Almost simple groups with Socle $G_2(q)$ acting on finite linear spaces

S. Li, X. Li and W. Liu
ALMOST SIMPLE GROUPS WITH SOCLE $G_2(q)$ ACTING ON FINITE LINEAR SPACES

S. LI, X. LI AND W. LIU*

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Abstract. After the classification of the flag-transitive linear spaces, the attention has been turned to line-transitive linear spaces. In this article, we present a partial classification of the finite linear spaces $S$ on which an almost simple group $G$ with the socle $G_2(q)$ acts line-transitively.

Keywords: Line-transitive, linear space, almost simple group.


1. Introduction

A linear space $S$ is an incidence structure consisting of a set of points $P$ and a set of lines $L$ such that any two points are incident with exactly one line. The linear space is called non-trivial if every line contains at least three points and there are at least two lines. Write $v = |P|$ and $b = |L|$.

The classification of the finite linear spaces admitting a line-transitive automorphism group has been already investigated by Camina, et al, (see [5] and [7]). We continue this investigation by considering the case where the socle of a line-transitive automorphism group is $G_2(q)$. The statement of our theorem is as follows:

Theorem A Let $G$ be an almost simple group and let $S$ be a finite linear space on which $G$ acts as a line-transitive automorphism group. Suppose that $T = Soc(G)$ is isomorphic to $G_2(q)$, where $q = p^a$ and $a \not\equiv 0 \pmod{6}$. Then either

(a) $T$ is line-transitive; or
(b) $T_L$ is isomorphic to one subgroup of $(SL_2(q) \circ SL_2(q)) \cdot 2$, where $T_L$ is the line-stabilizer of $T$.

In the case (b) of the theorem it has not been made further progress without adding an extra hypothesis and a complete classification seems to be out of

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reach with our present methods. The restriction \( a \neq 0 \pmod{6} \) will play a role in our proof of the theorem and it can’t be removed with our present methods.

If a linear space \( S \) is line-transitive, then every line has the same number of points and every point lies on the same number of lines. We call such a linear space a regular linear space. Let \( G \) be a group acting on a linear space \( S \). We will write \( \alpha \) to be a point of \( S \) and \( G_\alpha \) to be the stabilizer of \( \alpha \) under the action of \( G \). Similarly \( L \) is a line of \( S \) and \( GL_L \) is the corresponding line-stabilizer.

2. Preliminary results

Let \( F = GF(q) \) be a finite field of order \( q = p^e \) (\( p \) a characteristic). Let \( T \) be the Chevalley group of type \( G_2 \) over \( F \). Then the order of \( T \) is

\[ q^6(q^2 - 1)(q^6 - 1) = q^6(q - 1)^2(q + 1)^2(q^2 + q + 1)(q^2 - q + 1). \]

We know that no prime greater than three divides more than one of these factors. Let \( V \) and \( F \) be as defined in [1]. We define \( \Gamma(V) \) to be the group of all semilinear maps on \( V \). Write \( \Gamma(V, F) \) to be the subgroup of \( \Gamma(V) \) preserving \( F \). Then \( \Gamma(V, F) \) is an extension of \( T \) by a field automorphism of order \( a \), and \( Aut(T) = \Gamma(V, F) \) unless \( p = 3 \). Let \( H_1 = \Gamma(V, F) \) and \( H_0 = G_2(q) \). If \( p = 3 \), then \( |Aut(H_0) : H_1| = 2 \), and if \( p \geq 5 \), we have \( H_1 = Aut(H_0) \).

We need some information about the subgroup of \( G_2(q) \).

**Lemma 2.1.** ([11]) Assume that \( H_0 \leq H \leq H_1 \), where \( H_0 \cong G_2(q) \) (\( q = p^e \) is odd) and \( H_1 \) are as above. Let \( M \) be a maximal subgroup of \( H \) not containing \( H_0 \). Then \( M_0 = M \cap H_0 \) is \( H_0 \)-conjugate to one of the following groups:

<table>
<thead>
<tr>
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<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>([q^3] : GL_2(q))</td>
<td>( q^6(q - 1)^2(q + 1) )</td>
<td>parabolic centralizer</td>
</tr>
<tr>
<td>((SL_2(q) \circ SL_2(q)) \cdot 2)</td>
<td>( 2q^2(q^2 - 1)^2 )</td>
<td>involution centralizer</td>
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<td></td>
</tr>
<tr>
<td>(G_2(q_0))</td>
<td>( q_0^6(q_0^2 - 1)(q_0^6 - 1) )</td>
<td>( q = q_0^m ), ( m ) prime</td>
</tr>
<tr>
<td>(2G_2(q))</td>
<td>( q^3(q^3 + 1)(q - 1) )</td>
<td>( p = 3, n ) odd</td>
</tr>
<tr>
<td>(PGL_2(q))</td>
<td>( q(q^2 - 1) )</td>
<td>( p \geq 7, q \geq 11 )</td>
</tr>
<tr>
<td>(2^3 L_3(2))</td>
<td>( 2^6 \cdot 3 \cdot 7 )</td>
<td>( q = p )</td>
</tr>
<tr>
<td>(L_2(8))</td>
<td>( 2^3 \cdot 3^2 \cdot 7 )</td>
<td>( p \geq 5, F = F_p[\omega] )</td>
</tr>
<tr>
<td>(L_2(13))</td>
<td>( 2^2 \cdot 3 \cdot 7 \cdot 13 )</td>
<td>( p \neq 13, F = F_p[\sqrt{13}] )</td>
</tr>
<tr>
<td>(G_2(2))</td>
<td>( 2^6 \cdot 3 \cdot 7 )</td>
<td>( q = p \geq 5 )</td>
</tr>
<tr>
<td>(J_1)</td>
<td>( 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 )</td>
<td>( q = 11 )</td>
</tr>
</tbody>
</table>

Conversely, if \( K \leq H_0 \) is \( H_0 \)-conjugate to one of these groups, then \( NC_G(K) \) is maximal in \( H \).

**Lemma 2.2.** ([11]) Assume that \( H_0 \leq H \leq H_1 \), where \( H_0 \cong G_2(q) \) (\( q = 3^e \)) and \( H \) contains a graph automorphism of \( H_0 \). Let \( M \) be a maximal subgroup of \( H \) not containing \( H_0 \). Then \( M_0 = M \cap H_0 \) is \( H_0 \)-conjugate to one of the following groups:
Lemma 2.3. (8) Let $G = G_2(q)$, where $q = 2^a$ with $a > 2$. The maximal subgroups of $G$ are listed as follows:

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<td>parabolic</td>
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<td>$2q^2(q - 1)^2$</td>
<td>involution centralizer</td>
</tr>
<tr>
<td>$2^3 \cdot L_3(20)$</td>
<td>$2^6 \cdot 3 \cdot 7$</td>
<td>$q = 3$</td>
</tr>
<tr>
<td>$(Z_{q^2-e})^2 \cdot D_{12}$</td>
<td>$12(q^2 - 1)^2$</td>
<td>$p \geq 9, \epsilon = \pm 1$</td>
</tr>
<tr>
<td>$(Z_{q^2+e+1})^2 \cdot Z_6$</td>
<td>$6(q^2 + cq + 1)$</td>
<td>$p \geq 9, \epsilon = \pm 1$</td>
</tr>
<tr>
<td>$G_2(q_0)$</td>
<td>$q_0^5(q_0^2 - 1)(q_0^1 - 1)$</td>
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Conversely, if $K \leq H_0$ is $H_0$-conjugate to one of these groups, then $N_G(K)$ is maximal in $H$.

Lemma 2.4. (8 and 11)

(i) If $q$ is even, then $G_2(q)$ has exactly two conjugacy class of involutions, and the order of the centralizer of a non-central involution is $q^4(q^2 - 1)$.

(ii) If $q$ is odd, then $G_2(q)$ has a unique conjugacy class of involutions, and the order of the centralizer of an involution is $q^2(q^2 - 1)^2$.

Lemma 2.5. (Lemma 3.3 of [13]) Some subgroups of $G = G_2(q)$ are being shown in the following:

1. $T_1 = Z_{q-1} \times Z_{q-1}$ and $N_G(T_1) = T_1 \cdot D_{12}$;
2. $T_2 = Z_{q+1} \times Z_{q+1}$ and $N_G(T_2) = T_2 \cdot D_{12}$;
3. $T_3 = Z_{q^2+q+1}$ and $N_G(T_3) = T_3 : Z_6$;
4. $T_4 = Z_{q^2-q+1}$ and $N_G(T_4) = T_4 : Z_6$;
5. $T_5 = Z_{q-1}$ and $N_G(T_5) = T_5 \cdot (Z_2 \times Z_2)$.

Lemma 2.6. (Lemma 3.5 of [13]) Let $q = q_0^m$, where $m$ is an odd prime and let $\epsilon = \pm$. Then the following hold:

1. $(q - \epsilon 1)^2$ does not divide $|G_2(q_0)|$;
2. $(q^2 + cq + 1)$ does not divide $|G_2(q_0)|$.

Lemma 2.7. Let $q = q_0^m$, where $m$ is a positive integer with $m \geq 4$ and let $\epsilon = \pm$. Then the following hold:

1. $(q - \epsilon 1)^2$ does not divide $|G_2(q_0)|$;
Almost simple group 594

(2) \((q^2 + \epsilon q + 1)\) does not divide \(|G_2(q_0)|\).
(3) \((q^2 - 1)\) does not divide \(|G_2(q_0)|\).

Proof. (1) If \(m > 6\), then there exists a \(p\)-primitive divisor of \(p^m - 1\), denoted by \(t\). Hence \(t \nmid q_0^6\) and \((q - \epsilon 1)^2\) does not divide \(|G_2(q_0)|\); if \(m = 4\) or \(6\), then \((q - \epsilon 1)^2 \nmid |G_2(q_0)|\). The proof is finished.

Similarly we can prove that the assertions (2) and (3) are true. \(\Box\)

We assume that \(G\) is a automorphism group acting line-transitively on a linear space \(S\) with parameters \(b, v, k, r\), where \(b\) is the number of lines, \(v\) is the number of points, \(r\) is the number of lines through a point and \(k\) is the number of points on a line. Recall the basic counting lemmas for regular linear spaces.

\[ v = r(k - 1) + 1, \]
\[ v(v - 1) = bk(k - 1). \]

Let \(b_1 = (b, v), b_2 = (b, v - 1), k_1 = (k, v), \) and \(k_2 = (k, v - 1).\) Then \(k = k_1k_2, b = b_1b_2, r = b_2k_2, \) and \(v = b_1k_1.\)

In [5], the authors defined a significant prime which divides \(b\) but not \(v.\) Observe that every prime divisor of \(b_2\) is a significant prime. It is well-known that a linear space is a projective plane if and only if \(b = v\), namely, \(b_2 = 1.\) Thus every linear space other than the projective plane has significant primes.

There is a fact that we shall use throughout this article. Observe that if an involution in \(G\) does not fix a point then \(G\) acts flag-transitively (see [6]). But the flag-transitive linear spaces are classified by Buekenhout, Delandtsheer, Doyen et al (see [4] and [3]), and so we assume that every involution fixes at least a point.

We state here a number of basic results which will be used repeatedly throughout the paper.

Lemma 2.8. \(\) Let \(G\) act line-transitively on a linear space \(S,\) and \(b_2\) be defined as above. Then the followings hold:
(1) \((b_2, v) = 1;\)
(2) \(b_2\) divides \(|G_\alpha|\).

Proof. (1) Note that \(b_2 = (b, v - 1),\) it is clear that \((b_2, v) = 1.

(2) Since \(G\) is line-transitive, by the theorem of R. E. Block in [2] we have \(G\) is point-transitive. Hence \(b = |G : G_L|\) and \(v = |G : G_\alpha|,\) where \(L \in \mathcal{L}\) and \(\alpha \in \mathcal{P}.\) Since \(rv = bk,\) it follows that \(b_2|G_L| = k_1|G_\alpha|,\) Note that \((b_2, k_1) = 1\) and hence \(b_2\) divides \(|G_\alpha|\). \(\Box\)

Lemma 2.9. \(\) (Zhou, Li and Liu [18]) Let \(G\) act line-transitively on a linear space \(S.\) Let \(K\) be a subgroup of \(G\). If \(K \not\subseteq G_L\) for any line \(L \in \mathcal{L},\) and \(K \leq G_\alpha\) for some point \(\alpha \in \mathcal{P},\) then \(N_G(K) \leq G_\alpha.\)
Lemma 2.10. (Lemma 2.8 of [16]) Let $G$ act line-transitively on a linear space $S$. If there exists a prime $p$ such that $p | b$ but $p \nmid v$, then for some $\alpha \in \mathcal{P}$, $N_G(P) \leq G_\alpha$, where $P$ is a Sylow $p$-subgroup of $G$.

Lemma 2.11. (Liu [12] and [13]) Let $G$ act line-transitively on a linear space $S$. Assume that $P$ is a Sylow $p$-subgroup of $G_\alpha$ for some $\alpha \in \mathcal{P}$. If $P$ is not a Sylow $p$-subgroup of $G$, then there exists a line $L$ through $\alpha$ such that $P \leq G_L$.

Lemma 2.12. ([14] and [15]) Let $G$ be a transitive group on $\Omega$, and $K$ be a conjugacy class of an element of $G$. Let $x \in K$ and $\text{Fix}_\Omega(\langle x \rangle)$ denote the fixed points set of $\langle x \rangle$ acting on $\Omega$. Then

$$|\text{Fix}_\Omega(\langle x \rangle)| = |G_\alpha K| \cdot |\Omega|/|K|,$$

where $\alpha \in \Omega$. In particular, if $G$ has a unique conjugacy class of involutions, then

$$|\text{Fix}_\Omega(\langle i \rangle)| = e(G_\alpha) \cdot |\Omega|/e(G),$$

where $i$ is an involution of $G$ and $e(G)$ denotes the number of involutions of $G$.

Lemma 2.13. ([14] and [15]) Let $G$ act line-transitively on a linear space $S$. Let $i$ be an involution of $G_L$, where $L$ is a line of $S$. Set $f_1 = |\text{Fix}_\Omega(\langle i \rangle)|$ and $f_2 = |\text{Fix}_S(\langle i \rangle)|$. If $S$ is not a projective plane and $f_1 \geq 2$, then $v \leq f_2^2$.

In order to do our work, we need to introduce the concept of exceptional triple. Let $G$ and $H$ be finite groups acting transitively on a finite set $\Omega$; with $H$ a normal subgroup of $G$. Then the triple $(G, H, \Omega)$ is called exceptional if the only common orbit of $G$ and $H$ on $\Omega \times \Omega$ is the diagonal. This definition is equivalent to the following: Let $\alpha \in \Omega$, then every $G_\alpha$-orbit except $\{\alpha\}$ breaks up into strictly smaller $H_\alpha$-orbits.

We call the triple $(G, H, \Omega)$ arithmetically exceptional, if there is a subgroup $B$ of $G$ which contains $H$, such that $(B, H, \Omega)$ is exceptional, and $B/H$ is cyclic. When $G$ is a primitive permutation group of almost simple type, Guralnick, Muller and Saxl have obtained their classification (see [10]). In particular, when $\text{Soc}(G) = G_2(q)$, there is the following lemma:

Lemma 2.14. ([10, Theorem 1.5 (g)]) Let $G$ be a primitive permutation group of almost simple type, so $L \leq G \leq \text{Aut}(L)$ with $L$ a simple nonabelian group. Suppose that there are subgroups $B$ and $H$ of $G$ with $H \leq G$ and $B/H$ cyclic, such that $(B, H)$ is exceptional. Let $M$ be a point stabilizer in $G$. Suppose that $L$ has Lie rank $\geq 2$, $L \neq \text{Sp}_4(2)^\prime \cong \text{PSL}_2(9)$. Then $M \cap L$ is a subfield subgroup, the centralizer in $L$ of a field automorphism of odd prime order $r$. Moreover,

(i) $r \neq p$ (with $p$ the defining characteristic of $L$),

(ii) if $r = 3$, then $L$ is of type $\text{Sp}_4(q)$ with $q$ even, and

(iii) there are no $\text{Aut}(L)$-stable $L$-conjugacy classes of $r$-elements.
Lemma 2.15. ([10, Lemma 3.3]) Let $H$ be a normal subgroup of the finite group $G$ with $G/H$ cyclic generated by $xH$. Let $\Omega$ be a transitive $G$-set. Assume that $H$ is also transitive on $\Omega$. Let $\chi(g)$ be the number of fixed points of $g \in G$ on $\Omega$. The following are equivalent:

1. $(G, H, \Omega)$ is exceptional;
2. $\chi(xh) \leq 1$ for all $h \in H$;
3. $\chi(xh) = 1$ for all $h \in H$;
4. $\chi(xh) \geq 1$ for all $h \in H$.

Lemma 2.16. ([9]) Let $G$ act line-transitively on a linear space $S$. Let $H$ be a subgroup of $G$ such that $H \leq G$ and $|G : H| = s$, a prime. If $H$ is line-intransitive and $S$ is not a projective plane, then $(G, H, \mathcal{P})$ is an exceptional triple.

3. The proof of Theorem

Since $T = G_2(q) \leq G \leq \text{Aut}(T)$ with $q = p^a$, we have $|\text{Out}(T)| = 2a$ or $a$ according as $p = 3$ or not, and $G = T : \langle x \rangle$, where $x \in \text{Out}(T)$. Let $o(x) = m$. Then we have when $p = 3$, $m \mid 2a$; when $p \neq 3$, $m \mid a$. Moreover, $|G| = q^6(q^2 - 1)(q^6 - 1)m$.

By [9], we know that almost simple groups cannot act line-transitively on non-Desarguesian projective planes. Hence we can assume that $S$ is not a projective plane. Suppose that $T$ is not line-transitive on $S$.

Let $s$ be a prime divisor of $m$. There exists a normal subgroup $H$ in $G$ such that $|G/H| = s$ and $H$ is not line-transitive (otherwise replacing $G$ by $H$). By Lemma 2.16 we have that the triple $(G, H, \mathcal{P})$ is exceptional. Let $\chi(g)$ be the number of fixed points of $g \in G$ on $\mathcal{P}$ and $y = x^{m/s}$. Then $G/H = (yH)$. We show that $H$ is point-transitive. In fact, if $H$ is not point-transitive, by the proof of Lemma 2.16 (see [9, Lemma 26]) we have $S$ is a projective plane, a contradiction. Now appealing to Lemma 2.15 we have $\chi(yh) = 1$ for all $h \in H$. It follows that $y$ has a unique fixed point of $\mathcal{P}$, says $\alpha$. Considering the cycle decomposition of $x$ acting on $\mathcal{P}$, we find that $x$ fixes no other points of $\mathcal{P}$ than $\alpha$. Hence $x \in G_\alpha$ and $G = TG_\alpha$. Then $T$ acts transitively on $\mathcal{P}$.

Since $T$ is not line-transitive, it follows that $x \notin G_L$ for any $L \in \mathcal{L}$. Note that $x \in G_\alpha$, and we appeal to Lemma 2.9 to conclude that $N_{G_\alpha}((x)) \leq G_\alpha$. By Lemmas 2.1 and 2.2 and 2.3, we find that the overgroup of $G_2(p)$ is only the subfield subgroup. Since $C_T(x) \leq N_{G_\alpha}((x))$ and $G_2(p) \leq C_T(x)$, we may assume that $G_\alpha \cap T = T_\alpha \cong G_2(q_0)$ for $q = q_0^c$ with a positive integer $c$. If $c$ is prime, then $G_\alpha$ is a maximal subgroup of $G$ and hence $G$ is primitive on $\mathcal{P}$. Thus by the proof of Lemma 2.14 (see [10, Theorem 1.5 (g)]) we have $c \nmid |2G_2(3)|$ for $p = 3$ and $c \nmid |G_2(p)|$ for $p \neq 3$, and hence 1) if $p = 2$, then $c \nmid |G_2(2)| = 2^6 \cdot 3^3 \cdot 7$, and hence $c \neq 2, 3, 7$; 2) if $p = 3$, then $c \nmid |2G_2(3)| = 2^3 \cdot 3^3 \cdot 7$ and hence $c \neq 2, 3, 7$; 3) if $p \geq 5$, then $c \nmid |G_2(p)| = p^6(p^6 - (p^2 - 1)$, it follows that $c \neq p, 2$ and $c \nmid (p - 1)^2, (p + 1)^2, p^2 + p + 1$ and $p^2 - p + 1$. 
Let $i$ be an involution of $T$ and $K$ be a conjugate class of $i$ in $G$. If $q$ is even, then we let $i$ be a non-central involution. Thus by Lemma 2.4 we have $|C_G(i)| \leq q^2(q^2 - 1)m$. If $q$ is even, then $v$ is even and so $f_1$ is even. Note that $i$ fixes at least a point, hence $f_1 \geq 2$. If $q$ is odd, then by Lemma 2.12 we have

$$f_1 = |\text{Fix}_P(i)| = \frac{v \cdot |G_a \cap K|}{|K|} = \frac{|C_G(i)|}{|C_{G_a}(i)|} \geq \frac{q^2(q^2 - 1)^2}{q_0^2(q_0^2 - 1)} \geq 2.$$ 

Since

$$f_2 = |\text{Fix}_L(i)| = \frac{b \cdot |G_L \cap K|}{|K|} < |C_G(i)|,$$

by Lemma 2.13 we can get the following inequality

$$v - q^6(q^6 - 1)(q^2 - 1) < (|C_G(i)|)^2 = q^8(q^2 - 1)^2m^2.$$ 

This implies that

$$q^6(q^6 - 1)(q^2 - 1) < q^8(q^2 - 1)^2m^2q_0^{14}.$$ 

It follows that

$$q^2 < \frac{q^4 + q^2 + 1}{q^2} = \frac{q^6(q^6 - 1)(q^2 - 1)}{q^8(q^2 - 1)^2} < q_0^{14}m^2.$$ 

Let $q_0 = p^\lambda$. Then $a = \lambda c$ for a positive integer $\lambda$ and $q = p^\lambda c$. We can get

(3.1)  

$$p^{\lambda(c-7)} < m.$$ 

Recall that $m|2a$ for $p = 3$ and $m|a$ for $p \neq 3$, so we have

1) if $p = 3$, then $m|2a$ and it follows from (3) that

(3.2)  

$$3^{\lambda(c-7)} < 2\lambda c,$$

which forces that $c \leq 9$.

2) if $p = 2$, then $m|a$ and hence

(3.3)  

$$2^{\lambda(c-7)} < \lambda c,$$

which forces that $c \leq 10$.

3) if $p \geq 5$, then $m|a$ and it follows that

(3.4)  

$$5^{\lambda(c-7)} \leq p^{\lambda(c-7)} < \lambda c,$$

which forces that $c \leq 8$.

1. If $p = 2$, $c = 9$, then it follows from (5) that $4^\lambda < 9\lambda$. Hence $\lambda = 1$, $a = 9$ or $\lambda = 2$, $a = 18$.

2. If $p = 2$, $c = 10$, then again by (5) we have $8^\lambda < 10\lambda$, and hence $\lambda = 1$, $a = 10$.

3. If $p = 3$, $c = 9$, then it follows from (4) that $9^\lambda < 18\lambda$. We have $\lambda = 1$, $a = 9$.

Recall that $a \not\equiv 0 \pmod{6}$. Hence the cases we should examine are in the following table.
It is clear that $G_\alpha$ contains no Sylow $p$-subgroups of $G$. Let $Q_0$ be a Sylow $p$-subgroup of $G_2(q_0)$. It follows by Lemma 2.11 that $Q_0 \leq G_L$. Then $Q_0 \leq T \cap G_L = T_L$. Examining the subgroup of $G_2(q)$ in Lemmas 2.1 and 2.2 and 2.3, we find that $G_L \cap T$ is isomorphic to $G_2(q_2)$ or a subgroup of some maximal subgroup of $T$, where $q$ is a power of $q_2$ and $q_0|q_2$.

### 3.1 Case: $T_L \cong G_2(q_2)$.

By Lemma 2.4 we have $|C_{G_L}(i)| \geq q_2^4(q_2^4 - 1) \geq q_0^4(q_0^4 - 1)$ and hence

$$|\text{Fix}_L((i))| = |G_L \cap K| = \frac{|C_{G_L}(i)|}{|C_{G_L((i))}|} \leq \frac{q^4(q^2 - 1)^m}{q_0^4(q_0^4 - 1)}.$$  

Appealing to Lemma 2.13, we have

$$v = \frac{q^6(q^6 - 1)(q^2 - 1)}{q_0^6(q_0^6 - 1)(q_0^2 - 1)} < \frac{(2^4(q^2 - 1)m)^2}{q_0^4(q_0^4 - 1)}.$$  

Since $q^6(q^6 - 1)(q^2 - 1) > q^2(q^4(q^2 - 1))^2$ and $q_0 \geq 2$, we have

$$q^2 < \frac{q_0^6(q_0^6 - 1)(q_0^2 - 1)}{(q_0^4(q_0^2 - 1))^2} = \frac{q_0^2(1 + q_0^2 + q_0^4)}{(1 + q_0^2)^2} < \frac{21}{16} q_0^2 m^2.$$  

It follows that

$$q < \sqrt{\frac{21}{16} q_0 m} < \frac{8}{5} q_0 m.$$  

Thus

$$p^a < \frac{8}{5} p^{a/c} \cdot m.$$  

Recall that $q_0 = p^\lambda$ and $a = \lambda c$. Then we have

$$p^{\lambda(c-1)} < \frac{8}{5} \cdot m.$$  

- if $p = 2$, then $2^{\lambda(c-1)} \leq 8 \lambda c$, which is a contradiction for $c \geq 4$.
- if $p = 3$, then $3^{\lambda(c-1)} \leq \frac{9}{5} \lambda c$. This is impossible for $c \geq 4$.
- if $p \geq 5$, then $5^{\lambda(c-1)} \leq \frac{5}{5} \lambda c$. This is a contradiction.  

Hence the case where $T_L \cong G_2(q_2)$ is excluded.

### 3.2 Case: $G_L \cap T$ is conjugate to a subgroup of some maximal subgroup of $T$.

<table>
<thead>
<tr>
<th>$c$</th>
<th>$a$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$3\lambda$, where $\lambda$ is a positive integer</td>
<td>$p \geq 5$ is a prime</td>
</tr>
<tr>
<td>7</td>
<td>$7\lambda$, where $\lambda$ is a positive integer</td>
<td>$p \geq 5$ is a prime</td>
</tr>
<tr>
<td>4, 5, 8</td>
<td>$\lambda c$, where $\lambda$ is a positive integer</td>
<td>$p$ is a prime</td>
</tr>
<tr>
<td>9</td>
<td>9</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
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<tr>
<td>10</td>
<td>10</td>
<td>2</td>
</tr>
</tbody>
</table>
Let \( M \) be a maximal subgroup of \( T \) such that \( T_L \leq M \). Let \( b = |G : G_L| = \frac{|T'|}{|T_L|} \) (where \( m_1 | m \) and \( a_1 = |M|/|T_L| \) and let \( b' = |T'|/|M| \), then we have \( b = b'a_1 \frac{m}{m_1} \). Thus if a prime divides \( b' \), it also divides \( b \). We can assume that \( T_L \) is not isomorphic to one subgroup of \((SL_2(q) \circ SL_2(q)) \cdot 2 \). If \( G \) contains a graph automorphism of \( G_2(q) \), then \( M \) lies inside \([q^3] : GL_2(q)\), or \( 2G_2(q) \) or \( PGL_2(q) \). Otherwise, \( M \) lies in \([q^3] : GL_2(q), SL_3(q) : 2, \) or \( 2G_2(q) \), or \( PGL_2(q) \), where \( \epsilon = \pm 1 \).

3.2.1 Case: \( c = 9, p = 2 \)
In this case we have \( v = 2^{48} \cdot 3 \cdot 7 \cdot 3249 \cdot 5329 \cdot 261633 \cdot 262657 \) and \( q = 2^9 \). Hence \( b_2 \) divides \( 2^6(2^2 - 1)(2^6 - 1) \cdot 9 \) by Lemma 2.8. Since \( (b_2, v) = 1 \), we have \( b_2 = 1 \). This implies that \( S \) is a projective plane which is a contradiction.

3.2.2 Case: \( c = 10, p = 2 \)
We have \( v = 2^{54} \cdot 5^2 \cdot 11^2 \cdot 13 \cdot 31^2 \cdot 205^2 \cdot 151 \cdot 331 \cdot 80581 \) and \( q = 2^{10} \). Let \( S \) be a Sylow 7 -subgroup of \( T \). Since \( (7, a) = 1 \) and hence \( S \) is also the Sylow 7-subgroup of \( G \). Note that \( 7 \nmid v \), and so we may split the proof into 2 or 3 cases.

If \( 7 \nmid b \), it follows from Lemmas 2.10 and 2.5 that \( Z_{q^2+q+1} \leq G_\alpha \), which is impossible by Lemma 2.7.

If \( 7 \mid b \), this implies that \( 7 \mid b_2 \) and \( G_L \) contains a Sylow 7-subgroup of \( G \). Hence the only case to occur is that \( M = SL_3(q) : 2 \) and \( \epsilon = + \). It follows from Lemma 2.8 that \( b_2 | 2^7 \cdot 3^3 \cdot 5 \cdot 7 \). Note that \( (b_2, v) = 1 \) and hence \( b_2 | 3^3 \). Since \( v \) is even, then \( k_1 \) is even. If \( b_2 = 3 \), then \( k_1 = 2 \) and hence \( |G_L| = \frac{9}{4} |G_\alpha| = 2^8 \cdot 3^2 \cdot 5 \cdot 7 \). There is no such group. Thus \( 3^2 | b \). Note that \( 3^3 || T \) and hence \( Z_3 \times Z_3 \nsubseteq T_L \). Applying Lemma 2.9 and 2.5 it yields the fact that \( N_G(Z_3 \times Z_3) \leq G_\alpha \), which implies \( Z_{q-1} \times Z_{q-1} \leq G_\alpha \), a contradiction.

3.2.3 Case: \( c = 9, p = 3 \)
We have \( v = 3^{48} \cdot 7 \cdot 13 \cdot 703^2 \cdot 757^2 \cdot 387400807 \cdot 387440173 \) and \( q = 3^9 \). By Lemma 2.4 we have that \( |C_G(i)| \leq q^3(q^2 - 1)^2 m \). Note that \( |C_G(i)| \geq 2 \) and hence it follows from Lemma 2.13 that \( v \leq \frac{|C_G(i)|^2}{4 |C_G(i)|^2} \leq \frac{1}{4} |C_G(i)|^2 \). This implies that \( 3^{48} \cdot 7 \cdot 13 \cdot 703^2 \cdot 757^2 \cdot 387400807 \cdot 387440173 \leq 2^{10} \cdot 3^{40} \cdot 7^4 \cdot 13^4 \cdot 703^4 \cdot 757^4 \), which is a contradiction.

3.2.4 Case: \( c = 3, p \geq 5 \)
In this case we have \( Z_{q_0+1} \times Z_{q_0+1} \leq G_\alpha \). We may consider two cases.

If \( Z_{q_0+1} \times Z_{q_0+1} \leq G_L \), it follows by Lemmas 2.1 that \( M = SL_3(q) : 2 \) where \( \epsilon = - \). Then \( Z_{q_0} \times Z_{q_0-1} \nsubseteq G_L \). It follows from Lemma 2.9 that \( N_G(Z_{q_0} \times Z_{q_0-1}) \leq G_\alpha \), which implies that \( Z_{q-1} \times Z_{q-1} \leq G_\alpha \). This is a contradiction.

If \( Z_{q_0+1} \times Z_{q_0+1} \nsubseteq G_L \), it follows from Lemma 2.9 that \( N_G(Z_{q_0+1} \times Z_{q_0+1}) \leq G_\alpha \). This implies that \( Z_{q+1} \times Z_{q+1} \leq G_\alpha \) which is a contradiction.

3.2.5 Case: \( c = 7, p \geq 5 \)
In this case we can find a prime $t$ greater than 5 such that $t$ divides $q_0^2 + c_0 + 1$. Let $U^c$ be a cyclic subgroup of $G_2(q_0)$ of order $q_0^2 + c_0 + 1$, where $c = \pm$ Then $U^c \leq Z_{q^2 + c_0 + 1}$. If $U^c \leq G_L$, by Lemmas 2.1 and 2.3 we find that the possibility for $M$ is the case $M = SL_3^L(q) = 2$ or $2G_2(q)$ where $c = \mp$. Hence $U^c \not\leq M$. It follows by Lemmas 2.9 and 2.5 that $Z_{q^2 + q + 1} : Z_6 \leq G_\alpha$, which gives a contradiction. If $U^c \not\leq G_L$, then by Lemma 2.9 we have $N_G(Z_{q^2 + q + 1}) \leq G_\alpha$ and hence $Z_{q^2 + q + 1} : Z_6 \leq G_\alpha$. This is a contradiction.

3.2.6 Case: $c = 4$ or 8

We have $v = q_0^{18}(q_0^2 + 1)^2(q_0^4 + 1)^2(q_0^8 - q_0^4 + 1)(q_0^4 - q_0^4 + 1)(q_0^8 - q_0^4 + 1)$ for $c = 4$ and $q_0^{22}(q_0^2 + 1)^2(q_0^4 + 1)^2(q_0^8 - q_0^4 + 1)(q_0^4 - q_0^4 + 1)(q_0^16 - q_0^8 + 1)$ for $c = 8$.

First suppose that $G$ contains a graph automorphism of $G_2(q)$. Then $M$ lies inside $[q^2] : (Z_{q-1})^2$, or $2G_2(q)$ or $PGL_3(q)$. Hence $Z_{q_0^2 + q + 1} \not\leq M$ and $Z_{q_0^2 + q + 1} \not\leq G_L$. Note that $Z_{q_0^2 + q + 1} \leq Z_{q^2 + q + 1}$. It follows by Lemma 2.9 that $N_G(Z_{q_0^2 + q + 1}) \leq G_\alpha$, which implies $Z_{q^2 + q + 1} \leq G_\alpha$. Thus $Z_{q^2 + q + 1} \leq T \cap G_\alpha = T_\alpha$. This contradicts Lemma 2.7.

Now assume that $G$ contains no graph automorphism of $G_2(q)$. We consider several cases.

Suppose first that $q_0 = 2$. We have $v = 2^{18} \cdot 5^2 \cdot 13 \cdot 17^2 \cdot 241$ and $q = 2^4$ for $c = 4$ and $v = 2^{17} \cdot 5^2 \cdot 13 \cdot 17^2 \cdot 241 \cdot 257^2 \cdot 65281$ and $q = 2^8$ for $c = 8$. Hence the Sylow 7-subgroup of $T$ is also of $G$. Note that $7 \nmid v$ and so we may split the proof into 2 or 3 cases.

If $7 \nmid b$, it follows from Lemmas 2.9 and 2.5 that $Z_{q^2 + q + 1} \leq G_\alpha$, which is impossible by Lemma 2.7.

If $7 \nmid b$, this implies that $7 \nmid b_2$ and $G_L$ contains a Sylow 7-subgroup of $G$. Hence the only case to occur is that $M = SL_3(q) = 2$. It follows from Lemma 2.8 that $b_2[2^8 \cdot 3^3 \cdot 7$ for $c = 4$ and $b_2[2^9 \cdot 3^3 \cdot 7$ for $c = 8$. Note that $(b_2, v) = 1$ and hence $b_2[d_3^3]$. Since $v$ is even, then $k_1$ is even. If $b_2 = 3$, then $k_1 = 2$ and hence $|G_L| = \frac{3}{2}|G_\alpha| = 2^9 \cdot 3^2 \cdot 5 \cdot 7$ for $c = 4$ and $|G_L| = \frac{3}{2}|G_\alpha| = 2^{10} \cdot 3^2 \cdot 5 \cdot 7$ for $c = 8$. There are no subgroups of such orders. Thus $3^2 | [T]$ and hence $Z_3 \times Z_3 \not\leq T_L$. Applying Lemma 2.9 and 2.5 it yields the fact that $N_G(Z_3 \times Z_3) \leq G_\alpha$, which implies $Z_{q-1} \times Z_{q-1} \leq G_\alpha$, a contradiction.

Suppose next that $q_0 \neq 2$. Then we calculate that there exists a prime divisor $s$ greater than 3 of $q_0^2 + c_0 + 1$. It follows that $s | b'$ and hence $s \nmid b$. We may have

$$(q_0^2 + c_0 + 1, q_0^2 + 1) = 1, \quad (q_0^2 + c_0 + 1, q_0^4 + 1) = 1$$

$$(q_0^2 + c_0 + 1, q_0^8 + 1) = 1, \quad (q_0^2 + c_0 + 1, q_0^4 - q_0^4 + 1) = 1$$

$$(q_0^2 + c_0 + 1, q_0^8 - q_0^4 + 1) = 1, \quad (q_0^2 + c_0 + 1, q_0^{16} - q_0^8 + 1) = 1$$

Then it follows that $q_0^2 + c_0 + 1, v) = 1$ and hence $s \nmid v$.

Let $S$ be a Sylow $s$-subgroup of $T$. Then $S$ is also a $s$-subgroup of $T_\alpha$. We have $S \not\leq T_L$. In fact, if $S \leq T_L$, then $S \leq M$ and hence $s \nmid b' = \frac{|T|}{|M|}$, a
contradiction.) It follows from Lemma 2.10 that $N_G(S) \leq G_\alpha$, which implies that $Z_{q^2+q+1} \leq G_\alpha$. This is impossible.

3.2.7 Case: $c = 5$

Suppose first that $q_0 = 2$. Then $v = 2^{24} \cdot 11^2 \cdot 31^2 \cdot 151 \cdot 331$ and $q = 2^5$. The Sylow 7-subgroups of $T$ are also those of $G$. We may split the proof into 2 cases.

If 7 divides $b$, it follows from Lemmas 2.10 and 2.5 that $Z_{q^2+q+1} \leq G_6$. This is impossible.

If 7 does not divide $b$, then $G_L$ contains a Sylow 7-subgroup of $G$. Thus the only case to occur is that $M = SL_3(q) : 2$. We may get $b' = q^3(q^3 + 1)/2 = 2^{14}(2^{15} + 1)$ and hence $9|b$. Since $27|||T|$, it follows that $Z_3 \times Z_3 \not\leq T_L$. Applying Lemmas 2.9 and 2.5 it yields the fact that $N_G(Z_3 \times Z_3) \leq G_\alpha$, and hence $(Z_{q+1} \times Z_{q+1}) \cdot D_{12} \leq G_\alpha$, which is a contradiction.

Now we can assume that $q_0 \neq 2$. Then there exists a prime divisor $t$ greater than 3 of $q_0^2 + \epsilon q_0 + 1$ by a direct calculation. Let $U^\epsilon$ be a cyclic subgroup of $G_2(q_0)$ of order $q_0^2 + \epsilon q_0 + 1$, where $\epsilon = \pm$. Then $U^\epsilon \leq Z_{q^2+q+1}$. If $U^\epsilon \leq G_L$, by Lemmas 2.1 and 2.2 and 2.3 we find that $M = SL_3(q) : 2$ or $2G_2(q)$, where $\epsilon = -$. Hence $U^+ \not\leq M$. It follows from Lemmas 2.9 and 2.5 that $Z_{q^2+q+1} : Z_6 \leq G_\alpha$. This is a contradiction. If $U^\epsilon \not\leq G_L$, then by Lemma 2.9 we have $N_G(Z_{q^2-q+1}) \leq G_\alpha$ and hence $Z_{q^2-q+1} : Z_6 \leq G_\alpha$. This is a contradiction.

Thus we have that $T$ is line-transitive on $S$. The proof of the Theorem is complete.

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References

Almost simple group


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