ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 41 (2015), No. 3, pp. 633–638

Title:

The metric dimension and girth of graphs

Author(s):

M. Jannesari

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 41 (2015), No. 3, pp. 633–638
Online ISSN: 1735-8515

THE METRIC DIMENSION AND GIRTH OF GRAPHS

M. JANNESARI

(Communicated by Ebadollah S. Mahmoodian)

ABSTRACT. A set $W \subseteq V(G)$ is called a resolving set for G, if for each two distinct vertices $u, v \in V(G)$ there exists $w \in W$ such that $d(u, w) \neq d(v, w)$, where $d(x, y)$ is the distance between the vertices x and y. The minimum cardinality of a resolving set for G is called the metric dimension of G, and denoted by $\dim(G)$. In this paper, it is proved that in a connected graph G of order n which has a cycle, $\dim(G) \leq n - g(G) + 2$, where $g(G)$ is the length of the shortest cycle in G, and the equality holds if and only if G is a cycle, a complete graph or a complete bipartite graph $K_{s,t}$, $s, t \geq 2$.
Keywords: Resolving set, metric dimension, girth.
MSC(2010): Primary: 05C12; Secondary: 05C35.

1. Introduction

Throughout the paper, G is a finite, simple, and connected graph of order n with vertex set V and edge set E. The distance between two vertices u and v, denoted by $d(u, v)$, is the length of ¡the¿ shortest path between u and v in G. The diameter of G, denoted by $diam(G)$ is $\max\{d(u, v)\colon u, v \in V\}$. The degree of a vertex v, $\deg(v)$, is the number of its neighbors. The notations $\sim$ and $\nsim$ denote the adjacency and non-adjacency relations, respectively. The notations $P_n = (v_1, v_2, \ldots, v_n)$ and $C_n = (v_1, v_2, \ldots, v_n, v_1)$ are used for the path and the cycle of order n, respectively. The number of edges in a cycle is its length. If G has a cycle, then the length of the shortest cycle in G is called the girth of G and denoted by $g(G)$. For a subset S of $V(G)$, $G \setminus S$ is the induced subgraph $\langle V(G) \setminus S\rangle$. A vertex $v \in V(G)$ is a *cut vertex* in G if $G \setminus \{v\}$ has at least two components. If $G \neq K_2$ has no cut vertex, then G is called a 2-*connected* graph.

For an ordered set $W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)$ and a vertex v of G, the k-vector

$$r(v|W) := (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$$

Article electronically published on June 15, 2015.
Received: 2 May 2012, Accepted: 6 April 2014.

©2015 Iranian Mathematical Society

is called the *metric representation* of v with respect to W. The set W is called a *resolving set* for G if distinct vertices have different representations. A resolving set for G with minimum cardinality is called a *metric basis*, and its cardinality is the *metric dimension* of G, denoted by $\dim(G)$. It is obvious that to see whether a given set W is a resolving set, it is sufficient to consider the vertices in $V(G)\backslash W$, because $w \in W$ is the unique vertex of G for which $d(w, w) = 0$.

In [14], Slater introduced the idea of a resolving set and used a *locating set* and the *location number* for a resolving set and the metric dimension, respectively. He described the usefulness of these concepts when working with U.S. Sonar and Coast Guard Loran stations. Independently, Harary and Melter [8] discovered the concept of the location number as well and called it the metric dimension. For more results related to these concepts see [3–5, 7, 10, 11]. The concept of a resolving set has various applications in diverse areas including coin weighing problems [13], network discovery and verification [2], robot navigation [11], mastermind game [4], problems of pattern recognition and image processing [12], and combinatorial search and optimization [13].

Chartrand et al. ‘ [6]’ obtained the following bound for the metric dimension in terms of the order and the diameter.

Theorem 1.1. ‘ [6]’ *If G is a connected graph of order n, then* $\dim(G) \leq n - diam(G)$.

Ten years later, Hernando et al. [9] characterized all graphs G of order n and metric dimension $n - diam(G)$. Also, Bagheri et al. [1] provided an upper bound for $\dim(G)$ in terms of the domination number and order of G and characterized all graphs that attain this bound. The main goal of this paper is to prove that for a connected graph G of order n and girth $g(G)$

$$\dim(G) \leq n - g(G) + 2$$

and characterize all graphs such that this bound is tight for them. In fact, it is proved that cycles, complete and complete bipartite graphs are all graphs with $\dim(G) = n - g(G) + 2$. To prove the main results the following known results are needed. It is obvious that for a graph G of order n, $1 \leq \dim(G) \leq n - 1$. Chartrand et al. [6] characterized all graphs of order n and metric dimension $n - 1$.

Theorem 1.2. *[6] Let G be a graph of order n. Then* $\dim(G) = n - 1$ *if and only if $G = K_n$.*

They also characterized all graphs of order n and metric dimension $n - 2$.

Theorem 1.3. *[6] Let G be a graph of order $n \geq 4$. Then* $\dim(G) = n - 2$ *if and only if $G = K_{s,t}$ $(s, t \geq 1), G = K_s \vee \overline{K}_t$ $(s \geq 1, t \geq 2)$, or $G = K_s \vee (K_t \cup K_1)$ $(s, t \geq 1)$.*

The following definition is needed to state some results in the next section.

Definition 1.4. An *ear* of a graph G is a maximal path whose internal vertices are of degree 2 in G. An *ear decomposition* of G is a decomposition $G_0, G_1, \ldots, G_k$ such that G_0 is a cycle and for each i, $1 \leq i \leq k$, G_i is an ear of $G_0 \cup G_1 \cup \ldots \cup G_i$.

Whitney [15] proved the following important characterization for 2-connected graphs.

Theorem 1.5. [15] A graph is 2-connected if and only if it has an ear decomposition. Moreover, every cycle in a 2-connected graph is the initial cycle in some ear decomposition.

2. Main results

The aim of this section is to find an upper bound for the metric dimension in terms of the order and the girth of a graph and to characterize all graphs which attain this bound. This bound is presented in the next theorem.

Theorem 2.1. *Let G be a graph of order n. If G has a cycle, then* $\dim(G) \leq n - g(G) + 2$.

Proof. Let $g(G) = g$ and $C_g = (v_1, v_2, \ldots, v_g, v_1)$ be a shortest cycle in G. Since $\{v_1, v_2\}$ is a metric basis of C_g, $V(G) \setminus \{v_3, \ldots, v_g\}$ is a resolving set for G of size $n - g(G) + 2$. Therefore, $\dim(G) \leq n - g(G) + 2$. □

There are several families that the bound $n - g(G) + 2$ is much better than the bound in Theorem 1.1 for their metric dimension. The trivial example is a cycle. Also, Moore graphs have diameter d and girth $2d + 1$, hence

$$n - g(G) + 2 = n - 2d + 1 \leq n - d.$$

Note that, $\dim(K_n) = n - 1 = n - g(K_n) + 2$, $\dim(C_n) = 2 = n - g(C_n) + 2$, and for $r, s \geq 2$, $\dim(K_{r,s}) = r + s - 2 = n - g(K_{r,s}) + 2$. Therefore, the bound in Theorem 2.1 is tight for these graphs. In the remainder of this section, it is proved that these are all graphs that this bound is tight for them. First some required results are presented.

Proposition 2.2. *Let v be a cut vertex in a graph G. Then each resolving set for G is disjoint from at most one component of $G \setminus \{v\}$. Moreover, if W is a resolving set for G which is not disjoint from at least two components of $G \setminus \{v\}$, then $W \setminus \{v\}$ is a resolving set for G.*

Proof. Let H and K be two components of $G \setminus \{v\}$ and W be a resolving set for G. If $W \cap V(H) = W \cap V(K) = \emptyset$, then let $x \in V(H)$ and $y \in V(K)$ such that $x \sim v$ and $y \sim v$. Therefore, for each $w \in W$,

$$d(x, w) = d(x, v) + d(v, w) = 1 + d(v, w) = d(y, v) + d(v, w) = d(y, w),$$

which contradicts the assumption that W is a resolving set for G. Thus W is disjoint from at most one component of $G \setminus \{v\}$.

To prove the secound part, let $h \in W \cap V(H)$ and $k \in W \cap V(K)$. If $W \setminus \{v\}$ is not a resolving set for G, then there exist vertices $a, b \in V(G)$ such that $d(a, v) \neq d(b, v)$ and for each $w \in W \setminus \{v\}$, $d(a, w) = d(b, w)$.
If $a, b \notin V(H)$, then

$$d(a, h) = d(a, v) + d(v, h) \neq d(b, v) + d(v, h) = d(b, h).$$

This gives $d(a, h) \neq d(b, h)$, which is a contradiction.
If $a \in V(H)$ and $b \notin V(H)$, then

$$d(b, v) + d(v, h) = d(b, h) = d(a, h) \leq d(a, v) + d(v, h).$$

Hence $d(b, v) < d(a, v)$, because $d(a, v) \neq d(b, v)$. On the other hand,

$$d(a, v) + d(v, k) = d(a, k) = d(b, k) \leq d(a, v) + d(v, k).$$

Therefore $d(a, v) < d(b, v)$, which is impossible.
If $a, b \in V(H)$, then

$$d(a, k) = d(a, v) + d(v, k) \neq d(b, v) + d(v, k) = d(b, k),$$

¡t¿hat is $d(a, k) \neq d(b, k)$. These contradictions imply that $W \setminus \{v\}$ is a resolving set for G. □

Corollary 2.3. *Let u be a vertex of degree 1 in a graph G and v be the neighbour of u. If W is a resolving set for G, then $(W \cup \{u\}) \setminus \{v\}$ is also a resolving set for G.*

Proof. Let W be a resolving set for G. Clearly $W \cup \{u\}$ is also a resolving set for G. Note that v is a cut vertex of G and $\langle \{u\} \rangle$ is a component of $G \setminus \{v\}$. If $W \cap (V(G) \setminus \{u, v\}) \neq \emptyset$, then by Proposition 2.2, $(W \cup \{u\}) \setminus \{v\}$ is also a resolving set for G. If $W \subseteq \{u, v\}$, then by Proposition 2.2, $G \setminus \{v\}$ has exactly two components. On the other hand, for each $x \in V(G) \setminus \{u\}$, $r(x|\{u, v\}) = (a, a-1)$, for some integer $a \geq 1$. Since $\{u, v\}$ is a resolving set for G, the first coordinates of the metric representation of all vertices in $V(G) \setminus \{u\}$ are different from each other. Therefore, $\{u\} = (W \cup \{u\}) \setminus \{v\}$ is also a resolving set for G. □

The following proposition states that all graphs G of order n with metric dimension $n - g(G) + 2$ are 2-connected.

Proposition 2.4. *Let G be a graph of order n which has a cycle. If $\dim(G) = n - g(G) + 2$, then G is 2-connected.*

Proof. Suppose on the contrary, that v is a cut vertex of G. Let $C_g = (v_1, v_2, \ldots, v_g, v_1)$ be the shortest cycle in G. Since v is a cut vertex, there exists a component H of $G \setminus \{v\}$, such that $V(C_g) \subseteq V(H) \cup \{v\}$. Since every pair of adjacent vertices is a basis in a cycle, $W = V(G) \setminus \{v_3, \ldots, v_g\}$ is ¡a¿ resolving set for G of size $\dim(G)$. Note that, W intersects at least two components of $G \setminus \{v\}$. Therefore, by Proposition 2.2, $W \setminus \{v\}$ is a resolving set for G

and hence $v \notin W$; otherwise $|W \setminus \{v\}| = \dim(G) - 1$. Thus, $v \in \{v_3, \ldots, v_g\}$, say $v = v_i$, $3 \leq i \leq g$. But $B = V(G) \setminus \{v_1, \ldots, v_{i-2}, v_{i+1}, \ldots, v_g\}$ is a basis of G which contains $v = v_i$ and intersects at least two components of $G \setminus \{v\}$. Thus, by Proposition 2.2, $B \setminus \{v\}$ is a resolving set for G of size smaller than $\dim(G)$. This contradiction implies that G is a 2-connected graph. $\square$

Theorem 2.5. *Let G be a graph of order n which has a cycle. Then* $\dim(G) = n - g(G) + 2$ *if and only if G is a cycle C_n, complete graph K_n, $n \geq 3$, or complete bipartite graph, $K_{r,s}$, $r, s \geq 2$.*

Proof. It is easy to see that if G is a cycle C_n, complete graph K_n, $n \geq 3$, or complete bipartite graph, $K_{r,s}$, $r, s \geq 2$, then $\dim(G) = n - g(G) + 2$. Now let $\dim(G) = n - g(G) + 2$ and $C_g = (v_1, v_2, \ldots, v_g, v_1)$ be the shortest cycle in G. By Proposition 2.4, G is a 2-connected graph. Therefore, by Theorem 1.5, G has an ear decomposition with initial cycle C_g. Assume that $C_g, G_1, G_2, \ldots, G_k$ be an ear decomposition of G with initial cycle C_g. If $G = C_g$, then G is a cycle. If $G \neq C_g$ and $G_1 = (x_0, x_1, \ldots, x_t)$, then $x_0, x_t \in V(C_g)$. We claim that in this case $g \leq 4$.

Without loss of generality one can assume that $x_0 = v_i$ and $x_t = v_j$, where $1 \leq i < j \leq g$. Since C_g is a shortest cycle in G, $j - i \leq t$ and $g + i - j \leq t$. If $t \leq 2$, then $j - i \leq 2$ and $g + i - j \leq 2$. Thus $g \leq 4$.

If $t \geq 3$, then the set

$$W = V(G) \setminus \{x_2, v_1, v_2, \ldots, v_{i-1}, v_{i+2}, \ldots, v_g\}$$

is not a resolving set for G, because $|W| = \dim(G) - 1$. Therefore, there exist vertices $a, b \in V(G) \setminus W$ such that $r(a|W) = r(b|W)$. Since $\{v_i, v_{i+1}\}$ is a basis for C_g and the distances in C_g and G are the same, $W \cap V(C_g)$ resolves C_g. Consequently, $x_2 \in \{a, b\}$, say $x_2 = a$, and $b \in V(C_g) \setminus \{v_i, v_{i+1}\}$. Hence, $d(b, x_1) = d(a, x_1) = 1$, because $x_1 \in W$. Note that x_1 is adjacent to vertices x_0, x_2 and b. Since $b \notin \{x_0, x_2\}$, x_1 is of degree at least 3 in $C_g \cup G_1$. Thus x_1 is adjacent to at least two vertices of C_g, say v_i and v_r. Therefore $d(v_i, v_r) \leq 2$. Since C_g is a shortest cycle and x_1 is not on C_g, $g \leq 4$.

If $g = 3$, then $\dim(G) = n - 3 + 2 = n - 1$ and by Theorem 1.2, $G = K_n$. If $g = 4$, then $\dim(G) = n - 4 + 2 = n - 2$ and by Theorem 1.3, G is $K_{r,s}$, $K_r \vee \overline{K}_s$, or $K_r \vee (K_s \cup K_1)$. But $K_{r,s}$, $r, s \geq 2$ is the only graph among these graphs whose girth is 4. $\square$

Acknowledgments

The author would like to cordially thank the anonymous referees for their valuable and helpful comments.

References

[1] B. Bagheri, M. Jannesari and B. Omoomi, Relations between metric dimension and domination number of graphs, *arXiv:1112.2326 [math.CO]*.

[2] Z. Beerliova, F. Eberhard, T. Erlebach, A. Hall, M. Hoffmann, M. Mihalak and L. S. Ram, Network dicovery and verification, *IEEE Journal On Selected Areas in Communications* **24** (2006), no. 12, 2168–2181.

[3] P. Buczkowski, G. Chartrand, C. Poisson and P. Zhang, On k-dimensional graphs and their bases, *Period. Math. Hungar.* **46** (2003), no. 1, 9–15.

[4] J. Caceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara and D. R. Wood, On the metric dimension of Cartesian products of graphs, *SIAM J. Discrete Math.* **21** (2007), no. 2, 423–441.

[5] G. G. Chappell, J. Gimbel and C. Hartman, Bounds on the metric and partition dimensions of a graph, *Ars Combin.* **88** (2008) 349–366.

[6] G. Chartrand, L. Eroh, M. A. Johnson and O. R. Ollermann, Resolvability in graphs and the metric dimension of a graph, *Discrete Appl. Math.* **105** (2000), no. 1-3, 99–113.

[7] G. Chartrand and P. Zhang, The theory and applications of resolvability in graphs, A survey, In Proc. 34th Southeastern International Conf. on Combinatorics, Graph Theory and Computing, *Congr. Numer.* **160** (2003) 47–68.

[8] F. Harary and R. A. Melter, On the metric dimension of a graph, *Ars Combin.* **2** (1976) 191–195.

[9] C. Hernando, M. Mora, I. M. Pelayo, C. Seara and D. R. Wood, Extremal graph theory for metric dimension and diameter, *Electron. J. Combin.* **17** (2010) no. 1, 28 pages.

[10] M. Jannesari and B. Omoomi, On randomly k-dimensional graphs, *Appl. Math. Lett.* **24** (2011), no. 10, 1625–1629.

[11] S. Khuller, B. Raghavachari and A. Rosenfeld, Landmarks in graphs, *Discrete Appl. Math.* **70**, no. 3, (1996) 217–229.

[12] R. A. Melter and I. Tomescu, Metric bases in digital geometry, *Comput. Gr. Image Process.* **25** (1984) 113–121.

[13] A. Sebo and E. Tannier, On metric generators of graphs, *Math. Oper. Res.* **29** (2004), no. 2, 383–393.

[14] P. J. Slater, Leaves of trees, Proceedings of the Sixth Southeastern Conference on Combinatorics, Graph Theory, and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1975), 549–559, Congressus Numerantium, No. 14, Utilitas Math., Winnipeg, 1975.

[15] H. Whitney, Congruent graphs and the connectivity of graphs, *Amer. J. Math.* **54** (1932), no. 1, 150–168.

(M. Jannesari) Shahreza High Education Center, 86149-56841, Shahreza, Iran
E-mail address: m.jannesari@math.iut.ac.ir