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ALMOST SIMPLE GROUPS WITH SOCLE $G_2(q)$ ACTING ON FINITE LINEAR SPACES

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Abstract. After the classification of the flag-transitive linear spaces, the attention has been turned to line-transitive linear spaces. In this article, we present a partial classification of the finite linear spaces $\mathcal{S}$ on which an almost simple group G with the socle $G_2(q)$ acts line-transitively.
Keywords: Line-transitive, linear space, almost simple group.
MSC(2010): Primary: 05B05; Secondary: 20B25.

1. Introduction

A linear space $\mathcal{S}$ is an incidence structure consisting of a set of points $\mathcal{P}$ and a set of lines $\mathcal{L}$ such that any two points are incident with exactly one line. The linear space is called non-trivial if every line contains at least three points and there are at least two lines. Write $v = |\mathcal{P}|$ and $b = |\mathcal{L}|$.

The classification of the finite linear spaces admitting a line-transitive automorphism group has been already investigated by Camina, et al, (see [5] and [7]). We continue this investigation by considering the case where the socle of a line-transitive automorphism group is $G_2(q)$. The statement of our theorem is as follows:

Theorem A Let G be an almost simple group and let $\mathcal{S}$ be a finite linear space on which G acts as a line-transitive automorphism group. Suppose that $T = Soc(G)$ is isomorphic to $G_2(q)$, where $q = p^a$ and $a \not\equiv 0\,(mod\,6)$. Then either

(a) T is line-transitive; or

(b) T_L is isomorphic to one subgroup of $(SL_2(q) \circ SL_2(q)) \cdot 2$, where T_L is the line-stabilizer of T.

In the case (b) of the theorem it has not been made further progress without adding an extra hypothesis and a complete classification seems to be out of

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reach with our present methods. The restriction $a \not\equiv 0 \pmod 6$ will play a role in our proof of the theorem and it can't be removed with our present methods.

If a linear space $\mathcal{S}$ is line-transitive, then every line has the same number of points and every point lies on the same number of lines. We call such a linear space a regular linear space. Let G be a group acting on a linear space $\mathcal{S}$. We will write α to be a point of $\mathcal{S}$ and G_α to be the stabilizer of α under the action of G. Similarly L is a line of $\mathcal{S}$ and G_L is the corresponding line-stabilizer.

2. **Preliminary results**

Let $\mathbf{F} = GF(q)$ be a finite field of order $q = p^a$ (p a characteristic). Let T be the Chevalley group of type G_2 over $\mathbf{F}$. Then the order of T is

$$q^6(q^2-1)(q^6-1) = q^6(q-1)^2(q+1)^2(q^2+q+1)(q^2-q+1).$$

We know that no prime greater than three divides more than one of these factors. Let V and $\mathcal{F}$ be as defined in [1]. We define $\Gamma(V)$ to be the group of all semilinear maps on V. Write $\Gamma(V,\mathcal{F})$ to be the subgroup of $\Gamma(V)$ preserving $\mathcal{F}$. Then $\Gamma(V,\mathcal{F})$ is an extension of T by a field automorphism of order a, and $Aut(T) = \Gamma(V,\mathcal{F})$ unless $p = 3$. Let $H_1 = \Gamma(V,\mathcal{F})$ and $H_0 = G_2(q)$. If $p = 3$, then $|Aut(H_0) : H_1| = 2$, and if $p \geq 5$, we have $H_1 = Aut(H_0)$.

We need some information about the subgroup of $G_2(q)$.

Lemma 2.1. ([11])) *Assume that $H_0 \leq H \leq H_1$, where $H_0 \cong G_2(q)$ ($q = p^n$ is odd) and H_1 are as above. Let M be a maximal subgroup of H not containing H_0. Then $M_0 = M \cap H_0$ is H_0-conjugate to one of the following groups:*

Structure	*Order*	*Remarks*
$[q^5] : GL_2(q)$	$q^6(q-1)^2(q+1)$	*parabolic*
$(SL_2(q) \circ SL_2(q)) \cdot 2$	$2q^2(q^2-1)^2$	*involution centralizer*
$SL_3^\epsilon(q) : 2$	$2q^3(q^3-\epsilon 1)(q^2-1)$	$\epsilon = \pm$
$G_2(q_0)$	$q_0^6(q_0^2-1)(q_0^6-1)$	$q = q_0^m$, *m prime*
${}^2G_2(q)$	$q^3(q^3+1)(q-1)$	$p = 3$, *n odd*
$PGL_2(q)$	$q(q^2-1)$	$p \geq 7, q \geq 11$
$2^{3\cdot}L_3(2)$	$2^6 \cdot 3 \cdot 7$	$q = p$
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	$p \geq 5, \mathbf{F} = \mathbf{F}_p[\omega]$ $\omega^3 - 3\omega + 1 = 0$
$L_2(13)$	$2^2 \cdot 3 \cdot 7 \cdot 13$	$p \neq 13, \mathbf{F} = \mathbf{F}_p[\sqrt{13}]$
$G_2(2)$	$2^6 \cdot 3^3 \cdot 7$	$q = p \geq 5$
J_1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	$q = 11$

Conversely, if $K \leq H_0$ is H_0-conjugate to one of these groups, then $N_G(K)$ is maximal in H.

Lemma 2.2. ([11])) *Assume that $H_0 \leq H \leq H_1$, where $H_0 \cong G_2(q)$ ($q = 3^n$) and H contains a graph automorphism of H_0. Let M be a maximal subgroup of H not containing H_0. Then $M_0 = M \cap H_0$ is H_0-conjugate to one of the following groups:*

Structure	*Order*	*Remarks*
$[q^5]:GL_2(q)$	$q^6(q-1)^2(q+1)$	*parabolic*
$(SL_2(q)\circ SL_2(q))\cdot 2$	$2q^2(q^2-1)^2$	*involution centralizer*
$2^3\cdot L_3(20)$	$2^6\cdot 3\cdot 7$	*q=3*
$(Z_{q^2-\epsilon 1})^2\cdot D_{12}$	$12(q^2-1)^2$	$p\geqslant 9, \epsilon=\pm 1$
$(Z_{q^2+\epsilon q+1})^2\cdot Z_6$	$6(q^2+\epsilon q+1)$	$p\geqslant 9, \epsilon=\pm 1$
$G_2(q_0)$	$q_0^6(q_0^2-1)(q_0^6-1)$	$q=q_0^m$, *m prime*
${}^2G_2(q)$	$q^3(q^3+1)(q-1)$	$p=3$, *n odd*
$PGL_2(q)$	$q(q^2-1)$	$p\geq 7, q\geq 11$
$L_2(13)$	$2^2\cdot 3\cdot 7\cdot 13$	$p\neq 13, F=F_p[\sqrt{13}]$

Conversely, if $K\leq H_0$ is H_0-conjugate to one of these groups, then $N_G(K)$ is maximal in H.

Lemma 2.3. ([8]) *Let $G=G_2(q)$, where $q=2^a$ with $a>2$. The maximal subgroups of G are listed as follows:*

Structure	*Order*	*Remarks*
$[q^5]:GL_2(q)$	$q^6(q-1)^2(q+1)$	*parabolic*
$((SL_2(q)\circ SL_2(q))\cdot 2$	$2q^2(q^2-1)^2$	*reducible*
$SL_3^\epsilon(q):2$	$2q^3(q^3-\epsilon 1)(q^2-1)$	$\epsilon=\pm$, *the normalizer of subgroup of order* 3
$SL_3^\epsilon(q):2$	$2q^3(q^3-\epsilon 1)(q^2-1)$	$\epsilon=\pm$, *irreducible*
$G_2(q_0)$	$q_0^6(q_0^2-1)(q_0^6-1)$	$q=q_0^m$, *m prime*

Lemma 2.4. ([8] and [11])

(i) *If q is even, then $G_2(q)$ has exactly two conjugacy class of involutions, and the order of the centralizer of a non-central involution is $q^4(q^2-1)$.*

(ii) *If q is odd, then $G_2(q)$ has a unique conjugacy class of involutions, and the order of the centralizer of an involution is $q^2(q^2-1)^2$.*

Lemma 2.5. (Lemma 3.3 of [13]) *Some subgroups of $G=G_2(q)$ are being shown in the following:*

(1) $T_1=Z_{q-1}\times Z_{q-1}$ *and* $N_G(T_1)=T_1\cdot D_{12}$*;*
(2) $T_2=Z_{q+1}\times Z_{q+1}$ *and* $N_G(T_2)=T_2\cdot D_{12}$*;*
(3) $T_3=Z_{q^2+q+1}$ *and* $N_G(T_3)=T_3:Z_6$*;*
(4) $T_4=Z_{q^2-q+1}$ *and* $N_G(T_4)=T_4:Z_6$*;*
(5) $T_5=Z_{q^2-1}$ *and* $N_G(T_5)=T_5\cdot(Z_2\times Z_2)$.

Lemma 2.6. (Lemma 3.5 of [13]) *Let $q=q_0^m$, where m is an odd prime and let $\epsilon=\pm$. Then the following hold:*

(1) $(q-\epsilon 1)^2$ *does not divide* $|G_2(q_0)|$*;*
(2) $(q^2+\epsilon q+1)$ *does not divide* $|G_2(q_0)|$.

Lemma 2.7. *Let $q=q_0^m$, where m is a positive integer with $m\geqslant 4$ and let $\epsilon=\pm$. Then the following hold:*

(1) $(q-\epsilon 1)^2$ *does not divide* $|G_2(q_0)|$*;*

(2) $(q^2+\epsilon q+1)$ *does not divide* $|G_2(q_0)|$.
(3) (q^2-1) *does not divide* $|G_2(q_0)|$.

Proof. (1) If $m>6$, then there exists a p-primitive divisor of p^m-1, denoted by t. Hence $t \nmid q_0^6$ and $(q-\epsilon 1)^2$ does not divide $|G_2(q_0)|$; if $m=4$ or 6, then $(q-\epsilon 1)^2 \nmid |G_2(q_0)|$. The proof is finished.

Similarly we can prove that the assertions (2) and (3) are true. $\square$

We assume that G is a automorphism group acting line-transitively on a linear space $\mathcal{S}$ with parameters b,v,k,r, where b is the number of lines, v is the number of points, r is the number of lines through a point and k is the number of points on a line. Recall the basic counting lemmas for regular linear spaces.

$$v=r(k-1)+1, \tag{2.1}$$
$$v(v-1)=bk(k-1). \tag{2.2}$$

Let $b_1=(b,v)$, $b_2=(b,v-1)$, $k_1=(k,v)$, and $k_2=(k,v-1)$. Then
$$k=k_1k_2,\ b=b_1b_2,\ r=b_2k_2, \text{ and } v=b_1k_1.$$

In [5], the authors defined a significant prime which divides b but not v. Observe that every prime divisor of b_2 is a significant prime. It is well-known that a linear space is a projective plane if and only if $b=v$, namely, $b_2=1$. Thus every linear space other than the projective plane has significant primes.

There is a fact that we shall use throughout this article. Observe that if an involution in G does not fix a point then G acts flag-transitively (see [6]). But the flag-transitive linear spaces are classified by Buekenhout, Delandtsheer, Doyen et al (see [4] and [3]), and so we assume that every involution fixes at least a point.

We state here a number of basic results which will be used repeatedly throughout the paper.

Lemma 2.8. *Let G act line-transitively on a linear space $\mathcal{S}$, and b_2 be defined as above. Then the followings hold:*
(1) $(b_2,v)=1$;
(2) b_2 *divides* $|G_\alpha|$.

Proof. (1) Note that $b_2=(b,v-1)$, it is clear that $(b_2,v)=1$.

(2) Since G is line-transitive, by the theorem of R. E. Block in [2] we have G is point-transitive. Hence $b=|G:G_L|$ and $v=|G:G_\alpha|$, where $L\in\mathcal{L}$ and $\alpha\in\mathcal{P}$. Since $rv=bk$, it follows that $b_2|G_L|=k_1|G_\alpha|$. Note that $(b_2,k_1)=1$ and hence b_2 divides $|G_\alpha|$. $\square$

Lemma 2.9. (Zhou, Li and Liu [18]) *Let G act line-transitively on a linear space $\mathcal{S}$. Let K be a subgroup of G. If $K \nleq G_L$ for any line $L\in\mathcal{L}$, and $K\leq G_\alpha$ for some point $\alpha\in\mathcal{P}$, then $N_G(K)\leq G_\alpha$.*

Lemma 2.10. (Lemma 2.8 of [16]) *Let G act line-transitively on a linear space $\mathcal{S}$. If there exists a prime p such that $p|b$ but $p \nmid v$, then for some $\alpha \in \mathcal{P}$, $N_G(P) \leq G_\alpha$, where P is a Sylow p-subgroup of G.*

Lemma 2.11. (Liu [12] and [13]) *Let G act line-transitively on a linear space $\mathcal{S}$. Assume that P is a Sylow p-subgroup of G_α for some $\alpha \in \mathcal{P}$. If P is not a Sylow p-subgroup of G, then there exists a line L through α such that $P \leq G_L$.*

Lemma 2.12. ([14] and [15]) *Let G be a transitive group on Ω, and K be a conjugacy class of an element of G. Let $x \in K$ and $\mathrm{Fix}_\Omega(\langle x \rangle)$ denote the fixed points set of $\langle x \rangle$ acting on Ω. Then*

$$|\mathrm{Fix}_\Omega(\langle x \rangle)| = |G_\alpha \mathcal{K}| \cdot |\Omega|/|K|,$$

where $\alpha \in \Omega$. In particular, if G has a unique conjugacy class of involutions, then

$$|\mathrm{Fix}_\Omega(\langle i \rangle)| = \frac{e(G_\alpha) \cdot |\Omega|}{e(G)},$$

where i is an involution of G and $e(G)$ denotes the number of involutions of G.

Lemma 2.13. ([14] and [15]) *Let G act line-transitively on a linear space $\mathcal{S}$. Let i be an involution of G_L, where L is a line of $\mathcal{S}$. Set $f_1 = |\mathrm{Fix}_\mathcal{P}(\langle i \rangle)|$ and $f_2 = |\mathrm{Fix}_\mathcal{L}(\langle i \rangle)|$. If $\mathcal{S}$ is not a projective plane and $f_1 \geq 2$, then $v \leq f_2^2$.*

In order to do our work, we need to introduce the concept of exceptional triple. Let G and H be finite groups acting transitively on a finite set Ω; with H a normal subgroup of G. Then the triple (G, H, Ω) is called *exceptional* if the only common orbit of G and H on $\Omega \times \Omega$ is the diagonal. This definition is equivalent to the following: Let $\alpha \in \Omega$, then every G_α-orbit except $\{\alpha\}$ breaks up into strictly smaller H_α-orbits.

We call the triple (G, H, Ω) *arithmetically exceptional*, if there is a subgroup B of G which contains H, such that (B, H, Ω) is exceptional, and B/H is cyclic. When G is a primitive permutation group of almost simple type, Guralnick, Muller and Saxl have obtained their classification (see [10]). In particular, when $Soc(G) = G_2(q)$, there is the following lemma:

Lemma 2.14. ([10, Theorem 1.5 (g)] *Let G be a primitive permutation group of almost simple type, so $L \trianglelefteq G \leq Aut(L)$ with L a simple nonabelian group. Suppose that there are subgroups B and H of G with $H \trianglelefteq G$ and B/H cyclic, such that (B, H) is exceptional. Let M be a point stabilizer in G. Suppose that L has Lie rank ≥ 2, $L \neq Sp_4(2)' \cong PSL_2(9)$. Then $M \cap L$ is a subfield subgroup, the centralizer in L of a field automorphism of odd prime order r. Moreover,*

(i) *$r \neq p$ (with p the defining characteristic of L),*
(ii) *if $r = 3$, then L is of type $Sp_4(q)$ with q even, and*
(iii) *there are no $Aut(L)$-stable L-conjugacy classes of r-elements.*

Lemma 2.15. ([10, Lemma 3.3]) *Let H be a normal subgroup of the finite group G with G/H cyclic generated by xH. Let Ω be a transitive G-set. Assume that H is also transitive on Ω. Let $\chi(g)$ be the number of fixed points of $g \in G$ on Ω. The following are equivalent:*
(1) (G, H, Ω) *is exceptional;*
(2) $\chi(xh) \leq 1$ *for all* $h \in H$*;*
(2) $\chi(xh) = 1$ *for all* $h \in H$*;*
(2) $\chi(xh) \geq 1$ *for all* $h \in H$*.*

Lemma 2.16. ([9]) *Let G act line-transitively on a linear space $\mathcal{S}$. Let H be a subgroup of G such that $H \trianglelefteq G$ and $|G : H| = s$, a prime. If H is line-intransitive and $\mathcal{S}$ is not a projective plane, then $(G, H, \mathcal{P})$ is an exceptional triple.*

3. **The proof of Theorem**

Since $T = G_2(q) \trianglelefteq G \leq Aut(T)$ with $q = p^a$, we have $|Out(T)| = 2a$ or a according as $p = 3$ or not, and $G = T : \langle x\rangle$, where $x \in Out(T)$. Let $o(x) = m$. Then we have when $p = 3$, $m|2a$; when $p \neq 3$, $m|a$. Moreover, $|G| = q^6(q^2 - 1)(q^6 - 1)m$.

By [9], we know that almost simple groups cannot act line-transitively on non-Desarguesian projective planes. Hence we can assume that $\mathcal{S}$ is not a projective plane. Suppose that T is not line-transitive on $\mathcal{S}$.

Let s be a prime divisor of m. There exists a normal subgroup H in G such that $|G/H| = s$ and H is not line-transitive(otherwise replacing G by H). By Lemma 2.16 we have that the triple $(G, H, \mathcal{P})$ is exceptional. Let $\chi(g)$ be the number of fixed points of $g \in G$ on $\mathcal{P}$ and $y = x^{m/s}$. Then $G/H = \langle yH\rangle$. We show that H is point-transitive. In fact, if H is not point-transitive, by the proof of Lemma 2.16 (see [9, Lemma 26]) we have $\mathcal{S}$ is a projective plane, a contradiction. Now appealing to Lemma 2.15 we have $\chi(yh) = 1$ for all $h \in H$. It follows that y has a unique fixed point of $\mathcal{P}$, says α. Considering the cycle decomposition of x acting on $\mathcal{P}$, we find that x fixes no other points of $\mathcal{P}$ than α. Hence $x \in G_\alpha$ and $G = TG_\alpha$. Then T acts transitively on $\mathcal{P}$.

Since T is not line-transitive, it follows that $x \notin G_L$ for any $L \in \mathcal{L}$. Note that $x \in G_\alpha$ and we appeal to Lemma 2.9 to conclude that $N_G(\langle x\rangle) \leq G_\alpha$. By Lemmas 2.1 and 2.2 and 2.3, we find that the overgroup of $G_2(p)$ is only the subfield subgroup. Since $C_T(x) \leq N_G(\langle x\rangle)$ and $G_2(p) \leq C_T(x)$, we may assume that $G_\alpha \cap T = T_\alpha \cong G_2(q_0)$ for $q = q_0^c$ with a positive integer c. If c is prime, then G_α is a maximal subgroup of G and hence G is primitive on $\mathcal{P}$. Thus by the proof of Lemma 2.14 (see [10, Theorem 1.5 (g)])we have $c \nmid |{}^2G_2(3)|$ for $p = 3$ and $c \nmid |G_2(p)|$ for $p \neq 3$, and hence 1) if $p = 2$, then $c \nmid |G_2(2)| = 2^6 \cdot 3^3 \cdot 7$, and hence $c \neq 2, 3, 7$; 2) if $p = 3$, then $c \nmid |{}^2G_2(3)| = 2^3 \cdot 3^3 \cdot 7$ and hence $c \neq 2, 3, 7$; 3) if $p \geq 5$, then $c \nmid |G_2(p)| = p^6(p^6 - 1)(p^2 - 1)$, it follows that $c \neq p, 2$ and $c \nmid (p - 1)^2, (p + 1)^2, p^2 + p + 1$ and $p^2 - p + 1$.

Let i be an involution of T and K be a conjugate class of i in G. If q is even, then we let i be a non-central involution. Thus by Lemma 2.4 we have $|C_G(i)| \leq q^4(q^2-1)m$. If q is even, then v is even and so f_1 is even. Note that i fixes at least a point, hence $f_1 \geq 2$. If q is odd, then by Lemma 2.12 we have

$$f_1 = |\mathrm{Fix}_{\mathcal{P}}(\langle i \rangle)| = \frac{v \cdot |G_\alpha \cap K|}{|K|} = \frac{|C_G(i)|}{|C_{G_\alpha}(i)|} \geq \frac{q^2(q^2-1)^2}{q_0^4(q_0^2-1)} \geq 2.$$

Since

$$f_2 = |\mathrm{Fix}_{\mathcal{L}}(\langle i \rangle)| = \frac{b \cdot |G_L \cap K|}{|K|} < |C_G(i)|,$$

by Lemma 2.13 we can get the following inequality

$$v{=}\tfrac{q^6(q^6-1)(q^2-1)}{q_0^6(q_0^6-1)(q_0^2-1)} < (|C_G(i)|)^2 = q^8(q^2-1)^2m^2.$$

This implies that

$$q^6(q^6-1)(q^2-1) < q^8(q^2-1)^2m^2q_0^{14}.$$

It follows that

$$q^2 < \frac{q^4+q^2+1}{q^2} = \frac{q^6(q^6-1)(q^2-1)}{q^8(q^2-1)^2} < q_0^{14}m^2.$$

Let $q_0 = p^\lambda$. Then $a = \lambda c$ for a positive integer λ and $q = p^{\lambda c}$. We can get

$$p^{\lambda(c-7)} < m. \tag{3.1}$$

Recall that $m|2a$ for $p = 3$ and $m|a$ for $p \neq 3$, so we have

1) if $p = 3$, then $m|2a$ and it follows from (3) that

$$3^{\lambda(c-7)} < 2\lambda c, \tag{3.2}$$

which forces that $c \leq 9$.

2) if $p = 2$, then $m|a$ and hence

$$2^{\lambda(c-7)} < \lambda c, \tag{3.3}$$

which forces that $c \leq 10$.

3) if $p \geq 5$, then $m|a$ and it follows that

$$5^{\lambda(c-7)} \leq p^{\lambda(c-7)} < \lambda c, \tag{3.4}$$

which forces that $c \leq 8$.

1. If $p = 2, c = 9$, then it follows from (5) that $4^\lambda < 9\lambda$. Hence $\lambda = 1$, $a = 9$ or $\lambda = 2, a = 18$.

2. If $p = 2, c = 10$, then again by (5) we have $8^\lambda < 10\lambda$, and hence $\lambda = 1, a = 10$.

3. If $p = 3, c = 9$, then it follows from (4) that $9^\lambda < 18\lambda$. We have $\lambda = 1, a = 9$.

Recall that $a \not\equiv 0\,(mod\,6)$. Hence the cases we should examine are in the following table.

c	a	p
3	3λ, where λ is is a positive integer	$p \geq 5$ is a prime
7	7λ, where λ is is a positive integer	$p \geq 5$ is a prime
$4, 5, 8$	λc, where λ is is a positive integer	p is a prime
9	9	2
9	9	3
10	10	2

It is clear that G_α contains no Sylow p-subgroups of G. Let Q_0 be a Sylow p-subgroup of $G_2(q_0)$. It follows by Lemma 2.11 that $Q_0 \leq G_L$. Then $Q_0 \leq T \cap G_L = T_L$. Examining the subgroup of $G_2(q)$ in Lemmas 2.1 and 2.2 and 2.3, we find that $G_L \cap T$ is isomorphic to $G_2(q_2)$ or a subgroup of some maximal subgroup of T, where q is a power of q_2 and $q_0|q_2$.

3.1 Case: $T_L \cong G_2(q_2)$.

By Lemma 2.4 we have $|C_{G_L}(i)| \geq q_2^2(q_2^4-1) \geq q_0^2(q_0^4-1)$ and hence

$$|\mathrm{Fix}_{\mathcal{L}}(\langle i \rangle)| = \frac{|G_L \cap K|}{|G_L|} = \frac{|C_G(i)|}{|C_{G_L(i)|}} \leq \frac{q^4(q^2-1)m}{q_0^2(q_0^4-1)}.$$

Appealing to Lemma 2.13, we have

$$v = \tfrac{q^6(q^6-1)(q^2-1)}{q_0^6(q_0^6-1)(q_0^2-1)} < (\tfrac{q^4(q^2-1)m}{q_0^2(q_0^4-1)})^2.$$

Since $q^6(q^6-1)(q^2-1) > q^2(q^4(q^2-1))^2$ and $q_0 \geq 2$, we have

$$q^2 < \frac{q_0^6(q_0^6-1)(q_0^2-1)}{(q_0^2(q_0^4-1))^2} = \frac{q_0^2(1+q_0^2+q_0^4)}{(1+q_0^2)^2} < \frac{21}{16}q_0^2m^2.$$

It follows that

$$q < \sqrt{\frac{21}{16}}q_0m < \frac{8}{5}q_0m.$$

Thus

$$p^a < \frac{8}{5}p^{a/c} \cdot m.$$

Recall that $q_0 = p^\lambda$ and $a = \lambda c$. Then we have

$$p^{\lambda(c-1)} < \frac{8}{5} \cdot m.$$

- if $p = 2$, then $2^{\lambda(c-1)} \leq \frac{8}{5}\lambda c$, which is a contradiction for $c \geq 4$.
- if $p = 3$, then $3^{\lambda(c-1)} \leq \frac{16}{5}\lambda c$. This is impossible for $c \geq 4$.
- if $p \geq 5$, then $5^{\lambda(c-1)} \leq \frac{8}{5}\lambda c$. This is a contradiction.

Hence the case where $T_L \cong G_2(q_2)$ is excluded.

3.2 Case: $G_L \cap T$ is conjugate to a subgroup of some maximal subgroup of T.

Let M be a maximal subgroup of T such that $T_L \leq M$. Let $b = |G : G_L| = \frac{|T|m}{|T_L|m_1}$(where $m_1|m$) and $a_1 = |M|/|T_L|$ and let $b' = |T|/|M|$, then we have $b = b'a_1\frac{m}{m_1}$. Thus if a prime divides b', it also divides b. We can assume that T_L is not isomorphic to one subgroup of $(SL_2(q) \circ SL_2(q)) \cdot 2$. If G contains a graph automorphism of $G_2(q)$, then M lies inside $[q^5] : GL_2(q)$, or ${}^2G_2(q)$ or $PGL_2(q)$. Otherwise, M lies in $[q^5] : GL_2(q)$, or $SL_3^{\epsilon}(q) : 2$, or ${}^2G_2(q)$, or $PGL_2(q)$, where $\epsilon = \pm$.

3.2.1 Case: $c = 9, p = 2$

In this case we have $v = 2^{48} \cdot 3 \cdot 7 \cdot 3249 \cdot 5329 \cdot 261633 \cdot 262657$ and $q = 2^9$. Hence b_2 divides $2^6(2^2-1)(2^6-1)\cdot 9$ by Lemma 2.8. Since $(b_2, v) = 1$, we have $b_2 = 1$. This implies that $\mathcal{S}$ is a projective plane which is a contradiction.

3.2.2 Case: $c = 10, p = 2$

We have $v = 2^{54} \cdot 5^2 \cdot 11^2 \cdot 13 \cdot 31^2 \cdot 205^2 \cdot 151 \cdot 331 \cdot 80581$ and $q = 2^{10}$. Let S be a Sylow 7 -subgroup of T. Since $(7, a) = 1$ and hence S is also the Sylow 7-subgroup of G. Note that $7 \nmid v$, and so we may split the proof into 2 or 3 cases. ¿.

If $7|b$, it follows from Lemmas 2.10 and 2.5 that $Z_{q^2+q+1} \leq G_\alpha$, which is impossible by Lemma 2.7.

If $7 \nmid b$, this implies that $7 \nmid b_2$ and G_L contains a Sylow 7-subgroup of G. Hence the only case to occur is that $M = SL_3^{\epsilon}(q) : 2$ and $\epsilon = +$. It follows from Lemma 2.8 that $b_2|2^7 \cdot 3^3 \cdot 5 \cdot 7$. Note that $(b_2, v) = 1$ and hence $b_2|3^3$. Since v is even, then k_1 is even. If $b_2 = 3$, then $k_1 = 2$ and hence $|G_L| = \frac{2}{3}|G_\alpha| = 2^8 \cdot 3^2 \cdot 5 \cdot 7$. There is no such group. Thus $3^2|b$. Note that $3^3|||T|$ and hence $Z_3 \times Z_3 \not\subseteq T_L$. Applying Lemma 2.9 and 2.5 it yields the fact that $N_G(Z_3 \times Z_3) \leq G_\alpha$, which implies $Z_{q-1} \times Z_{q-1} \leq G_\alpha$, a contradiction.

3.2.3 Case: $c = 9, p = 3$

We have $v = 3^{48} \cdot 7 \cdot 13 \cdot 703^2 \cdot 757^2 \cdot 387400807 \cdot 387440173$ and $q = 3^9$. By Lemma 2.4 we have that $|C_G(i)| \leq q^2(q^2-1)^2 m$. Note that $|C_{G_L}(i)| \geq 2$ and hence it follows from Lemma 2.13 that $v \leq \frac{|C_{G_L}(i)|^2}{|C_G(i)|^2} \leq \frac{1}{4}|C_G(i)|^2$. This implies that $3^{48} \cdot 7 \cdot 13 \cdot 703^2 \cdot 757^2 \cdot 387400807 \cdot 387440173 \leq 2^{10} \cdot 3^{40} \cdot 7^4 \cdot 13^4 \cdot 703^4 \cdot 757^4$, which is a contradiction.

3.2.4 Case: $c = 3$, $p \geq 5$

In this case we have $Z_{q_0+1} \times Z_{q_0+1} \leq G_\alpha$. We may consider two cases.

If $Z_{q_0+1} \times Z_{q_0+1} \leq G_L$, it follows by Lemmas 2.1 that $M = SL_3^{\epsilon}(q) : 2$ where $\epsilon = -$. Then $Z_{q_0-1} \times Z_{q_0-1} \not\leq G_L$. It follows from Lemma 2.9 that $N_G(Z_{q_0-1} \times Z_{q_0-1}) \leq G_\alpha$, which implies that $Z_{q-1} \times Z_{q-1} \leq G_\alpha$. This is a contradiction.

If $Z_{q_0+1} \times Z_{q_0+1} \not\leq G_L$, it follows from Lemma 2.9 that $N_G(Z_{q_0+1} \times Z_{q_0+1}) \leq G_\alpha$. This implies that $Z_{q+1} \times Z_{q+1} \leq G_\alpha$ which is a contradiction.

3.2.5 Case: $c = 7$, $p \geq 5$

In this case we can find a prime t greater than 5 such that t divides $q_0^2+\epsilon q_0+1$. Let U^ϵ be a cyclic subgroup of $G_2(q_0)$ of order $q_0^2+\epsilon q_0+1$, where $\epsilon=\pm$. Then $U^\epsilon \leq Z_{q^2+\epsilon q+1}$. If $U^- \leq G_L$, by Lemmas 2.1 and 2.3 we find that the possibility for M is the case $M=SL_3^\epsilon(q):2$ or ${}^2G_2(q)$ where $\epsilon=-$. Hence $U^+ \not\leq M$. It follows by Lemmas 2.9 and 2.5 that $Z_{q^2+q+1}:Z_6 \leq G_\alpha$, which gives a contradiction. If $U^- \not\leq G_L$, then by Lemma 2.9 we have $N_G(Z_{q^2-q+1}) \leq G_\alpha$ and hence $Z_{q^2-q+1}:Z_6 \leq G_\alpha$. This is a contradiction.

3.2.6 Case: $c=4$ or 8

We have $v=q_0^{18}(q_0^2+1)^2(q_0^4+1)^2(q_0^4-q_0^2+1)(q_0^8-q_0^4+1)$ for $c=4$ and $q_0^{42}(q_0^2+1)^2(q_0^4+1)^2(q_0^8+1)^2(q_0^4-q_0^2+1)(q_0^8-q_0^4+1)(q_0^{16}-q_0^8+1)$ for $c=8$.

First suppose that G contains a graph automorphism of $G_2(q)$. Then M lies inside $[q^5]:(Z_{q-1})^2$, or ${}^2G_2(q)$ or $PGL_2(q)$. Hence $Z_{q_0^2+q_0+1} \not\leq M$ and $Z_{q_0^2+q_0+1} \not\leq G_L$. Note that $Z_{q_0^2+q_0+1} \leq Z_{q^2+q+1}$. It follows by Lemma 2.9 that $N_G(Z_{q_0^2+q_0+1}) \leq G_\alpha$, which implies $Z_{q^2+q+1} \leq G_\alpha$. Thus $Z_{q^2+q+1} \leq T\cap G_\alpha = T_\alpha$. This contradicts Lemma 2.7.

Now assume that G contains no graph automorphism of $G_2(q)$. We consider several cases.

Suppose first that $q_0=2$. We have $v=2^{18}\cdot 5^2\cdot 13\cdot 17^2\cdot 241$ and $q=2^4$ for $c=4$ and $v=2^{42}\cdot 5^2\cdot 13\cdot 17^2\cdot 241\cdot 257^2\cdot 65281$ and $q=2^8$ for $c=8$. Hence the Sylow 7-subgroup of T is also of G. Note that $7\nmid v$ and so we may split the proof into 2 or 3 cases .

If $7|b$, it follows from Lemmas 2.9 and 2.5 that $Z_{q^2+q+1} \leq G_\alpha$, which is impossible by Lemma 2.7.

If $7\nmid b$, this implies that $7\nmid b_2$ and G_L contains a Sylow 7-subgroup of G. Hence the only case to occur is that $M=SL_3(q):2$. It follows from Lemma 2.8 that $b_2|2^8\cdot 3^3\cdot 7$ for $c=4$ and $b_2|2^9\cdot 3^3\cdot 7$ for $c=8$. Note that $(b_2,v)=1$ and hence $b_2|3^3$. Since v is even, then k_1 is even. If $b_2=3$, then $k_1=2$ and hence $|G_L|=\frac{2}{3}|G_\alpha|=2^9\cdot 3^2\cdot 5\cdot 7$ for $c=4$ and $|G_L|=\frac{2}{3}|G_\alpha|=2^{10}\cdot 3^2\cdot 5\cdot 7$ for $c=8$. There are no subgroups of such orders. Thus $3^2|b$. Note that $3^3|||T|$ and hence $Z_3\times Z_3 \nsubseteq T_L$. Applying Lemma 2.9 and 2.5 it yields the fact that $N_G(Z_3\times Z_3) \leq G_\alpha$, which implies $Z_{q-1}\times Z_{q-1} \leq G_\alpha$, a contradiction.

Suppose next that $q_0 \neq 2$. Then we calculate that there exists a prime divisor s greater than 3 of $q_0^2+\epsilon q_0+1$. It follows that $s|b'$ and hence $s|b$. We may have

$$(q_0^2+\epsilon q_0+1, q_0^2+1)=1, \qquad (q_0^2+\epsilon q_0+1, q_0^4+1)=1$$

$$(q_0^2+\epsilon q_0+1, q_0^8+1)=1, \qquad (q_0^2+\epsilon q_0+1, q_0^4-q_0^2+1)=1$$

$$(q_0^2+\epsilon q_0+1, q_0^8-q_0^4+1)=1, \qquad (q_0^2+\epsilon q_0+1, q_0^{16}-q_0^8+1)=1$$

Then it follows that $(q_0^2+\epsilon q_0+1, v)=1$ and hence $s\nmid v$.

Let S be a Sylow s-subgroup of T. Then S is also a s-subgroup of T_α. We have $S \nsubseteq T_L$ (In fact, if $S \leq T_L$, then $S \leq M$ and hence $s \nmid b' = \frac{|T|}{|M|}$, a contradiction.) It follows from Lemma 2.10 that $N_G(S) \leq G_\alpha$, which implies that $Z_{q^2+q+1} \leq G_\alpha$. This is impossible.

3.2.7 Case: $c = 5$

Suppose first that $q_0 = 2$. Then $v = 2^{24} \cdot 11^2 \cdot 31^2 \cdot 151 \cdot 331$ and $q = 2^5$. The Sylow $7-$subgroups of T are also those of G. We may split the proof into 2 cases

If 7 divides b, it follows from Lemmas 2.10 and 2.5 that $Z_{q^2+q+1} : Z_6 \leq G_\alpha$. This contradicts Lemma 2.7.

If 7 does not divide b, then G_L contains a Sylow 7-subgroup of G. Thus the only case to occur is that $M = SL_3(q) : 2$. We may get $b' = q^3(q^3+1)/2 = 2^{14}(2^{15}+1)$ and hence $9|b$. Since $27|||T|$, it follows that $Z_3 \times Z_3 \not\leq T_L$. Applying Lemmas 2.9 and 2.5 it yields the fact that $N_G(Z_3 \times Z_3) \leq G_\alpha$, and hence $(Z_{q+1} \times Z_{q+1}) \cdot D_{12} \leq G_\alpha$, which is a contradiction.

Now we can assume that $q_0 \neq 2$. Then there exists a prime divisor t greater than 3 of $q_0^2 + \epsilon q_0 + 1$ by a direct calculation. Let U^ϵ be a cyclic subgroup of $G_2(q_0)$ of order $q_0^2 + \epsilon q_0 + 1$, where $\epsilon = \pm$. Then $U^\epsilon \leq Z_{q^2+\epsilon q+1}$. If $U^- \leq G_L$, by Lemmas 2.1 and 2.2 and 2.3 we find that $M = SL_3^\epsilon(q) : 2$ or ${}^2G_2(q)$, where $\epsilon = -$. Hence $U^+ \not\leq M$. It follows from Lemmas 2.9 and 2.5 that $Z_{q^2+q+1} : Z_6 \leq G_\alpha$. This is a contradiction. If $U^- \not\leq G_L$, then by Lemma 2.9 we have $N_G(Z_{q^2-q+1}) \leq G_\alpha$ and hence $Z_{q^2-q+1} : Z_6 \leq G_\alpha$. This is a contradiction.

Thus we have that T is line-transitive on $\mathcal{S}$. The proof of the Theorem is complete.

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