \leq i \leq n-1$, then

$$W_i(K+L)^{1/(n-i)} \geq W_i(K)^{1/(n-i)} + W_i(L)^{1/(n-i)}, \tag{1.2}$$

with equality if and only if K and L are homothetic.

In 2004, the quermassintegral difference function was defined by Leng [13] as follows

$$Dw_i(K,D) = W_i(K) - W_i(D),$$

where K and D are convex bodies, $D \subseteq K$, and $0 \leq i \leq n-1$. Inequality 1.2 was extended to the quermassintegral difference of convex bodies as follows.

Theorem A. *K, L, and D be convex bodies in $\mathbb{R}^n$. If $D \subseteq K$, and D' is a homothetic copy of D, then*

$$Dw_i(K+L, D+D')^{1/(n-i)} \geq Dw_i(K,D)^{1/(n-i)} + Dw_i(L,D')^{1/(n-i)}, \tag{1.3}$$

with equality for $0 \leq i < n-1$ if and only if K and L are homothetic and $(W_i(K), W_i(D)) = \mu(W_i(L), W_i(D'))$, where μ is a constant. .

In 2010, the dual quermassintegral difference function was defined by Lv [24] as follows

$$D\tilde{w}_i(K,D) = \tilde{W}_i(K) - \tilde{W}_i(D),$$

where K and D are star bodies and $D \subseteq K$. Dual Brunn-Minkowski-type inequality for the dual quermassintegral difference was also established.

Motivated by the work of Leng and Lv, we give the following definition:

Definition 1.1. *Let K be a convex body and D be a star body in $\mathbb{R}^n$, with $D \subseteq K$, the mixed volumes difference function of $\mathbf{\Pi}K$ and $\mathbf{I}D$ is defined for $0 \leq i < n$ by*

$$Dw_i^*(\mathbf{\Pi}K, \mathbf{I}D) = W_i(\mathbf{\Pi}K) - \tilde{W}_i(\mathbf{I}D). \tag{1.4}$$

Taking $i = 0$ in 1.4, will change it to $Dv^*(\mathbf{\Pi}K, \mathbf{I}D) = V(\mathbf{\Pi}K) - V(\mathbf{I}D)$, which is called volume difference function of a projection body $\mathbf{\Pi}K$ and an intersection body $\mathbf{I}D$.

In 1993, Lutwak established the Brunn-Minkowski inequality for mixed projection bodies as follows:

Theorem B ([15]) *Let K and L be convex bodies in $\mathbb{R}^n$. If $0 \leq j < n-2$ and $0 \leq i < n$, then*

$$\begin{aligned} &W_i(\mathbf{\Pi}_j(K+L))^{1/(n-i)(n-j-1)} \\ &\geq W_i(\mathbf{\Pi}_jK)^{1/(n-i)(n-j-1)} + W_i(\mathbf{\Pi}_jL)^{1/(n-i)(n-j-1)}, \end{aligned} \tag{1.5}$$

with equality if and only if K and L are homothetic.

The first aim of this paper is to establish the Brunn-Minkowski inequality for volume difference function of the mixed projection and the mixed intersection bodies.

Theorem 1.2. *Let K and L be convex bodies, and D and D' be star bodies in $\mathbb{R}^n$. If $D \subseteq K$, $D' \subseteq L$, $0 \leq j < n-2$ and $0 \leq i < n-1$, then*

$$(W_i(\mathbf{\Pi}_j(K+L)) - \tilde{W}_i(\mathbf{I}_j(D \tilde{+} D')))^{1/(n-i)(n-j-1)}$$
$$\geq (W_i(\mathbf{\Pi}_j K) - \tilde{W}_i(\mathbf{I}_j D))^{1/(n-i)(n-j-1)}$$
$$+(W_i(\mathbf{\Pi}_j L) - \tilde{W}_i(\mathbf{I}_j D'))^{1/(n-i)(n-j-1)}, \tag{1.6}$$

with equality if and only if K and L are homothetic, D and D' are dilates, and $(W_i(\mathbf{\Pi}_j K), \tilde{W}_i(\mathbf{I}_j D)) = \mu(W_i(\mathbf{\Pi}_j L), \tilde{W}_i(\mathbf{I}_j D'))$, where μ is a constant.

Remark 1.3. *Here, $\tilde{+}$ is the radial Minkowski sum, $\mathbf{\Pi}_j K$ denotes jth mixed projection bodies of a convex body K and $\mathbf{I}_j D$ denotes jth mixed intersection bodies of a star body K (see Section 2).*

In case D and D' are single points, 1.6 reduces to inequality 1.5.

In [15], Lutwak established the Minkowski inequality for mixed projection bodies as follows.

Theorem C ([15]) *Let K and L be convex bodies in $\mathbb{R}^n$. If $0 \leq j < n-2$ and $0 \leq i < n$, then*

$$W_i(\mathbf{\Pi}_j(K,L))^{n-1} \geq W_i(\mathbf{\Pi} K)^{n-j-1} W_i(\mathbf{\Pi} L)^j, \tag{1.7}$$

with equality if and only if K and L are homothetic.

The another aim of this paper is to establish the Minkowski inequality for volume difference function of the mixed projection and the mixed intersection bodies.

Theorem 1.4. *Let K and L be convex bodies, and D and D' be star bodies in $\mathbb{R}^n$. If $D \subseteq K, D' \subseteq L$, $0 \leq j < n-2$ and $0 \leq i < n-1$, then*

$$(W_i(\mathbf{\Pi}_j(K,L)) - \tilde{W}_i(\mathbf{I}_j(D,D')))^{n-1}$$
$$\geq (W_i(\mathbf{\Pi} K) - \tilde{W}_i(\mathbf{I} D)))^{n-j-1}(W_i(\mathbf{\Pi} K) - \tilde{W}_i(\mathbf{I} D)))^j, \tag{1.8}$$

with equality if and only if K and L are homothetic, D and D' are dilates, and $(W_i(\mathbf{\Pi} K), \tilde{W}_i(\mathbf{I} D)) = \mu(W_i(\mathbf{\Pi} L), \tilde{W}_i(\mathbf{I} D'))$, where μ is a constant.

Remark 1.5. *In case D and D' are single points, 1.8 reduces to inequality 1.7.*

We refer the reader to the next section for above interrelated notations, definitions and background materials.

2. Background materials

The setting for this paper is the n-dimensional Euclidean space $\mathbb{R}^n (n > 2)$. Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. We reserve the letter u for unit vectors and the letter B for the unit ball centered at the origin. The boundary of B is S^{n-1}. For $u \in S^{n-1}$, let E_u denote the hyperplane, through the origin, that is orthogonal to u. We will use K^u to denote the image of K under an orthogonal projection onto the hyperplane E_u. We use $V(K)$ for the n-dimensional volume of convex body K. The support function of $K \in \mathcal{K}^n$, $h(K,\cdot)$, is defined on $\mathbb{R}^n$ by $h(K,\cdot) = Max\{x \cdot y : y \in K\}$. Let δ denote the Hausdorff metric on $\mathcal{K}^n$, namely, for $K, L \in \mathcal{K}^n$, $\delta(K,L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$.

Associated with a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin, is its radial function $\rho(K,\cdot) : S^{n-1} \to \mathbb{R}$ defined for $u \in S^{n-1}$, by $\rho(K,u) = Max\{\lambda \geq 0 : \lambda u \in K\}$. If $\rho(K,\cdot)$ is positive and continuous, K will be called a star body. Let $\mathcal{S}^n$ denote the set of star bodies in $\mathbb{R}^n$.

2.1. **Quermassintegral of convex body.** For $K_1, \ldots, K_r \in \mathcal{K}^n$ and $\lambda_1, \ldots, \lambda_r \geq 0$, the volume of the Minkowski liner combination $\lambda_1 K_1 + \cdots + \lambda_r K_r$ is a homogeneous nth-degree polynomial in the λ_i,

$$V(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum V_{i_1,\ldots,i_n} \lambda_{i_1} \cdots \lambda_{i_n}, \tag{2.1}$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ whose entries are positive integers not exceeding r. If we require the coefficients of the polynomial in 2.1 to be symmetric in their arguments, then they are uniquely determined. The coefficient $V_{i_1,\ldots,i_n}$ is nonnegative and depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$. It is written as $V(K_{i_1}, \ldots, K_{i_n})$ and is called the mixed volume of $K_{i_1}, \ldots, K_{i_n}$. If $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = L$, the mixed volume is written as $V_i(K,L)$. The mixed volume $V_i(K,B)$ is written as $W_i(K)$ and called the quermassintegral of a convex body K.

2.2. **Dual quermassintegral of star body.** We introduce a vector addition on $\mathbb{R}^n$, which we call radial addition, as follows. If $x_1, \ldots, x_r \in \mathbb{R}^n$, then $x_1 \tilde{+} \ldots \tilde{+} x_r$ is defined to be the usual vector sum of $x_1, \ldots, x_r$, provided that $x_1, \ldots, x_r$ all lie in a 1-dimensional subspace of $\mathbb{R}^n$, and as the zero vector otherwise.

If $K_1, \ldots, K_r \in \mathcal{S}^n$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$, then *the radial Minkowski linear combination*, $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r$, is defined by $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r = \{\lambda_1 x_1 \tilde{+} \cdots \tilde{+} \lambda_r x_r : x_i \in K_i\}$. It has the following important property that for $K, L \in \mathcal{S}^n$ and $\lambda, \mu \geq 0$,

$$\rho(\lambda K \tilde{+} \mu L, \cdot) = \lambda \rho(K,\cdot) + \mu \rho(L,\cdot) \tag{2.2}$$

For $K_1, \ldots, K_r \in \mathcal{S}^n$ and $\lambda_1, \ldots, \lambda_r \geq 0$, the volume of the radial Minkowski liner combination $\lambda_1 K_1 \tilde{+} \ldots \tilde{+} \lambda_r K_r$ is a homogeneous nth-degree polynomial in the λ_i, defined by

$$V(\lambda_1 K_1 \tilde{+} \ldots \tilde{+} \lambda_r K_r) = \sum \tilde{V}_{i_1,\ldots,i_n} \lambda_{i_1} \cdots \lambda_{i_n}, \tag{2.3}$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ whose entries are positive integers not exceeding r. If we require the coefficients of the polynomial in 2.3 to be symmetric in their arguments, then they are uniquely determined. The coefficient $\widetilde{V}_{i_1,\ldots,i_n}$ is nonnegative and depends only on the bodies $K_{i_1},\ldots,K_{i_n}$. It is written as $\widetilde{V}(K_{i_1},\ldots,K_{i_n})$ and is called the dual mixed volume of $K_{i_1},\ldots,K_{i_n}$. If $K_1=\cdots=K_{n-i}=K$, $K_{n-i+1}=\cdots=K_n=L$, the dual mixed volumes is written as $\widetilde{V}_i(K,L)$. The dual mixed volumes $\widetilde{V}_i(K,B)$ is written as $\widetilde{W}_i(K)$ and called as the dual quermassintegral of the star body K.

2.3. **Mixed projection bodies.** If $K_1,\ldots,K_r\in\mathcal{K}^n$ and $\lambda_1,\ldots,\lambda_r\geq 0$, then the projection body of the Minkowski linear combination $\lambda_1K_1+\cdots+\lambda_rK_r\in\mathcal{K}^n$ can be written as a symmetric homogeneous polynomial of degree $(n-1)$ in the λ_i ([15]):

$$\mathbf{\Pi}(\lambda_1K_1+\cdots+\lambda_rK_r)=\sum\lambda_{i_1}\ldots\lambda_{i_{n-1}}\mathbf{\Pi}_{i_1\cdots i_{n-1}} \tag{2.4}$$

where the sum is a Minkowski sum taken over all $(n-1)$-tuples $(i_1,\ldots,i_{n-1})$ of positive integers not exceeding r. The body $\mathbf{\Pi}_{i_1\ldots i_{n-1}}$ depends only on the bodies $K_{i_1},\ldots,K_{i_{n-1}}$, and is uniquely determined by (2.3.1), it is called *the mixed projection bodies* of $K_{i_1},\ldots,K_{i_{n-1}}$, and is written as $\mathbf{\Pi}(K_{i_1},\ldots,K_{i_{n-1}})$. If $K_1=\cdots=K_{n-1-i}=K$ and $K_{n-i}=\cdots=K_{n-1}=L$, then $\mathbf{\Pi}(K_{i_1},\ldots,K_{i_{n-1}})$ will be written as $\mathbf{\Pi}_i(K,L)$. If $L=B$, then $\mathbf{\Pi}_i(K,L)$ is denoted by $\mathbf{\Pi}_iK$ and when $i=0$, $\mathbf{\Pi}_iK$ is denoted by $\mathbf{\Pi}K$, where $\mathbf{\Pi}K$ is the projection body of K (see [15]).

2.4. **Mixed intersection bodies.** For $K\in\mathcal{S}^n$, there is a unique star body $\mathbf{I}K$ whose radial function satisfies for $u\in S^{n-1}$,

$$\rho(\mathbf{I}K,u)=v(K\cap E_u). \tag{2.5}$$

It is called the *intersection bodies* of K. From a result of Busemann, it follows that $\mathbf{I}K$ is convex if K is convex and origin symmetric. Clearly any intersection body is centred.

The volume of intersection bodies is given by

$$V(\mathbf{I}K)=\frac{1}{n}\int_{S^{n-1}}v(K\cap E_u)^n dS(u).$$

The mixed intersection bodies of $K_1,\ldots,K_{n-1}\in\mathcal{S}^n$, $\mathbf{I}(K_1,\ldots,K_{n-1})$, whose radial function is defined by

$$\rho(\mathbf{I}(K_1,\ldots,K_{n-1}),u)=\tilde{v}(K_1\cap E_u,\ldots,K_{n-1}\cap E_u), \tag{2.6}$$

where $\tilde{v}$ is the $(n-1)$-dimensional dual mixed volume. If $K_1=\cdots=K_{n-i-1}=K,K_{n-i}=\cdots=K_{n-1}=L$, then $\mathbf{I}(K_1,\ldots,K_{n-1})$ is written as $\mathbf{I}_i(K,L)$. If $L=B$, then $\mathbf{I}_i(K,L)$ is written $\mathbf{I}_iK$ and is called the ith intersection body of K. For $\mathbf{I}_0K$ simply write $\mathbf{I}K$.

3. Main results

Theorem 3.1. *Let K and L be convex bodies, and D and D' be star bodies in $\mathbb{R}^n$. If $D \subseteq K$, $D' \subseteq L$, $0 \leq j < n-2$ and $0 \leq i < n-1$, then*

$$Dw_i^*(\mathbf{\Pi}_j(K+L), \mathbf{I}_j(D\tilde{+}D'))^{1/(n-i)(n-j-1)}$$

$$\geq Dw_i^*(\mathbf{\Pi}_j K, \mathbf{I}_j D)^{1/(n-i)(n-j-1)} + Dw_i^*(\mathbf{\Pi}_j L, \mathbf{I}_j D')^{1/(n-i)(n-j-1)}, \tag{3.1}$$

with equality if and only if K and L are homothetic, D and D' are dilates, and $(W_i(\mathbf{\Pi}_j K), \tilde{W}_i(\mathbf{I}_j D)) = \mu(W_i(\mathbf{\Pi}_j L), \tilde{W}_i(\mathbf{I}_j D'))$, where μ is a constant.

We need the following lemmas to prove Theorem 3.1.

Lemma 3.2 ([9]). *If K is a convex body in $\mathbb{R}^n$ with $o \in intK$, then*

$$\mathbf{I}K \subseteq \mathbf{\Pi}K. \tag{3.2}$$

Lemma 3.3 ([16]). *If K is a convex body in $\mathbb{R}^n$ with $o \in intK$, then*

$$W_i(K) \geq \tilde{W}_i(K), \tag{3.3}$$

with equality if and only if K is a n-ball.

Lemma 3.4. *(see [2]) Let*

$$\phi(x) = (x_1^p - x_2^p - \cdots - x_n^p)^{1/p}, \ p > 1,$$

for x_i in the region $\mathbb{R}$ defined by

$$(a) \ x_i \geq 0,$$

$$(b) \ x_1 \geq (x_2^p + x_3^p + \cdots + x_n^p)^{1/p}.$$

Then for $x, y \in \mathbb{R}^n$, we have

$$\phi(x+y) \geq \phi(x) + \phi(y), \tag{3.4}$$

with equality if and only if $x = \mu y$ where μ is a constant.

Proof of Theorem 3.1. If $K, L \in \mathcal{K}^n$, $0 \leq j < n-2$ and $0 \leq i < n$, then

$$W_i(\mathbf{\Pi}_j(K+L))^{1/(n-i)(n-j-1)} \tag{3.5}$$

$$\geq W_i(\mathbf{\Pi}_j K)^{1/(n-i)(n-j-1)} + W_i(\mathbf{\Pi}_j L)^{1/(n-i)(n-j-1)}, \tag{3.6}$$

with equality if and only if K and L are homothetic. □

Moreover, in view of the following inequality (see [23]), if $D, D' \in \mathcal{S}^n$, $0 \leq j < n-2$ and $0 \leq i < n$, then

$$\tilde{W}_i(\mathbf{I}_j(D\tilde{+}D'))^{1/(n-i)(n-j-1)} \tag{3.7}$$

$$\leq \tilde{W}_i(\mathbf{I}_j D)^{1/(n-i)(n-j-1)} + \tilde{W}_i(\mathbf{I}_j D')^{1/(n-i)(n-j-1)}, \tag{3.8}$$

with equality if and only if D and D' are dilates.

From 3.5 and 3.7, we have

$$(3.9)\quad W_i(\mathbf{\Pi}_j(K+L)) - \tilde{W}_i(\mathbf{I}_j(D\tilde{+}D'))$$

$$(3.10)\quad \geq \left[W_i(\mathbf{\Pi}_jK)^{1/(n-i)(n-j-1)} + W_i(\mathbf{\Pi}_jL)^{1/(n-i)(n-j-1)}\right]^{(n-i)(n-j-1)}$$

$$(3.11)\quad -\left[\tilde{W}_i(\mathbf{I}_jD)^{1/(n-i)(n-j-1)} + \tilde{W}_i(\mathbf{I}_jD')^{1/(n-i)(n-j-1)}\right]^{(n-i)(n-j-1)},$$

with equality if and only if K and L are homothetic, and D and D' are dilates.

From Lemma 3.1 and 3.2, we have

$$W_i(\mathbf{\Pi}_j(K+L)) \geq \tilde{W}_i(\mathbf{\Pi}_j(K+L)) \geq \tilde{W}_i(\mathbf{I}_j(D\tilde{+}D')),$$
$$W_i(\mathbf{\Pi}_jK) \geq \tilde{W}_i(\mathbf{\Pi}_jK) \geq \tilde{W}_i(\mathbf{I}_jD),$$

and

$$W_i(\mathbf{\Pi}_jL) \geq \tilde{W}_i(\mathbf{\Pi}_jL) \geq \tilde{W}_i(\mathbf{I}_jD').$$

By using Lemma 3.3, we have

$$\left[W_i(\mathbf{\Pi}_j(K+L)) - \tilde{W}_i(\mathbf{I}_j(D\tilde{+}D'))\right]^{1/(n-i)(n-j-1)}$$
$$\geq \Bigg\{ \left[W_i(\mathbf{\Pi}_jK)^{1/(n-i)(n-j-1)} + W_i(\mathbf{\Pi}_jL)^{1/(n-i)(n-j-1)}\right]^{(n-i)(n-j-1)}$$
$$-\left[\tilde{W}_i(\mathbf{I}_jD)^{1/(n-i)(n-j-1)} + \tilde{W}_i(\mathbf{I}_jD')^{1/(n-i)(n-j-1)}\right]^{(n-i)(n-j-1)} \Bigg\}^{1/(n-i)(n-j-1)}$$
$$\geq (W_i(\mathbf{\Pi}_jK) - \tilde{W}_i(\mathbf{I}_jD))^{1/(n-i)(n-j-1)} + (W_i(\mathbf{\Pi}_jL) - \tilde{W}_i(\mathbf{I}_jD'))^{1/(n-i)(n-j-1)}.$$

In view of the equality conditions of 3.9 and Lemma 3.3, it follows that this equality holds if and only if K and L are homothetic, D and D' are dilates, and $(W_i(\mathbf{\Pi}_jK), \tilde{W}_i(\mathbf{I}_jD)) = \mu(W_i(\mathbf{\Pi}_jL), \tilde{W}_i(\mathbf{I}_jD'))$, where μ is a constant.

Taking $i = 0, j = 0$ in 3.1, 3.1 reduces to

$$(3.12)\quad Dv^*(\mathbf{\Pi}(K+L), \mathbf{I}(D\tilde{+}D'))^{1/n(n-1)}$$

$$(3.13)\quad \geq Dv^*(\mathbf{\Pi}K, \mathbf{I}D)^{1/n(n-1)} + Dv^*(\mathbf{\Pi}L, \mathbf{I}D')^{1/n(n-1)},$$

with equality if and only if K and L are homothetic, D and D' are dilates, and $(V(\mathbf{\Pi}K), V(\mathbf{I}D)) = \mu(V(\mathbf{\Pi}L), V(\mathbf{I}D'))$, where μ is a constant.

In case D and D' are single points, 3.12 reduces to the classical Brunn-Minkowski inequality for mixed projection bodies.

Theorem 3.5. *Let K and L be convex bodies, and D and D' be star bodies in $\mathbb{R}^n$. If $D \subseteq K$, $D' \subseteq L$, $0 \leq j < n-2$ and $0 \leq i < n-1$, then*

$$(3.14)\quad Dw_i^*(\mathbf{\Pi}_j(K,L), \mathbf{I}_j(D,D'))^{n-1} \geq Dw_i^*(\mathbf{\Pi}K, \mathbf{I}D)^{n-j-1} Dw_i^*(\mathbf{\Pi}L, \mathbf{I}D')^j,$$

with equality if and only if K and L are homothetic, D and D' are dilates and $(W_i(\mathbf{\Pi}K), \tilde{W}_i(\mathbf{I}D)) = \mu(W_i(\mathbf{\Pi}L), \tilde{W}_i(\mathbf{I}D'))$, where μ is a constant.

We need the following lemmas to prove Theorem 3.2.

Lemma 3.6 ([24]). *If $K, L \in \mathcal{S}^n$, $0 \le i < n$ and $0 \le j < n-1$, then*

$$\tilde{W}_i(\mathbf{I}_j(K,L))^{n-1} \le \tilde{W}_i(\mathbf{I}K)^{n-j-1}\tilde{W}_i(\mathbf{I}L)^j,$$

with equality if and only if K and L are dilates.

Lemma 3.7 ([25]). *Let $a, b, c, d \ge 0, 0 < \alpha < 1, 0 < \beta < 1$ and $\alpha + \beta = 1$. If $a \ge b$ and $c \ge d$, then*

$$a^\alpha c^\beta - b^\alpha d^\beta \ge (a-b)^\alpha (c-d)^\beta, \tag{3.15}$$

with equality if and only if $ad = bc$.

Proof of Theorem 3.2. If $K, L \in \mathcal{K}^n$, then

$$W_i(\mathbf{\Pi}_j(K,L))^{n-1} \ge W_i(\mathbf{\Pi}K)^{n-j-1}W_i(\mathbf{\Pi}L)^j, \tag{3.16}$$

with equality if and only if K and L are homothetic. □

If $D, D' \in \mathcal{S}^n$, then

$$\tilde{W}_i(\mathbf{I}_j(D,D'))^{n-1} \le \tilde{W}_i(\mathbf{I}D)^{n-j-1}\tilde{W}_i(\mathbf{I}D')^j, \tag{3.17}$$

with equality if and only if D and D' are dilates.

From 3.16 and 3.17, we have

$$\begin{aligned} &W_i(\mathbf{\Pi}_j(K,L)) - \tilde{W}_i(\mathbf{I}_j(D,D')) \\ &\ge W_i(\mathbf{\Pi}K)^{(n-j-1)/(n-1)}W_i(\mathbf{\Pi}L)^{j/(n-1)} \\ &\quad -\tilde{W}_i(\mathbf{I}D)^{(n-j-1)/(n-1)}\tilde{W}_i(\mathbf{I}D')^{j/(n-1)}, \end{aligned} \tag{3.18}$$

with equality if and only if K and L are homothetic, and D and D' are dilates

Notice that

$$W_i(\mathbf{\Pi}_j(K,L)) \ge \tilde{W}_i(\mathbf{I}_j(D,D')),$$
$$W_i(\mathbf{\Pi}K) \ge \tilde{W}_i(\mathbf{I}D),$$

and

$$W_i(\mathbf{\Pi}L) \ge \tilde{W}_i(\mathbf{I}D').$$

By using Lemma 3.5, we have

$$\begin{aligned} &\left[W_i(\mathbf{\Pi}_j(K,L)) - \tilde{W}_i(\mathbf{I}_j(D,D'))\right]^{n-1} \\ &\ge \left[W_i(\mathbf{\Pi}K)^{(n-j-1)/(n-1)}W_i(\mathbf{\Pi}L)^{j/(n-1)} \right. \\ &\quad \left. -\tilde{W}_i(\mathbf{I}D)^{(n-j-1)/(n-1)}\tilde{W}_i(\mathbf{I}D')^{j/(n-1)}\right]^{n-1} \\ &\ge \left[W_i(\mathbf{\Pi}K) - \tilde{W}_i(\mathbf{I}D)\right]^{n-j-1}\left[W_i(\mathbf{\Pi}L) - \tilde{W}_i(\mathbf{I}D')\right]^j. \end{aligned}$$

In view of the equality conditions of 3.18 and Lemma 3.5, it follows that this equality holds if and only if K and L are homothetic, D and D' are dilates, and $(W_i(\mathbf{\Pi}K), \tilde{W}_i(\mathbf{I}D)) = \mu(W_i(\mathbf{\Pi}L), \tilde{W}_i(\mathbf{I}D'))$, where μ is a constant.

Taking $i = 0, j = 1$ in the inequality 3.14, 3.14 changes to the following result:

$$Dv^*(\mathbf{\Pi}_1(K, L), \mathbf{I}_1(D, D'))^{n-1} \geq Dv^*(\mathbf{\Pi}K, \mathbf{I}D)^{n-2} Dv^*(\Pi L, \mathbf{I}D'), \tag{3.19}$$

with equality if and only if K and L are homothetic, D and D' are dilates, and $(V(\mathbf{\Pi}K), V(\mathbf{I}D)) = \mu(V(\mathbf{\Pi}L), V(\mathbf{I}D'))$, where μ is a constant.

In case D and D' are single points, 3.19 reduces to the classical Minkowski inequality for mixed projection bodies.

Acknowledgments

The first author's research is supported by Natural Science Foundation of China (11371334). The second author's research is partially supported by a HKU Seed Grant for Basic Research. The authors express their grateful thanks to the referee for very good suggestions and comments.

References

[1] I. J. Bakelman, Convex Analysis and Nonlinear Geometric Elliptic Equations, Springer-Verlag, Berlin, 1994.
[2] E. F. Beckenbach, R. Bellman, Inequalities, 2nd edition, Springer-Verlag, New York, Inc. 1965.
[3] C. Borell, The Brunn-Minkowski inequality in Gauss space, *Invent. Math.* **30** (1975), no. 2, 207–216.
[4] C. Borell, Capacitary inequalities of the Brunn-Minkowski type, *Math. Ann.* **263** (1983), no. 2, 179-184.
[5] Y. D. Burago and V. A. Zalgaller, Geometric Inequalities, Springer-Verlag, New York, 1988.
[6] L. A. Caffarelli, D. Jerison and E. Lieb, On the case of equality in the Brunn-Minkowski inequality for capacity, *Adv. Math.* **117** (1996), no. 2, 193–207.
[7] G. Ewald, Combinatorial Convexity and Algebraic Geometry, Springer-Verlag, New York, 1996.
[8] W. J. Firey, p-means of convex bodies, *Math. Scand.* **10** (1962) 17–24.
[9] R. J. Gardner, Geometric Tomography, Cambridge University Press, Cambridge, 1995.
[10] R. J. Gardner and P. Gronchi, A Brunn-Minkowski inequality for the integer lattice, *Trans. Amer. Math. Soc.* **353** (2001), no. 10, 3995–4024.
[11] R. J. Gardner, The Brunn-Minkowski inequality, *Bull. Amer. Math. Soc.* **39** (2002), no. 3, 355–405.
[12] D. Jerison, A Minkowski problem for electrostatic capacity, *Acta Math.* **176** (1996), no. 1, 1–47.
[13] G. S. Leng, The Brunn-Minkowski inequality for volume difference, *Adv. Appl. Math.* **32** (2004), no. 3, 615–624.
[14] E. Lutwak, The Brunn-Minkowski-Firey theory I: Mixed volumes and the Minkowski problem, *J. Differential Geom.* **38** (1993), no. 1, 131–150.
[15] E. Lutwak, Inequalities for mixed projection bodies, *Trans. Amer. Math. Soc.* **339** (1993), no. 2, 901–916.
[16] E. Lutwak, Dual mixed volume, *Pacific J. Math.* **58** (1975), no. 2, 531–538.
[17] S. Lv, Dual Brunn-Minkowski inequality for volume differences, *Geom. Dedicata* **145** (2010) 169–180.
[18] A. Okounkov, Brunn-Minkowski inequality for multiplicities, *Invent Math.* **125** (1996), no. 3, 405–411.

[19] R. Schneider, Convex Bodies, The Brunn-Minkowski Theory, Cambridge University Press, Cambridge, 1993.
[20] R. Schneider and W. Weil, Zonoids and Related Topics, Convexity and its Applications, 296–317, Birkhäuser, Basel, 1983.
[21] W. Süss, Zusummensetzung von Eikörpen und homothetische Eiflächen, *Tôhoku. Math. J.* **35** (1932) 47–50.
[22] G. Y. Zhang, The affine Sobolev inequality, *J. Differential Geom.* **53** (1999), no. 1, 183-202.
[23] C. J. Zhao, G. Leng, Brunn-Minkowski inequality for mixed intersection bodies, *J. Math. Anal. Appl.* **301** (2005), no. 1, 115–123.
[24] C. J. Zhao, G. Leng, Inequalities for dual quermassintegrals of mixed intersection bodies, *Proc. Indian Acad. Sci.* **115** (2005), no. 1, 79-91.
[25] C. J. Zhao, On polars of Blaschke-Minkowski homomorphisms, *Math. Scand.* **111** (2013), no. 1, 147-160.

(Chang-Jian Zhao) Department of Mathematics, China Jiliang University, Hangzhou, 310018, China
E-mail address: chjzhao@163.com, chjzhao@aliyun.com, chjzhao315@sohu.com

(Wing-Sum Cheung) Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong
E-mail address: wscheung@hku.hk