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A REMARK ON ASYMPTOTIC ENUMERATION OF HIGHEST WEIGHTS IN TENSOR POWERS OF A REPRESENTATION

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Abstract. We consider the semigroup S of highest weights appearing in tensor powers $V^{\otimes k}$ of a finite dimensional representation V of a connected reductive group. We describe the cone generated by S as the cone over the weight polytope of V intersected with the positive Weyl chamber. From this we get a description for the asymptotic of the number of highest weights appearing in $V^{\otimes k}$ in terms of the volume of this polytope.
Keywords: Reductive group representation, tensor power, semigroup of integral points, weight polytope, moment polytope.
MSC(2010): Primary: 05E10; Secondary: 20G05.

1. Introduction

Let G be a connected reductive algebraic group over an algebraically closed field $\mathbf{k}$ of characteristic 0, and let V be a finite dimensional G-module. We consider the semigroup of dominant weights

$$S = S(V) = \{(k, \lambda) \mid V_\lambda \text{ appears in } V^{\otimes k}\},$$

where V_λ is the irreducible representation with highest weight λ. In this note we describe the cone $C(S)$ of this semigroup, i.e., the smallest closed convex cone (with apex at the origin) containing S (in other words, the closure of the convex hull of $S \cup \{0\}$). We use this to describe the asymptotic of the number of highest weights λ appearing in $V^{\otimes k}$.

This work is in the spirit of the general theory of semigroups of integral points and Newton-Okounkov bodies developed in [6] and [4].

Let A denote the finite set of highest weights in V, i.e., the dominant weights λ where V_λ appears in V. Consider the union of all the Weyl group orbits of $\lambda \in A$ and let $P^+(V)$ be its convex hull intersected with the positive Weyl

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chamber. We show that the slice of the cone $C(S)$ at $k = 1$ coincides with the polytope $P^+(V)$ (Theorem 3.2). The main tool in the proof will be the PRV theorem on the tensor product of irreducible representations. Beside the PRV theorem The rest of arguments are elementary in nature.

Let $H_V(k)$ denote the number of dominant weights λ where V_λ appears in $V^{\otimes k}$. From general statements about semigroups of integral points we then conclude that $H_V(k)$ grows of degree $q = \dim(P^+(V))$, i.e., the limit

$$a_q = \lim_{k\to\infty} H_V(k)/k^q$$

exists and is non-zero. In addition a_q is equal to the (properly normalized) volume of the polytope $P^+(V)$.

In the last section we discuss the connection between the semigroup $S(V)$, its associated polytope $P^+(V)$ and the moment polytope of G-varieties.

At the end, we would like to mention the related paper of Tate and Zeldtich [11] in which the authors address the (more difficult and independent) question of describing the asymptotic behavior of multiplicities of irreducible representations appearing in tensor powers of an irreducible representation. This note is also related to [5] (we should point out that the first version of this note appeared in arXiv before [5]).

To make this note accessible to a wider range of audience we have tried to cover most of the background material.

Notation: Throughout the paper we will use the following notation. G denotes a connected reductive algebraic group over an algebraically closed field $\mathbf{k}$ of characteristic 0.

- We fix a Borel subgroup B and a maximal torus T in G. The Weyl group of (G,T) is denoted by W. It contains a unique longest element denoted by w_0.
- Λ denotes the weight lattice of G (that is, the character group of T), and Λ^+ is the subset of dominant weights (for the choice of B). Put $\Lambda_\mathbb{R} = \Lambda \otimes_\mathbb{Z} \mathbb{R}$. Then the convex cone generated by Λ^+ in $\Lambda_\mathbb{R}$ is the positive Weyl chamber $\Lambda^+_\mathbb{R}$.
- For a weight $\lambda \in \Lambda$, the irreducible G-module corresponding to λ will be denoted by V_λ and a highest weight vector in V_λ will be denoted by v_λ. Finally for a dominant weight λ, we put $\lambda^* = -w_0(\lambda)$ which is again a dominant weight. One has $V_{\lambda^*} \cong V^*_\lambda$ as G-modules.

2. Semigroups of integral points and convex bodies

Let $S \subset \mathbb{N} \times \mathbb{Z}^n$ be a semigroup of integral points (i.e., S is closed under addition).

Let $C(S)$ be the smallest closed convex cone (with apex at the origin) containing S. Also let $G(S)$ be the subgroup of $\mathbb{Z}^{n+1}$ generated by S and, $L(S)$ the linear subspace of $\mathbb{R}^{n+1}$ spanned by S. The sets $C(S)$ and $G(S)$ lie in $L(S)$. To S we associate its *regularization* which is the semigroup $\mathrm{Reg}(S) = C(S) \cap G(S)$. The regularization $\mathrm{Reg}(S)$ is a simpler semigroup with more points which contains the semigroup S. In [6, Section 1.1] it is proved that *the regularization* $\mathrm{Reg}(S)$ *asymptotically approximates the semigroup* S. More precisely:

Theorem 2.1 (Approximation Theorem)**.** *Let $C' \subset C(S)$ be a strongly convex cone which intersects the boundary (in the topology of the linear space $L(S)$) of the cone $C(S)$ only at the origin. Then there exists a constant $N > 0$ (depending on C') such that each point in $G(S) \cap C'$ whose distance from the origin is bigger than N belongs to S.*

Let $\pi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ denote the projection on the first factor. We call a semigroup $S \subset \mathbb{N} \times \mathbb{Z}^n$ a *non-negative semigroup* if it is not contained in the hyperplane $\pi^{-1}(0)$. If in addition the cone $C(S)$ intersects the hyperplane $\pi^{-1}(0)$ only at the origin, S is called a *strongly non-negative semigroup.*

Let $S_k = S \cap \pi^{-1}(k)$ be the set of points in S at level k. For simplicity throughout this section we assume that $S_1 \neq \emptyset$.

We denote the group $G(S) \cap \pi^{-1}(0)$ by $\Lambda(S)$ and call it the *lattice associated to the non-negative semigroup* S. Finally, the number of points in S_k is denoted by $H_S(k)$. H_S is called the *Hilbert function of the semigroup* S.

Definition 2.2 (Newton-Okounkov convex set)**.** We call the projection of the convex set $C(S) \cap \pi^{-1}(1)$ on $\mathbb{R}^n$ (under the projection on the second factor $(1, x) \mapsto x$), the *Newton-Okounkov convex set of the semigroup* S and denote it by $\Delta(S)$. In other words,

$$\Delta(S) = \overline{\mathrm{conv}(\bigcup_{k>0} \{x/k \mid (k, x) \in S_k\})}.$$

If S is strongly non-negative then $\Delta(S)$ is compact and hence a convex body.

Let $\Lambda \subset \mathbb{R}^n$ be a lattice of full rank n. Let $E \subset \mathbb{R}^n$ be a subspace of dimension q which is rational with respect to Λ. The *Lebesgue measure normalized with respect to the lattice* Λ in E is the Lebesgue measure $d\gamma$ in E normalized such that the smallest measure of a q-dimensional parallelepiped with vertices in $E \cap \Lambda$ is equal to 1. The measure of a subset $A \subset E$ will be called its *normalized volume* and denoted by $\mathrm{Vol}_q(A)$ (whenever the lattice Λ is clear from the context).

Let H_S and $H_{\mathrm{Reg}(S)}$ be the Hilbert functions of S and its regularization respectively. From Theorem 2.1 it follows that $H_S(k)$ and $H_{\mathrm{Reg}(S)}(k)$ have the same asymptotic as k goes to infinity. Thus the Newton-Okounkov convex set $\Delta(S)$ is responsible for the asymptotic behavior of the Hilbert function of S ([6, Section 1.4]):

Theorem 2.3. *The function $H_S(k)$ grows like $a_q k^q$ where q is the dimension of the convex body $\Delta(S)$. This means that the limit*

$$a_q = \lim_{k\to\infty} H_S(k)/k^q$$

exists and is non-zero. Moreover, the q-th growth coefficient a_q is equal to $\mathrm{Vol}_q(\Delta(S))$, where the volume is normalized with respect to the lattice $\Lambda(S)$.

Finally we make an observation which will be used later in proof of the main result (Theorem 3.2).

Proposition 2.4. *Let $S \subset \mathbb{N}\times\mathbb{Z}^n$ be a non-negative semigroup and $C = C(S)$ the cone associated to S. Let $C' \subset C$ be a convex cone of full dimension (centered at the origin) and $S' = S \cap C'$ the subsemigroup consisting of all the points of S contained in C'. Then the cone $C(S')$ associated to S' coincides with C'.*

Proof. Clearly $C(S') \subset C'$. By contradiction suppose $C(S')$ is not equal to C'. Then there is a convex cone $\tilde{C} \subset C'$ of full dimension which intersects $C(S')$ and the boundary of C' (in the topology of the subspace $L(S)$) only at the origin. Since $\tilde{C}$ has full dimension it contains a rational point (with respect to the lattice $\Lambda(S)$) which then implis that it contains a point in $\Lambda(S)$. Now applying Theorem 2.1 we see that the convex cone $\tilde{C}$ should contain a point in S' which contradicts that $C(S') \cap \tilde{C} = \emptyset$. □

In the rest of the paper we will deal with semigroups and convex polytopes naturally associated to a reductive group G and its representations.

Remark 2.5. The proof of Theorem 2.1 relies on the proof of the special case when S is a finitely generated semigroup. The semigroups appearing in this note turn out to be in fact finitely generated, although we will not use this fact.

3. Main result

Let V be a finite dimensional G-module. Define the set $S(V) \subset \mathbb{N}\times\Lambda^+$ by

$$S(V) = \{(k,\lambda) \mid V_\lambda \text{ appears in } V^{\otimes k}\}.$$

If v_λ and v_μ are highest weight vectors in $V^{\otimes k}$ and $V^{\otimes \ell}$ with weights λ and μ respectively, then $v_\lambda \otimes v_\mu$ is a highest weight vector in $V^{\otimes k+\ell}$ of weight $\lambda+\mu$. It follows that $S(V)$ is a semigroup with respect to addition. Let $\Delta(V)$ denote the Newton-Okounkov body of the semigroup $S(V)$. In other words,

$$\Delta(V) = \overline{\mathrm{conv}(\bigcup_{k>0}\{\lambda/k \mid V_\lambda \text{ appears in } V^{\otimes k}\})}.$$

Also let A be the collection of γ's where V_γ appears in V.

Definition 3.1. The *weight polytope of* V is defined by $P(V) = \text{conv}\{w(\gamma) | w \in W, \gamma \in A\}$, i.e., the convex hull of the union of Weyl orbits of $\gamma \in A$. We will denote the intersection of $P(V)$ with the positive Weyl chamber $\Lambda^+_{\mathbb{R}}$ by $P^+(V)$ and call it the *moment polytope of* V.

Theorem 3.2. *$\Delta(V)$ coincides with the moment polytope $P^+(V)$.*

We will need the following well-known fact:

Lemma 3.3. *Let λ_1, λ_2 be dominant weights and let V_γ appear in $V_{\lambda_1} \otimes V_{\lambda_2}$. Then $\gamma = \lambda_1 + \lambda_2 - \sum_{\alpha \in R^+} c_\alpha \alpha$ where $c_\alpha \geq 0$ (R^+ denotes the set of positive roots). From this it follows that γ belongs to the convex hull of the Weyl orbit of $\lambda_1 + \lambda_2$.*

Our tool to prove Theorem 3.2 is the well-known PRV-Kumar theorem regarding the tensor product of two irreducible representations. It was conjectured by Parthasarathy, Ranga Rao and Varadarajan in [10]. Later it was proved by Kumar in [9]. We briefly recall its statement.

Theorem 3.4 (PRV-Kumar theorem)**.** *Let $\lambda_1, \lambda_2 \in \Lambda^+$ be two dominant weights. Suppose for two Weyl group elements $w_1, w_2 \in W$ we have $\gamma = w_1(\lambda_1) + w_2(\lambda_2)$ is a dominant weight. Then V_γ appears in the decomposition of the tensor product $V_{\lambda_1} \otimes V_{\lambda_2}$ into irreducible representations.*

Define the set $\tilde{S}(V) \subset \mathbb{N} \times \Lambda$ by:

$$\tilde{S}(V) = \{(k, w(\lambda)) \mid w \in W,\ V_\lambda \text{ appears in } V^{\otimes k}\}.$$

Roughly, speaking $\tilde{S}(V)$ is the union of W-otbits of λ for which V_λ appears in some tensor power $V^{\otimes k}$. Notice that $S(V) = \tilde{S}(V) \cap (\mathbb{N} \times \Lambda^+)$. The following is a straight forward corollary of Theorem 3.4

Corollary 3.5. *1) $\tilde{S}(V)$ is a semigroup. 2) The convex body $\Delta(\tilde{S}(V))$ associated to this semigroup coincides with $P(V)$.*

Proof. 1) Let $(k, w_1(\lambda_1))$, $(\ell, w_2(\lambda_2))$ be two elements in $\tilde{S}(V)$. We can write $w_1(\lambda_1) + w_2(\lambda_2)$ as $w(\lambda)$ for some $\lambda \in \Lambda^+$, $w \in W$. By Theorem 3.4, V_λ appears in $V_{\lambda_1} \otimes V_{\lambda_2}$ and hence it appears in $V^{\otimes \ell + k}$. This shows that $(\ell + k, w_1(\lambda_1) + w_2(\lambda_2)) = (\ell + k, w(\lambda))$ belongs to $\tilde{S}(V)$ which proves 1). 2) Since $\tilde{S}(V)$ is a semigroup and $P(V)$ is by definition the convex hull of $\tilde{S}(V)_1$, it follows that $kP(V)$ is contained in the convex hull of $\tilde{S}(V)_k$. On the other hand, by Lemma 3.3, for any integer $k > 0$, the convex hull of $\tilde{S}(V)_k$ is contained in $kP(V)$ and hence $\Delta(\tilde{S}(V))$ coincides with $P(V)$. □

Proof of Theorem 3.2. From Proposition 2.4, the convex body $\Delta(V)$ associated to the semigroup $S(V) \subset \tilde{S}(V)$ is just the intersection of $\Delta(\tilde{S}(V))$ with $\Lambda^+_{\mathbb{R}}$. By Corollary 3.5(2) we know that $\Delta(\tilde{S}(V))$ is the weight polytope $P(V)$ which finishes the proof. □

Corollary 3.6. *Let $H_V(k)$ be the number of λ such that V_λ appears in $V^{\otimes k}$. Then $H_V(k)$ grows of degree $q = \dim P^+(V)$. That is, the limit*

$$a_q = \lim_{k\to\infty} H_V(k)/k^q$$

exists and is non-zero. Moreover, a_q is equal to $\mathrm{Vol}_q(P^+(V))$, where volume is the Lebesgue measure in $\Lambda_{\mathbb{R}}$ normalized with respect to the lattice $\Lambda(S(V)) \subset \Lambda$.

Proof. Follows directly from Theorem 3.2 and Theorem 2.3. □

Example 3.7. Perhaps the simplest case of Theorem 3.2 and Corollary 3.6 is $G = \mathrm{SL}(2,\mathbb{C})$. One knows that the irreducible representations of $\mathrm{SL}(2,\mathbb{C})$ are enumerated by $n \in \mathbb{Z}_{\geq 0}$ as $V_n = \mathrm{Sym}^n(\mathbb{C}^2)$, where $\mathrm{SL}(2,\mathbb{C})$ acts on $\mathbb{C}^2$ in the usual way. It is well-known that for $n, m \geq 0$:

$$V_n \otimes V_m = V_{|n-m|} \oplus V_{|n-m+2|} \oplus \cdots \oplus V_{n+m}. \tag{1}$$

Let $V = m_1 V_{n_1} \oplus \cdots \oplus m_r V_{n_r}$ be the decomposition of a finite dimensional representation of $\mathrm{SL}(2,\mathbb{C})$ into irreducible representations where $0 \leq n_1 < \cdots < n_r$. From (1) we see that $\Delta(V) = [0, n_r]$. Corollary 3.6 then states that the number of irreducible representations appearing in $V^{\otimes k}$ is asymptotically equal to kn_r. For $G = \mathrm{SL}(n,\mathbb{C})$ the decomposition of tensor product of two irreducible representations $V_\lambda \otimes V_\gamma$ into irreducible representations is more complicated. Exactly what irreducible representations appear in $V_\lambda \otimes V_\gamma$ is related to the so-called Horn's conjecture/theorem (see for example [2, Section 3]).

4. Relation with moment polytope of group actions

In this section we see how the moment polytope $P^+(V)$ appears as a moment polytope for the action of $G \times G$ on G.

Let V be a finite dimensional G-module, and $X \subset \mathbb{P}(V)$ an irreducible closed G-invariant subvariety. Let $R = \bigoplus_{k\geq 0} R_k$ denote the homogeneous coordinate ring of X. It is a graded G-algebra. Following Brion, one defines the *moment polytope* $\Delta(X)$ to be:

$$\Delta(X) = \overline{\mathrm{conv}(\bigcup_{k>0} \{\lambda/k \mid V_\lambda \text{ appears in } R_k\})}.$$

One shows that $\Delta(X) \subset \Lambda^+_{\mathbb{R}}$ is a polytope (see [1]). Moreover, when $\mathbf{k} = \mathbb{C}$ and X is smooth, the polytope $\Delta(X)$ can be identified with the moment polytope of X regarded as a Hamiltonian space for the action of a maximal compact subgroup K of G and the symplectic structure induced from the projective space (see for example [3]).

Let $G\times G$ act on G via multiplication from left and right. Let $\mathbf{k}[G]$ denote the algebra of regular functions on the variety G. It is a rational $(G\times G)$-module. It is well-known that for each dominant weight λ, the (λ^*, λ)-isotypic component $\mathbf{k}[G]_{(\lambda^*,\lambda)}$ is isomorphic to $V_{\lambda^*} \otimes V_\lambda$. Moreover, any isotypic component of $\mathbf{k}[G]$

is of this form for some λ. In fact any $(G\times G)$-isotypic component $\mathbf{k}[G]_{(\lambda^*,\lambda)}$ in $\mathbf{k}[G]$ is the linear span of the matrix entries corresponding to the representation of G in V_λ (see [8]). Now if $\lambda_1,\lambda_2\in\Lambda^+$ are two dominant weights, the product $\mathbf{k}[G]_{(\lambda_1^*,\lambda_1)}\mathbf{k}[G]_{(\lambda_2^*,\lambda_2)}$ is the linear span of the matrix entries corresponding to $V_{\lambda_1}\otimes V_{\lambda_2}$. This shows that we have the following decomposition for the product of isotypic components in the algebra $\mathbf{k}[G]$:

$$\mathbf{k}[G]_{(\lambda_1^*,\lambda_1)}\mathbf{k}[G]_{(\lambda_2^*,\lambda_2)} = \bigoplus_{\gamma\in\chi(\lambda_1,\lambda_2)} V_{\gamma^*}\otimes V_\gamma, \tag{2}$$

where $\chi(\lambda_1,\lambda_2)$ denotes the collection of all $\gamma\in\Lambda^+$ for which V_γ appears in $V_{\lambda_1}\otimes V_{\lambda_2}$.

Now let $\pi : G\to \mathrm{GL}(V)$ be a finite dimensional representation. Then $\mathrm{End}(V)$ is naturally a $(G\times G)$-module where $G\times G$ acts via π by multiplication from left and right. Let $\tilde{\pi}: G\to\mathbb{P}(\mathrm{End}(V))$ be the induced map to projective space and let X be the closure of the image of G in $\mathbb{P}(\mathrm{End}(V))$. It is a $(G\times G)$-invariant closed irreducible subvariety.

From (2) one can see the following:

Proposition 4.1. *Let $R=\bigoplus_k R_k$ denote the homogeneous coordinate ring of X in $\mathbb{P}(\mathrm{End}(V))$. Then for $k>0$ we have: $V_{(\lambda^*,\lambda)}$ appears in R_k if and only if V_λ appears in $V^{\otimes k}$. It follows that, under the projection on the second factor, $\Delta(X)\subset\Lambda^+_{\mathbb{R}}\times\Lambda^+_{\mathbb{R}}$ is identified by $P^+(V)$.*

Remark 4.2. The relation between the moment polytope of X (i.e., a group compactification) and the polytope $P^+(V)$ has also been shown in [7] using methods from symplectic geometry.

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