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COHERENCE IN AMALGAMATED ALGEBRA ALONG AN IDEAL

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Abstract. Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B. In this paper, we investigate the transfer of the property of coherence to the amalgamation $A \bowtie^f J$. We provide necessary and sufficient conditions for $A \bowtie^f J$ to be a coherent ring.
Keywords: Amalgamated algebra, coherent ring.
MSC(2010): Primary: 13D05; Secondary: 13D02.

1. **Introduction**

Throughout this paper, all rings are commutative with identity element, and all modules are unitary. Let R be a commutative ring. We say that an ideal is regular if it contains a regular element, i.e; a non-zerodivisor element. For a nonnegative integer n, an R-module E is called n-presented if there is an exact sequence of R-modules:

$$F_n \to F_{n-1} \to \dots F_1 \to F_0 \to E \to 0$$

where each F_i is a finitely generated free R-module. In particular, 0-presented and 1-presented R-modules are, respectively, finitely generated and finitely presented R-modules.

A ring R is coherent if every finitely generated ideal of R is finitely presented; equivalently, if $(0 : a)$ and $I \cap J$ are finitely generated for every $a \in R$ and any two finitely generated ideals I and J of R. Examples of coherent rings are Noetherian rings, Boolean algebras, von Neumann regular rings, and Prüfer/semihereditary rings. For instance see [15].
Recall that an R-module M is called a coherent R-module if it is finitely generated and every finitely generated submodule of M is finitely presented.

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Let A and B be two rings, let J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J = \{(a, f(a) + j) \diagup a \in A, j \in J\}$$

called the amalgamation of A with B along J with respect to f (introduced and studied by D'Anna, Finocchiaro, and Fontana in [6, 7]). This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana in [8–10] and denoted by $A \bowtie I$). Moreover, other classical constructions (such as the $A + XB[X]$, $A + XB[[X]]$, and the $D + M$ constructions) can be studied as particular cases of the amalgamation [6, Examples 2.5 & 2.6] and other classical constructions, such as the Nagata's idealization and the CPI extensions (in the sense of Boisen and Sheldon [2]) are strictly related to it (see [6, Example 2.7 & Remark 2.8]).

One of the key tools for studying $A \bowtie^f J$ is based on the fact that the amalgamation can be studied in the frame of pullback constructions [6, Section 4]. This point of view allows the authors in [6, 7] to provide an ample description of various properties of $A \bowtie^f J$, in connection with the properties of A, J and f. Namely, in [6], the authors studied the basic properties of this construction (e.g., characterizations for $A \bowtie^f J$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as an amalgamation.

This paper investigates a property of coherence in amalgamated algebra along an ideal. Our results generate original examples which enrich the current literature with new families of non-Noetherian coherent rings.

2. Main results

This section characterizes the amalgamated algebra along an ideal $A \bowtie^f J$ to be a coherent ring. The main result (Theorem 2.2) examines the property of coherence that the amalgamation $A \bowtie^f J$ might inherit from the ring A for some classes of ideals J and homomorphisms f, and hence generates new examples of non-Noetherian coherent rings.

Let $f : A \to B$ be a ring homomorphism, J be an ideal of B and let n be a positive integer. Consider the function $f^n : A^n \to B^n$ to defined by $f^n((\alpha_i)_{i=1}^{i=n}) = (f(\alpha_i))_{i=1}^{i=n}$. Obviously, f^n is a ring homomorphism and J^n is an ideal of B^n. This allows us to define $A^n \bowtie^{f^n} J^n$. Moreover, let $\phi : (A \bowtie^f J)^n \to A^n \bowtie^{f^n} J^n$ defined by $\phi((a_i, f(a_i) + j_i)_{i=1}^{i=n}) = ((a_i)_{i=1}^{i=n}, f^n((a_i)_{i=1}^{i=n}) + (j_i)_{i=1}^{i=n})$. It is easily checked that ϕ is a ring isomorphism. So $(A \bowtie^f J)^n$ and $A^n \bowtie^{f^n} J^n$ are isomorphic as rings. Let U be a submodule of A^n. Then

$U \bowtie^{f^n} J^n := \{(u, f^n(u) + j) \in A^n \bowtie^{f^n} J^n / u \in U, j \in J^n\}$ is a submodule of $A^n \bowtie^{f^n} J^n$.

Next, before we announce the main result of this section (Theorem 2.2), we make the following useful remark.

Remark 2.1. *Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B. Then $f^n(\alpha a) = f(\alpha)f^n(a)$ for all $\alpha \in A$ and $a \in A^n$.*

Now, to the main result:

Theorem 2.2. *Let $f : A \to B$ be a ring homomorphism and let J be a proper ideal of B.*

(1) *If $A \bowtie^f J$ is a coherent ring, then so is A.*
(2) *Assume that J and $f^{-1}(J)$ are finitely generated ideals of $f(A) + J$ and A respectively. Then $A \bowtie^f J$ is a coherent ring if and only if A and $f(A) + J$ are coherent rings.*
(3) *Assume that J is a regular finitely generated ideal of $f(A) + J$. Then $A \bowtie^f J$ is a coherent ring if and only if A and $f(A) + J$ are coherent rings and $f^{-1}(J)$ is a finitely generated ideal of A.*

Before proving Theorem 2.2, we establish the following lemmas.

Lemma 2.3. *Let $f : A \to B$ be a ring homomorphism and let J be a proper ideal of B. Then:*

(1) *$\{0\} \times J$ (resp., $f^{-1}\{J\} \times \{0\}$) is a finitely generated ideal of $A \bowtie^f J$ if and only if J (resp., $f^{-1}\{J\}$) is a finitely generated ideal of $f(A) + J$ (resp., A).*
(2) *If $A \bowtie^f J$ is a coherent ring and $f^{-1}(J)$ is a finitely generated ideal of A, then $f(A) + J$ is a coherent ring.*

Proof. **(1)** Assume that $J := \sum_{i=1}^{i=n} (f(A) + J)k_i$ is a finitely generated ideal of $f(A) + J$, where $k_i \in J$. It is clear that $\sum_{i=1}^{i=n} (A \bowtie^f J)(0, k_i) \subset \{0\} \times J$. Let $x := (0, \sum_{i=1}^{i=n} (f(\alpha_i) + j_i)k_i) \in \{0\} \times J$, where $\alpha_i \in A$ and $j_i \in J$. Hence, $x = \sum_{i=1}^{i=n} (0, (f(\alpha_i) + j_i)k_i) = \sum_{i=1}^{i=n} (\alpha_i, f(\alpha_i) + j_i)(0, k_i) \in \sum_{i=1}^{i=n} (A \bowtie^f J)((0, k_i)$. Therefore, $\{0\} \times J \subset \sum_{i=1}^{i=n} (A \bowtie^f J)(0, k_i)$ and so $\{0\} \times J = \sum_{i=1}^{i=n} (A \bowtie^f J)(0, k_i)$. Conversely, Assume that $\{0\} \times J := \sum_{i=1}^{i=n} (A \bowtie^f J)(0, k_i)$ is a finitely generated ideal of $A \bowtie^f J$, where $k_i \in J$. It is readily seen that $J = \sum_{i=1}^{i=n} (f(A) + J)k_i$, as desired.

Assume that $f^{-1}\{J\} := \sum_{i=1}^{i=n} Ak_i$ is a finitely generated ideal of A, where $k_i \in f^{-1}\{J\}$. It is obvious that $\sum_{i=1}^{i=n} A \bowtie^f J(k_i, 0) \subset f^{-1}\{J\} \times \{0\}$. Let $x =: (\sum_{i=1}^{i=n} \alpha_i k_i, 0) \in f^{-1}\{J\} \times \{0\}$, where $\alpha_i \in A$. Then $x = \sum_{i=1}^{i=n} (\alpha_i k_i, 0) = \sum_{i=1}^{i=n} (\alpha_i, f(\alpha_i))(k_i, 0) \in \sum_{i=1}^{i=n} (A \bowtie^f J)(k_i, 0)$. Therefore, $f^{-1}\{J\} \times \{0\} \subset \sum_{i=1}^{i=n} (A \bowtie^f J)(k_i, 0)$ and so $f^{-1}\{J\} \times \{0\} = \sum_{i=1}^{i=n} A \bowtie^f J(k_i, 0)$. Conversely, Assume that $f^{-1}\{J\} \times \{0\} := \sum_{i=1}^{i=n} (A \bowtie^f J)(a_i, 0)$ is a finitely generated ideal of $A \bowtie^f J$, where $a_i \in f^{-1}\{J\}$. It is easy to check that $f^{-1}\{J\} = \sum_{i=1}^{i=n} Aa_i$, as desired.

(2) Assume that $A \bowtie^f J$ is a coherent ring and $f^{-1}\{J\} \times \{0\}$ is a finitely generated ideal of $A \bowtie^f J$. Then $f(A) + J \cong \frac{A\bowtie^f J}{f^{-1}\{J\}\times\{0\}}$ is a coherent ring by [15, Theorem 2.4.1], as desired. □

Lemma 2.4. *Let $f : A \to B$ be a ring homomorphism, J be an ideal of B, and let U be a submodule of A^n. Then:*

1. *Assume that U is a finitely generated A-module and J is a finitely generated ideal of $f(A) + J$. Then $U \bowtie^{f^n} J^n$ is a finitely generated $(A \bowtie^f J)$-module.*
2. *Assume that $f^n(U) \subset J^n$. Then $U \bowtie^{f^n} J^n$ is a finitely generated $(A \bowtie^f J)$-module if and only if U is a finitely generated A-module and J is a finitely generated ideal of $f(A) + J$.*

Proof. **(1)** Assume that $U := \sum_{i=1}^{i=n} Au_i$ is a finitely generated A-module, where $u_i \in U$ for all $i \in \{1, ...n\}$ and $J^n := \sum_{i=1}^{i=n}(f(A)+J)e_i$ is a finitely generated $(f(A) + J)$-module, where $e_i \in J^n$ for all $i \in \{1, ...n\}$. We claim that $U \bowtie^{f^n} J^n = \sum_{i=1}^{i=n}(A \bowtie^f J)(u_i, f^n(u_i)) + \sum_{i=1}^{i=n}(A \bowtie^f J)(0, e_i)$. Indeed, $\sum_{i=1}^{i=n}(A \bowtie^f J)(u_i, f^n(u_i)) + \sum_{i=1}^{i=n}(A \bowtie^f J)(0, e_i) \subset U \bowtie^{f^n} J^n$ since $(u_i, f^n(u_i)) \in U \bowtie^{f^n} J^n$ for all $i \in \{1, ...n\}$ and $(0, e_i) \in U \bowtie^{f^n} J^n$ for all $i \in \{1, ...n\}$. Conversely, let $(x, f^n(x) + k) \in U \bowtie^{f^n} J^n$, where $x \in U$ and $k \in J^n$. Hence, $x = \sum_{i=1}^{i=n} \alpha_i u_i \in U$, for some $\alpha_i \in A$ $(i \in \{1, ...n\})$ and $k = \sum_{i=1}^{i=n}(f(\beta_i) + j_i)e_i \in J^n$, for some $\beta_i \in A$ and $j_i \in J$ $(i \in \{1, ...n\})$. We obtain

$$\begin{aligned}(x, f^n(x) + k) &= (\sum_{i=1}^{i=n}\alpha_i u_i, \sum_{i=1}^{i=n} f(\alpha_i)f^n(u_i)) + (0, \sum_{i=1}^{i=n}(f(\beta_i) + j_i)e_i) \\ &= \sum_{i=1}^{i=n}(\alpha_i, f(\alpha_i))(u_i, f^n(u_i)) + \sum_{i=1}^{i=n}(0, f(\beta_i) + j_i)(0, e_i) \\ &= \sum_{i=1}^{i=n}(\alpha_i, f(\alpha_i))(u_i, f^n(u_i)) + \sum_{i=1}^{i=n}(\beta_i, f(\beta_i) + j_i)(0, e_i).\end{aligned}$$

Consequently, $(x, f^n(x) + k) \in \sum_{i=1}^{i=n}(A \bowtie^f J)(u_i, f^n(u_i)) + \sum_{i=1}^{i=n}(A \bowtie^f J)(0, e_i)$ since $(\alpha_i, f(\alpha_i)) \in (A \bowtie^f J)$ for all $i \in \{1, ...n\}$ and $(\beta_i, f(\beta_i) + j_i) \in A \bowtie^f J$ for all $i \in \{1, ...n\}$ and hence $U \bowtie^{f^n} J^n = \sum_{i=1}^{i=n}(A \bowtie^f J)(u_i, f^n(u_i)) + \sum_{i=1}^{i=n}(A \bowtie^f J)(0, e_i)$ is a finitely generated $(A \bowtie^f J)$-module, as desired.

(2) Assume that $f^n(U) \subset J^n$. If U is a finitely generated A-module and J is a finitely generated ideal of $f(A) + J$, then $U \bowtie^{f^n} J^n$ is a finitely generated $(A \bowtie^f J)$-module by (1). Conversely, assume that $U \bowtie^{f^n} J^n := \sum_{i=1}^{i=n}(A \bowtie^f$

$J)(u_i, f^n(u_i)+k_i)$ is a finitely generated $(A \bowtie^f J)$-module, where, $u_i \in U$ and $k_i \in J^n$ for all $1 \le i \le n$. It is clear that $U = \sum_{i=1}^{i=n} Au_i$. On the other hand, we claim that $J^n = \sum_{i=1}^{i=n}(f(A)+J)(f^n(u_i)+k_i)$. Indeed, let $j \in J^n$. Then $(0,j) = \sum_{i=1}^{i=n}(\alpha_i, f(\alpha_i)+j_i)(u_i, f^n(u_i)+k_i)$ for some $\alpha_i \in A$ and $j_i \in J$. So $j = \sum_{i=1}^{i=n}(f(\alpha_i)+j_i)(f^n(u_i)+k_i) \in \sum_{i=1}^{i=n}(f(A)+J)(f^n(u_i)+k_i)$. Thus $J^n \subset \sum_{i=1}^{i=n}(f(A)+J)(f^n(u_i)+k_i)$. But $f^n(u_i) \in J^n$ for all $i = 1, ...n$ since $f^n(U) \subset J^n$. Hence, $(f^n(u_i)+k_i) \in J^n$ $\forall i$ and so $\sum_{i=1}^{i=n}(f(A)+J)(f^n(u_i)+k_i) \subset J^n$. Therefore, $J^n = \sum_{i=1}^{i=n}(f(A)+J)(f^n(u_i)+k_i)$ is a finitely generated $(f(A)+J)$-module and so J is a finitely generated ideal of $(f(A)+J)$, completing the proof of Lemma 2.4. □

Lemma 2.5. *Let $f : A \to B$ be a ring homomorphism, and J be an ideal of B. Assume that J and $f^{-1}(J)$ are finitely generated ideals of $f(A)+J$ and A respectively. Then $f^{-1}\{J\} \times \{0\}$ is a coherent $(A \bowtie^f J)$-module provided A is a coherent ring.*

Proof. Since $f^{-1}\{J\} \times \{0\}$ is a finitely generated $(A \bowtie^f J)$-module, it remains to show that every finitely generated submodule of $f^{-1}\{J\} \times \{0\}$ is finitely presented. Assume that A is a coherent ring and let N be a finitely generated submodule of $f^{-1}\{J\} \times \{0\}$. It is clear that $N = I \times \{0\}$, where $I = \sum_{i=1}^{i=n} Aa_i$ for some positive integer n and $a_i \in I$. Consider the exact sequence of A-modules:

$$0 \to Kerv \to A^n \to I \to 0 \qquad (1)$$

where $v((\alpha_i)_{i=1}^{i=n}) = \sum_{i=1}^{i=n} \alpha_i a_i$. Then $Kerv = \{(\alpha_i)_{i=1}^{i=n} \in A^n \diagup \sum_{i=1}^{i=n} \alpha_i a_i = 0\}$. On the other hand, it is easily verified that $N = \sum_{i=1}^{i=n} A \bowtie^f J(a_i, 0)$. Consider the exact sequence of $(A \bowtie^f J)$-modules:

$$0 \to Keru \to (A \bowtie^f J)^n \to N \to 0 \qquad (2)$$

where $u((\alpha_i, f(\alpha_i)+j_i)_{i=1}^{i=n}) = \sum_{i=1}^{i=n}(\alpha_i, f(\alpha_i)+j_i)(a_i, 0)$. Then, $Keru = \{(\alpha_i, f(\alpha_i)+j_i)_{i=1}^{i=n} \in (A \bowtie^f J)^n \diagup \sum_{i=1}^{i=n} \alpha_i a_i = 0\}$. So $Keru = \{((\alpha_i)_{i=1}^{i=n}, f^n((\alpha_i)_{i=1}^{i=n}) + (j_i)_{i=1}^{i=n}) \in A^n \bowtie^{f^n} J^n \diagup (\alpha_i)_{i=1}^{i=n} \in Kerv\}$ and hence $Keru = Kerv \bowtie^{f^n} J^n$. But I is a finitely presented ideal of A since A is a coherent ring, so $Kerv$ is a finitely generated A-module (by a sequence (1)) and hence $Keru = Kerv \bowtie^{f^n} J^n$ is a finitely generated $(A \bowtie^f J)$-module (by lemma 2.4 (1)). Therefore, N is a finitely presented $(A \bowtie^f J)$-module by a sequence (2) and hence $f^{-1}\{J\} \times \{0\}$ is a coherent $A \bowtie^f J$-module, to complete the proof of Lemma 2.5. □

Lemma 2.6. *Let $f : A \to B$ be a ring homomorphism, and J be an ideal of B. If $A \bowtie^f J$ is a coherent ring and J is a regular ideal of $f(A) + J$, then $f^{-1}(J)$ is a finitely generated ideal of A.*

Proof. Assume that $A \bowtie^f J$ is a coherent ring and J contains a regular element k. Set $c = (0, k) \in A \bowtie^f J$. One can easily check that:

$$\begin{array}{rcl} (0 : c) & = & \{(a, f(a) + j) \in A \bowtie^f J \,/\, (a, f(a) + j)(0, k) = 0\} \\ & = & \{(a, f(a) + j) \in A \bowtie^f J \,/\, (f(a) + j)k = 0\} \\ & = & \{(a, f(a) + j) \in A \bowtie^f J \,/\, f(a) + j = 0\} \\ & = & \{(a, 0) \in A \bowtie^f J \,/\, a \in f^{-1}\{J\}\} \\ & = & f^{-1}\{J\} \times \{0\}. \end{array}$$

Since $A \bowtie^f J$ is a coherent ring, then $(0 : c) = f^{-1}\{J\} \times \{0\}$ is a finitely generated ideal of $A \bowtie^f J$. Therefore, $f^{-1}\{J\}$ is a finitely generated ideal of A, as desired. □

Proof of Theorem 2.2

Proof. **(1)** If $A \bowtie^f J$ is a coherent ring, then A is a coherent ring by [15, Theorem 4.1.5] since A is a module retract of $A \bowtie^f J$.

(2) Assume that J and $f^{-1}(J)$ are finitely generated ideals of $f(A) + J$ and A respectively. Then A and $f(A) + J$ are coherent rings since $A \bowtie^f J$ is a coherent ring (by Theorem 2.2 (1) and Lemma 2.3 (2)). Conversely, assume that A and $f(A)+J$ are coherent rings. Since $\frac{A \bowtie^f J}{f^{-1}\{J\} \times \{0\}} \cong f(A)+J$, $f(A)+J$ is a coherent ring and $f^{-1}\{J\} \times \{0\}$ is a coherent $A \bowtie^f J$-module (by Lemma 2.5), then $A \bowtie^f J$ is a coherent ring (by [15, Theorem 2.4.1]).

(3) Follows immediately from Theorem 2.2 (2) and Lemma 2.6. This completes the proof of the main Theorem. □

The following Corollary is an immediate consequence of Theorem 2.2 (3).

Corollary 2.7. *Let $f : A \to B$ be a ring homomorphism, B be an integral domain and let J be a proper and finitely generated ideal of $f(A) + J$. Then $A \bowtie^f J$ is a coherent ring if and only if A and $f(A) + J$ are coherent rings and $f^{-1}(J)$ is a finitely generated ideal of A.*

The Corollary below follows immediately from Theorem 2.2 (2) which examines the case of the amalgamated duplication.

Corollary 2.8. *Let A be a ring and I be a proper ideal of A.*

(1) *If $A \bowtie I$ is a coherent ring, then so is A.*
(2) *Assume that I is a finitely generated ideal of A. Then $A \bowtie I$ is a coherent ring if and only if A is a coherent ring.*

The next Corollary is an immediate consequence of Theorem 2.2 (2).

Corollary 2.9. *Let A be a ring, I be an ideal of A, $B := \frac{A}{I}$, and let $f : A \to B$ be the canonical homomorphism ($f(x) = \overline{x}$).*

(1) *Assume that J and $f^{-1}(J)$ are finitely generated ideals of B and A respectively. Then $A \bowtie^f J$ is a coherent ring if and only if A and B are coherent rings.*
(2) *Assume that J is a regular finitely generated ideal of B. Then $A \bowtie^f J$ is a coherent ring if and only if A and B are coherent rings and $f^{-1}(J)$ is a finitely generated ideal of A.*

The aforementioned result enriches the literature with new examples of coherent rings which are non-Noetherian rings.

Example 2.10. *Let A be a non-Noetherian coherent ring, I be a finitely generated ideal of A, $f : A \to B(= \frac{A}{I})$ be the canonical homomorphism, and let J be a finitely generated ideal of A. Then $A \bowtie^f \overline{J}$ is a non-Noetherian coherent ring.*

Proof. By Corollary 2.9, $A \bowtie^f \overline{J}$ is a coherent ring since A and B are both coherent rings and J is a finitely generated ideal of A. On the other hand, $A \bowtie^f \overline{J}$ is a non-Noetherian ring by [6, Proposition 5.6, p. 167] since A is a non-Noetherian ring. □

Example 2.11. *Let $A := \mathbb{Z} + X\mathbb{Q}[X]$, where $\mathbb{Z}$ is the ring of integers, and $\mathbb{Q}$ is the field of rational numbers. Let $I := X\mathbb{Q}[X]$, $B := \frac{A}{I}(\cong \mathbb{Z})$, $f : A \to B$ be the canonical homomorphism and let J be a nonzero ideal of B. Then $A \bowtie^f J$ is a non-Noetherian coherent ring.*

Proof. By Corollary 2.9, $A \bowtie^f J$ is a coherent ring since A and B are both coherent rings and J (resp., $f^{-1}(J) = n_0\mathbb{Z} + X\mathbb{Q}[X]$ for some positive integer n_0) is a finitely generated ideal of B (resp., A). On the other hand, $A \bowtie^f J$ is a non-Noetherian ring by [6, Proposition 5.6, p. 167] since A is a non-Noetherian ring. □

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