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A POSTERIORI $L^2(L^2)$-ERROR ESTIMATES WITH THE NEW VERSION OF STREAMLINE DIFFUSION METHOD FOR THE WAVE EQUATION

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Abstract. In this article, we study the new streamline diffusion finite element for treating the linear second order hyperbolic initial-boundary value problem. We prove a posteriori $L^2(L^2)$ and error estimates for this method under minimal regularity hypothesis. Test problem of an application of the wave equation in the laser is presented to verify the efficiency and accuracy of the method.

Keywords: Streamline diffusion method, finite element method, a posteriori error estimates.
MSC(2010): Primary: 65M60 Secondary: 65N1; 65N30; 76M10.

1. Introduction

Extreme a posteriori error estimates for approximate solutions of the wave equation is a large subject, despite its importance of these problems in the modeling of a number of physical phenomena. A posteriori error estimates may allow the acceleration of numerical methods via adaptive schemes powerful. There are various approaches to a posteriori error estimates and it has recently successfully applied to a variety of problems by several authors (see [2, 5, 14, 19, 21, 23]).
Georgoulis et al. used Galerkin discretization method for linear wave equation and obtained a posteriori error estimates in $L^\infty(L^2)$ norm in [14]. Johnson in [23] proved existence solution for second order hyperbolic problems and used discontinues Galerkin method for them and obtained a priori and a posteriori error estimates. In [6], we applied time discontinuous Galerkin (DG) scheme for a system of homogeneous coupled wave equations with a local damping and obtained a posteriori $L_2(L_2)-$ and $L_\infty(L_2)-$ error estimates. But in this

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paper, we use the *new streamline diffusion method* (NSDM) for solving a wave equation when this equation isn't homogeneous. Also, we obtain a posteriori error estimates for this equation in $L^2(L^2)$ norms.
Due to the fact that affected diffusion is added only in the characteristic direction so that internal layers are not spread, while the added diffusion removes amplifications near boundary layers.
On the other hand, we consider the linear second order hyperbolic initial boundary value problem(see [6, 12, 14, 16, 18, 23, 24, 26]) as follows:

$$\begin{cases} u_{tt} - \nabla.(a\nabla u) = f & \text{in} \quad \Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{for} \quad x \in \Omega, \\ u_t(x,0) = u_1(x) & \text{for} \quad x \in \Omega, \\ u(x,t) = 0 & \text{for} \quad (x,t) \in \partial\Omega \times (0,T]. \end{cases} \tag{1.1}$$

Here, $\Omega \subset \mathbb{R}^d$ is a bounded open polygonal domain with boundary $\partial\Omega$ and we have $u_0 \in H_0^1(\Omega)$,$u_1 \in L^2(\Omega)$, a is a scalar-value function in $C(\bar{\Omega})$ and $f \in L^2(0,T;L^2(\Omega))$.
For (1.1), we use one variable changing and apply a change for dependent variables to transform (1.1) into a new problem. We apply SD-method for new problem and obtain a posteriori error estimates. A posteriori error bound provides a computable upper bound on the error in some norm using the computed finite element solution (see [2, 5, 20]).
In order to make use of the theory of semigroups we write the system (1.1) in the following abstract form:

$$\begin{cases} \mathbf{w}_t + A\mathbf{w} = F & \text{in} \quad \Omega \times (0,T), \\ \mathbf{w}(x,0) = \mathbf{w}_0(x) & \text{for} \quad x \in \Omega, \\ \mathbf{w}(x,t) = 0 & \text{for} \quad (x,t) \in \partial\Omega \times (0,T]. \end{cases} \tag{1.2}$$

Here, we assume $v = u_t$ for $x \in \mathbb{R}^d$ and $t \in [0,T]$, also

$$\mathbf{w}(x,t) = (u(x,t), v(x,t))^T, \ \mathbf{w}_t(x,t) = (u_t(x,t), v_t(x,t))^T,$$

$$A = \begin{pmatrix} \mathbf{0} & -I \\ -\nabla.(a\nabla) & \mathbf{0} \end{pmatrix},$$

$$\mathbf{w}_0(x) = (u_0(x), u_1(x))^T \quad \text{and} \quad F(x,t) = (\mathbf{0}, f(x,t))^T,$$

where, I is the identity matrix. In [6], we solved (1.2) for $F = 0$ based on DG method.
But the rest of this work is organized as follows. In Sect. 2, we define slabs for space-time domain and obtain SD-method for (1.2). In Sect. 3, we prove a posteriori error estimates for SD-method of Sect. 2 and obtain dual problem. In the end of Sect. 3, we define interpolation estimates for dual problem. In Sect. 4, we complete proof for a posteriori error estimates by using definitions in Sect. 3. Computational results are given in Sect. 5.

2. The streamline diffusion method

In this section, we consider the new SD-method for solving (1.2) that is based on using finite element over the space-time domain $\Omega \times [0, T]$ and the Bézier elements.

2.1. Bézier elements.

The $p + 1$ Bernstein basis polynomial of degree k are defined for $t \in [0, T]$ as

$$\mathcal{B}_{i,k}(t) = \begin{pmatrix} k \\ i \end{pmatrix} t^i \frac{(T-t)^{k-i}}{T^m} \; , i = 0, \ldots, k. \tag{2.1}$$

These constitute a basis for the polynomials of degree k, moreover, they are pointwise non-negative.
The motivation for performing finite element computation using this basis comes from the fact that a piecewise Bernestein polynomial basis can be mapped onto a B-spline basis by invoking the Bézier extraction operator (see [9]).
This transformation enables the representation of a non-uniform rational basis spline (NURBS) or a T-spline by using a set of Bézier elements. Therefore, we consider a Bézier curve of degree p that it is defined by a linear combination of $k+1$ Bernstein polynomial basis functions. We define the set of basis functions as $B(x) = \{\mathcal{B}_{i,k}(x)\}_{i=1}^{k+1}$, and the corresponding set of vector-valued control points as $P = \{P_i\}_{i=1}^{p+1}$ where each $P_i \in \mathbb{R}^d$, d being the number of spatial dimensions, and P is a matrix of dimension $(k + 1)d$, viz. $P = \{P_i^j\}_{i,j=1}^{k+1,d}$. Hence, the Bézier curve can then be written as:

$$R_k(t) = \sum_{i=1}^{k+1} P_i \mathcal{B}_{i,k}(t), \qquad t \in [0, T]. \tag{2.2}$$

Also, we can repeat the Bézier curve for space as follows

$$R_k(x) = \sum_{i=1}^{k+1} P_i \prod_{j=1}^{d} \mathcal{B}_{j,k}(x_j), \tag{2.3}$$

$x = (x_1, \cdots, x_d)$. Therefore, we can represent the discrete solution u_k by a separation of variable viz:

$$u_h = \sum_{k=1}^{M} R_k(x) R_k(t), \tag{2.4}$$

where $M \sim \frac{1}{h}$.

2.2. **The new Streamline Diffusion.** To define this method, let $0 = t_0 < t_1 < \cdots < t_N = T$ be a subdivision of the time interval $[0, T]$ into intervals $I_n = (t_n, t_{n+1})$, with time steps $k_n = t_{n+1} - t_n$, $n = 0, 1, \cdots, N-1$ and introduce the corresponding space-time slabs, i.e.,

$$S_n = \{(x,t): \quad x \in \Omega, \quad t_n < t < t_{n+1}\}, \tag{2.5}$$

for $n = 0, 1, ..., N-1$. Further, for each n let $\mathbf{W}^n$ be a finite element subspace of $H^1(S_n) \times H^1(S_n)$, (see [1]) and let

$$\dot{\mathbf{W}}^n = \{\mathbf{w} \in \mathbf{W}^n |\ \mathbf{w}(0,t) = 0 \quad , \text{ for } \quad t \in I_n\}. \tag{2.6}$$

We can formulate SD-method on the slab S_n for (1.2), as follows:
for $n = 0, ..., N-1$, find $\mathbf{w}^n \in \dot{\mathbf{W}}^n$ such that

$$\begin{aligned} (\mathbf{w}_t^n + A\mathbf{w}^n, \mathbf{g} + \delta(\mathbf{g}_t + A\mathbf{g}))_n + \langle \mathbf{w}_+^n, \mathbf{g}_+ \rangle_n \\ = (F, \mathbf{g} + \delta(\mathbf{g}_t + A\mathbf{g}))_n + \langle \mathbf{w}_-^n, \mathbf{g}_+ \rangle_n, \end{aligned} \tag{2.7}$$

where $\delta = \bar{C}h$ with $0 < \bar{C} \le 1$ is a suitably chosen (sufficiently small, see [21]) positive constant and parameter h is defined in the below. Further, we define the following notations for (2.7):

$$(\mathbf{w}, \mathbf{g})_n = \int_{S_n} \mathbf{w}^T \cdot \mathbf{g} dx dt,$$

$$\langle \mathbf{w}, \mathbf{g} \rangle_n = \int_{\Omega} \mathbf{w}^T(x, t_n) \cdot \mathbf{g}(x, t_n) dx,$$

$$\mathbf{w}_+(x,t) = \lim_{s \to 0^+} \mathbf{w}(x, t+s),$$

$$\mathbf{w}_-(x,t) = \lim_{s \to 0^-} \mathbf{w}(x, t+s).$$

The terms including $\langle \quad , \quad \rangle$ in the above formula are jump conditions which imposes a weakly enforced continuity condition across the slab interfaces, at t_n and is the mechanism by which information is propagated from one slab to another. More concisely, after summing over n, we may rewrite (2.7) as follows: Summing over n, taking all the slabs together we get the function space $\dot{\mathbf{W}} = \prod_{n=0}^{N-1} \dot{\mathbf{W}}^n$. We find $\mathbf{w} \in \dot{\mathbf{W}}$, such that

$$B(\mathbf{w}, \mathbf{g}) = L(\mathbf{g}), \tag{2.8}$$

for $\mathbf{g} \in \dot{\mathbf{W}}$, and where the bilinear form $B(.,.)$ and the linear form $L(.)$ are defined by

$$\begin{aligned} B(\mathbf{w}, \mathbf{g}) = \sum_{n=0}^{N-1} (\mathbf{w}_t^n + A\mathbf{w}^n, g + \delta(\mathbf{g}_t + A\mathbf{g}))_n \\ + \sum_{n=1}^{N-1} \langle [\mathbf{w}^n], \mathbf{g}_+ \rangle_n + \langle \mathbf{w}_+^n, \mathbf{g}_+ \rangle_0, \end{aligned}$$

and,

$$L(g) = \sum_{n=0}^{N-1} (F, \mathbf{g} + \delta(\mathbf{g}_t + A\mathbf{g}))_n + \langle \mathbf{w}_0, \mathbf{g}_+ \rangle_0,$$

where,we define $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2)^T$ such that for $i = 1, 2$

$$[\mathbf{w}_i] = \mathbf{w}_{i,+} - \mathbf{w}_{i,-}, \; [\mathbf{w}] = ([\mathbf{w}_1], [\mathbf{w}_2])^T.$$

For $h > 0$, we define T_h^n to be a triangulation of the slab S_n into triangles K satisfying as usual the minimum angle condition (see [11]), and assume that the parameter h is represented with the maximum diameter of the triangles $K \in T_h^n$. We introduce

$$\mathbf{W}_h^n = \{\mathbf{w} \in [H^1(S_n)]^2 : \mathbf{w}|_K \in [R_k(K)]^2 \text{ for } K \in T_h^n, \mathbf{w}(0,t) = 0 \text{ for } t \in I_n\},$$

where $R_k(K)$ denotes the set of Bézier polynomials in K of degree less than or equal to k and we define the function space that summing over n, taking all the slabs together:

$$W_h = \prod_{n=0}^{N-1} W_h^n.$$

Thus (2.8) can be formulated as follows:
find $\mathbf{w}_h \in \mathbf{W}_h$ such that

$$B(\mathbf{w}_h, \mathbf{g}) = L(\mathbf{g}), \tag{2.9}$$

for $\mathbf{g} \in \mathbf{W}_h$. Moreover, we know that the exact solution of (2.8) satisfies

$$B(\mathbf{w}, \mathbf{g}) = L(\mathbf{g}),$$

for $\mathbf{g} \in \dot{\mathbf{W}}$, and by using (2.8) and (2.9), we have the Galerkin orthogonality relation

$$B(\mathbf{e}, \mathbf{g}) = 0, \tag{2.10}$$

where $\mathbf{e} = \mathbf{w} - \mathbf{w}_h$.

3. An a posteriori error estimate for the new SD-Method

In this section, we shall consider the new SD-method for (2.9):
Find $\mathbf{w}_h \in \mathbf{W}_h$, such that for $n = 0, 1, ..., N-1$:

$$\sum_{n=0}^{N-1} (\mathbf{w}_{h,t}^n + A\mathbf{w}_h^n, g + \delta(\mathbf{g}_t + A\mathbf{g}))_n + \sum_{n=1}^{N-1} \langle [\mathbf{w}_h^n], \mathbf{g}_+ \rangle_n + \langle \mathbf{w}_{h,+}^n, \mathbf{g}_+ \rangle_0 \tag{3.1}$$

$$= \sum_{n=0}^{N-1} (F, \mathbf{g})_n + \langle \mathbf{w}_0, \mathbf{g}_+ \rangle_0,$$

where $g \in \mathbf{W}_h$ and $\mathbf{w}_{h,-}^0 = 0$.

In this section, we shall introduce and analyze a posteriori estimates that are based on Bézier techniques proposed in (2.2), (2.3) and (2.4). The approach consists of viewing the computed solution $\mathbf{w}_h$ as coarse-mesh approximation to some function which is arguably a more accurate approximation to $\mathbf{w}$.

In order to obtain a representation of the error, we consider the following auxiliary problem, referred to as the linearized dual problem:
Find Φ such that

$$\begin{cases} L^*\Phi \equiv -\Phi_t + A^T\Phi = \Psi^{-1}\mathbf{e} & \text{in} \quad \Omega \times (0,T), \\ \Phi(0,t) = 0, & \text{for} \quad t \in (0,T], \\ \Phi(x,T) = \Phi_N, & \text{for} \quad x \in \Omega \end{cases} \tag{3.2}$$

and L^* denotes the adjoint of the operator L defined in (1.2) and Ψ is a positive weight function. Note that this problem is computed "backward", but there is a corresponding change in sign and we obtained the dual problem with boundary condition is different form (3.2) which is equal zero in [6]. Further, we shall first introduce the following notation:

$$\| \mathbf{w} \|_{L_2^{\psi}(\Omega)} = (\mathbf{w}, \Psi\mathbf{w})_{\Omega}^{1/2}. \tag{3.3}$$

Multiplying (3.2) by $\mathbf{e}$ and integrating by parts, and summing over n, we obtain the following error representation formula:

$$\begin{aligned} \| \mathbf{e} \|^2_{L_2^{\Psi^{-1}}(\Omega)} &= (\mathbf{e}, \Psi^{-1}\mathbf{e})_{\Omega} = (\mathbf{e}, L^*\Phi)_{\Omega} \\ &= \sum_{n=0}^{N-1} (\mathbf{e}, -\Phi_t + A^T\Phi)_n \\ &= \sum_{n=0}^{N-1} (\mathbf{e}, -\Phi_t)_n + \sum_{n=0}^{N-1} (\mathbf{e}, A^T\Phi)_n. \end{aligned} \tag{3.4}$$

We have for $n = 0, 1, ..., N-1$ by part integrating

$$\begin{aligned} \sum_{n=0}^{N-1} (\mathbf{e}, -\Phi_t)_n &= -\sum_{n=0}^{N-1} \int_{S_n} \mathbf{e}^T.\ \Phi_t dxdt \\ &= -\sum_{n=0}^{N-1} \int_{\Omega} \mathbf{e}^T(x,t_n).\ \Phi(x,t_n)dx + \int_{S_n} \mathbf{e}_t^T.\ \Phi dxdt \\ &= -\sum_{n=0}^{N-1} \int_{\Omega} \mathbf{e}^T(x,t_n).\ \Phi(x,t_n)dx + (\mathbf{e}_t, \Phi)_n. \end{aligned} \tag{3.5}$$

We define $\mathbf{e} = (e_1, e_2)^T$ and $\Phi = (\phi_1, \phi_2)^T$ and obtain for $n = 0, 1, ..., N-1$:

$$(\mathbf{e}, A^T\Phi)_n = \int_{S_n} \mathbf{e}^T.A^T\Phi dxdt \tag{3.6}$$

$$= \int_{S_n} \mathbf{e}^T . \begin{pmatrix} 0 & -\nabla.(a\nabla) \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} dxdt = \int_{S_n} (e_1, e_2). \begin{pmatrix} -\nabla.(a\nabla\phi_2) \\ -\phi_1 \end{pmatrix} dxdt$$

$$= \int_{S_n} (-e_1\nabla.(a\nabla\phi_2) - e_2\phi_1)dxdt = \int_{S_n} (a\nabla\phi_2\nabla e_1 - e_2\phi_1)dxdt$$

$$= \int_{S_n} (-\nabla.(a\nabla\phi_2)e_1 - e_2\phi_1)dxdt = \int_{S_n} (A\mathbf{e})^T.\Phi dxdt = (A\mathbf{e}, \Phi)_n.$$

Also, we have

$$J = \sum_{n=0}^{N-1} \int_\Omega \mathbf{e}^T(x, t_n).\ \Phi(x, t_n)dx$$

$$= (\langle \mathbf{e}_-, \Phi_- \rangle_1 - \langle \mathbf{e}_+, \Phi_+ \rangle_0) + (\langle \mathbf{e}_-, \Phi_- \rangle_2 - \langle \mathbf{e}_+, \Phi_+ \rangle_1)$$
$$+ \cdots + (\langle \mathbf{e}_-, \Phi_- \rangle_{N-1} - \langle \mathbf{e}_+, \Phi_+ \rangle_{N-2})$$
$$+ (\langle \mathbf{e}_-, \Phi_- \rangle_N - \langle \mathbf{e}_+, \Phi_+ \rangle_{N-1}).$$

To continue we write $\Phi_-^n = \Phi_-^n - \Phi_+^n + \Phi_+^n$, $n = 1, ..., N-1$, then we obtain

$$-J = -\langle \mathbf{e}_-, \Phi_- \rangle_N + \langle \mathbf{e}_+, \Phi_+ \rangle_0 + \sum_{n=0}^{N-1} \langle [\mathbf{e}], \Phi_+ \rangle_n + \sum_{n=0}^{N-1} \langle \mathbf{e}_-, [\Phi] \rangle_n.$$

According to (3.2), $\Phi(., t_N = T) = \Phi_N$ and since $\mathbf{e}_+^0 = \mathbf{w}_0$, we get

$$J = \sum_{n=0}^{N-1} \langle [\mathbf{w}_h], \Phi_+ \rangle_n + \langle \mathbf{e}_-, \Phi_N \rangle_N. \tag{3.7}$$

Then in (3.4), by using (3.5), (3.6) and (3.7), we have

$$\| \mathbf{e} \|^2_{L_2^{\Psi^{-1}}(\Omega)} = \sum_{n=0}^{N-1} (\mathbf{e}_t, \Phi) + \sum_{n=0}^{N-1} (A\mathbf{e}, \Phi)_n - \sum_{n=0}^{N-1} \langle [\mathbf{w}_h], \Phi_+ \rangle_n - \langle \mathbf{e}_-, \Phi_N \rangle_N$$

$$= \sum_{n=0}^{N-1} ((\mathbf{w} - \mathbf{w}_h)_t + A(\mathbf{w} - \mathbf{w}_h), \Phi)_n - \sum_{n=0}^{N-1} \langle [\mathbf{w}_h], \Phi_+ \rangle_n - \langle \mathbf{e}_-, \Phi_N \rangle_N$$

$$= \sum_{n=0}^{N-1} (F - \mathbf{w}_{h,t} - A\mathbf{w}_h, \Phi)_n - \sum_{n=0}^{N-1} \langle [\mathbf{w}_h], \Phi_+ \rangle_n - \langle \mathbf{e}_-, \Phi_N \rangle_N.$$

So that recalling (3.1) and using the Galerkin orthogonality (2.10), we obtain

$$\| \mathbf{e} \|^2_{L_2^{\Psi^{-1}}(\Omega)} = \sum_{n=0}^{N-1} (\mathbf{w}_{h,t} + A\mathbf{w}_h - F, \hat{\Phi} + \delta(\hat{\Phi}_t + A\hat{\Phi}) - \Phi)_n \tag{3.8}$$

$$+ \sum_{n=0}^{N-1} \langle [\mathbf{w}_h], (\hat{\Phi} - \Phi)_+ \rangle_n + \langle \mathbf{e}_-, \Phi_N \rangle_N \equiv I + II + III,$$

where $\hat{\Phi} \in \mathbf{W}_h$ is an interpolant of Φ. The idea is now to estimate $\hat{\Phi} + \delta(\hat{\Phi}_t + A\hat{\Phi}) - \Phi$ in terms of $\Psi^{-1}\mathbf{e}$ using a strong stability estimates for solution Φ

of the dual problem. In [6], we applied time discontinuous Galerkin method for our problem and obtained $\delta = 0$ but in this paper, we apply streamline diffusion method for (1.2) and obtain $\delta \neq 0$.

We shall now consider our interpolant $\hat{\Phi} \in \mathbf{W}_h$ in (3.8) to be the space-time L_2-projection of Φ, namely first, we define the L_2-projections:

$$P_n : L_2(\Omega) \to \mathbf{W}_h^n,$$

$$\pi_n : L_2(S_n) \to \Pi_{0,n} = \{\mathbf{w} \in L_2(S_n) : \mathbf{w}(x,.) \text{ that are constant on } I_n, x \in \Omega\},$$

in space and in space time, respectively, by

$$\int_\Omega (P_n\Phi)^T.\mathbf{w}dx = \int_\Omega \Phi^T.\mathbf{w}dx, \quad \forall \mathbf{w} \in \mathbf{W}_h^n,$$

$$\pi_n \mathbf{w} \mid_{S_n} = \frac{1}{k_n}\int_{I_n} \mathbf{w}(.,t)dt, \ \forall \mathbf{w} \in \Pi_{0,n}.$$

Then, we can define $\hat{\Phi} \mid_{S_n} \in \mathbf{W}_h^n$ by letting

$$\hat{\Phi} \mid_{S_n} = P_n\pi_n\Phi = \pi_n P_n \Phi \in \mathbf{W}_h^n,$$

where $\Phi = \Phi \mid_{S_n}$. Further, if we introduce P and π by

$$(P\Phi) \mid_{S_n} = P_n(\Phi \mid_{S_n}),$$

and

$$(\pi\Phi) \mid_{S_n} = \pi_n(\Phi \mid_{S_n}),$$

respectively, then we can let $\hat{\Phi} \in \mathbf{W}_h$ be

$$\hat{\Phi} = P\pi\Phi = \pi P \Phi \in \mathbf{W}_h.$$

Now, we define the residual of computed solution $\mathbf{w}_h$ by

$$R_0 = \mathbf{w}_{h,t} + A\mathbf{w}_h - F,$$

$$R_1 = \mathbf{w}_{h,+}^n - \mathbf{w}_{h,-}^n, \quad \text{on } S_n,$$

$$R_2 = \frac{(P_n - I)\mathbf{w}_{h,-}^n}{k_n}, \quad \text{on } S_n,$$

where I is the identity operator.
In the end of this section, we shall give a lemma for interpolation estimates by the projection operators P, leaving the overall of I and II to next section.

Lemma 3.1. *There is a constant C such that for residual $R \in L_2(\Omega)$,*

$$| \, (R, \Phi - P\Phi)_\Omega \, | = \mathcal{O}(h^2). \tag{3.9}$$

Proof. We claim that $\mathcal{O}(h^2) = C \parallel h^2(I-P)R \parallel_{L_2^{\Psi^{-1}}(\Omega)} \parallel \Phi_{xx} \parallel_{L_2^{\Psi}(\Omega)}$. For proof see [5, 6, 20]. □

4. **The completion of the proof of a posteriori error estimates**

This section is devoted to residual type a posteriori estimates. The estimators as well as the exposition follow the lines found. In this section, we state and prove a posteriori error estimates by estimating of the terms I and II in the error representation formula (3.8). To this approach we use the stability factors (see [3–7]) associated with discretization in time and space, defined by

$$\gamma_{\mathbf{e}}^{t} = \frac{\| \Phi_t \|_{L_2^{\Psi}(\Omega)}}{\| \mathbf{e} \|_{L_2^{\Psi^{-1}}(\Omega)}} \tag{4.1}$$

and

$$\gamma_{\mathbf{e}}^{x} = \frac{\| \Phi_{xx} \|_{L_2^{\Psi}(\Omega)}}{\| \mathbf{e} \|_{L_2^{\Psi^{-1}}(\Omega)}} \tag{4.2}$$

respectively. We now apply the result of the previous sections; using Cauchy-Schwarz inequality in (3.8) coupled with the interpolation estimate (3.9) and the strong stability factors (4.1) and (4.2), to derive the $L_2(L_2)$ a posteriori error estimates for the scheme (3.1).

Theorem 4.1. *The error* $\mathbf{e} = \mathbf{w} - \mathbf{w}_h$, *where* $\mathbf{w}$ *is the solution of the continuous problem* (1.2) *and* $\mathbf{w}_h$ *that of* (3.1), *satisfies the following stability estimate:*

$$\begin{aligned} \| \mathbf{e} \|_{L_2^{\Psi^{-1}}(\Omega)} \leq & \, C\gamma_{\mathbf{e}}^{x} \| h^2(I-P)R_0 \|_{L_2^{\Psi^{-1}}(\Omega)} \\ & +C \| h^2(I-A)R_0 \|_{L_2^{\Psi^{-1}}(\Omega)} \gamma_e^x \\ & +C\gamma_{\mathbf{e}}^{t} \| k_n R_1 \|_{L_2^{\Psi^{-1}}(\Omega)} +\gamma_{\mathbf{e}}^{x} \| h^2 R_2 \|_{L_2^{\Psi^{-1}}(\Omega)} \\ & +\gamma_{\mathbf{e}}^{t} \| k_n R_2 \|_{L_2^{\Psi^{-1}}(\Omega)} +||\Phi_N||_{L_2^{\Psi}(\Omega)}. \end{aligned} \tag{4.3}$$

Proof. Using the notation introduced above, we may write (3.8) as

$$\| \mathbf{e} \|^2_{L_2^{\Psi^{-1}}(\Omega)} = \sum_{n=0}^{N-1} (R_0, \hat{\Phi} + \delta(\hat{\Phi}_t + A\hat{\Phi}) - \Phi)_n$$

$$+ \sum_{n=0}^{N-1} \langle k_n \frac{[\mathbf{w}_h]}{k_n}, (\hat{\Phi} - \Phi)_+ \rangle_n + \langle \mathbf{e}_-, \Phi_N \rangle_N = I + II + III.$$

We shall estimate the terms I, II and III separately. The proof of I and III are modified version of the previous one and therefore are omitted (see [5, 6]). The a posteriori error estimates now follows immediately after collecting the terms and using the definition of the stability factors (4.1) and (4.2).

It remains to estimate the terms II, to this end, we need the following notation

$$\Phi_+^n(x) = \Phi(x,t) - \int_{t_n}^{t} \frac{\partial}{\partial \tau} \Phi(x,\tau) d\tau dt.$$

So that

$$k_n\Phi_+^n(x)=\int_{I_n}\Phi(x,t)dt-\int_{I_n}\int_{t_n}^t\Phi_\tau(x,\tau)d\tau dt, \tag{4.4}$$

where $\Phi_\tau=\frac{\partial\Phi}{\partial\tau}$ and $\hat{\Phi}_n=\hat{\Phi}(.,t_n)$. Also, we can write

$$\begin{aligned}II&=\sum_{n=0}^{N-1}\langle k_n\frac{[\mathbf{w}_h]}{k_n},(\hat{\Phi}-\Phi)_+\rangle_n\\&=\sum_{n=0}^{N-1}\langle k_n\frac{[\mathbf{w}_h]}{k_n},(\hat{\Phi}_n-P\Phi+P\Phi-\Phi)_+\rangle_n\\&=\sum_{n=0}^{N-1}\langle k_n\frac{[\mathbf{w}_h]}{k_n},(\hat{\Phi}_n-P\Phi)_+\rangle_n\\&+\sum_{n=0}^{N-1}\langle k_n\frac{[\mathbf{w}_h]}{k_n},(P\Phi-\Phi)_+\rangle_n:=II_1+II_2.\end{aligned}$$

To estimate II_1, we use (5.4) to get

$$\begin{aligned}II_1&=\sum_{n=0}^{N-1}\langle k_nR_1,(\hat{\Phi}_n)_+-P\Phi_+\rangle_n\\&=\sum_{n=0}^{N-1}\langle R_1,k_n\hat{\Phi}_n-Pk_n\Phi_+\rangle_n\\&=\sum_{n=0}^{N-1}\langle R_1,k_n\hat{\Phi}_n-\int_{I_n}P\Phi(.,t)dt+\int_{I_n}\int_{t_n}^tP\Phi_\tau(.,\tau)d\tau dt\rangle_n\\&=\sum_{n=0}^{N-1}\int_{I_n}\int_{t_n}^t\langle R_1,P\Phi_\tau(.,\tau)\rangle_nd\tau dt\\&\le\| k_nR_1\|_{L_2^{\Psi^{-1}}(\Omega)}\| P\Phi_t\|_{L_2^{\Psi}(\Omega)}\\&\le\| k_nR_1\|_{L_2^{\Psi^{-1}}(\Omega)}\| \Phi_t\|_{L_2^{\Psi}(\Omega)}\ .\end{aligned}$$

As for the II_2-terms we can write

$$\begin{aligned}II_2&=\sum_{n=0}^{N-1}\langle k_n\frac{[\mathbf{w}_h]}{k_n},(P\Phi-\Phi)_+\rangle_n\\&=\sum_{n=0}^{N-1}\langle\frac{\mathbf{w}_{h,+}^n-\mathbf{w}_{h,-}^n}{k_n},(P_n-I)k_n\Phi_+\rangle_n\\&=\sum_{n=0}^{N-1}\langle\frac{P_n\mathbf{w}_{h,-}^n-\mathbf{w}_{h,-}^n}{k_n},(P_n-I)(\int_{I_n}\Phi(.,t)dt-\int_{I_n}\int_{t_n}^t\Phi_\tau(.,\tau)d\tau dt)\rangle_n\end{aligned}$$

$$\leq \sum_{n=0}^{N-1} \int_{I_n} \langle \frac{(P_n - I)\mathbf{w}_{h,-}^n}{k_n}, (P_n - I)\Phi(.,t)\rangle_n dt$$

$$+ \sum_{n=0}^{N-1} \int_{I_n} \int_{t_n}^{t} \langle \frac{(P_n - I)\mathbf{w}_{h,-}^n}{k_n}, (P_n - I)\Phi_\tau(.,t) d\tau dt\rangle_n$$

$$\leq \| k_n R_2 \|_{L_2^{\Psi^{-1}}(\Omega)} \| \Phi_{xx} \|_{L_2^{\Psi}(\Omega)} + \| k_n R_2 \|_{L_2^{\Psi^{-1}}(\Omega)} \| \Phi_t \|_{L_2^{\Psi}(\Omega)} .$$

□

5. Numerical Implementation and realistic application

In this section, we implement the new streamline diffusion finite element method applied to solve a one-dimensional time dependent coupling of two hyperbolic equations. The wave equation plays the role of Newton's laws and conservation of energy in classical mechanics - i.e., it predicts the future behavior of a dynamical system [8,15,25]. It is a wave equation in terms of the wave function which predicts analytically and precisely the probability of events or outcomes. Below we discuss computational aspects of the approximate solution for (1.1) using the SD method: For $n = 0, ..., N-1$, find $\mathbf{w}^n \in \mathbf{W}^n$ such that

$$(5.1) \qquad (\mathbf{w}_{h,t}^n + A\mathbf{w}_h^n, \mathbf{g}_h + \delta(\mathbf{g}_{h,t} + A\mathbf{g}_h))_n + < \mathbf{w}_{h,+}^n, \mathbf{g}_{h,+} >_n$$
$$= < \mathbf{w}_{h,-}^n, \mathbf{g}_{h,+} >_n + (F, \mathbf{g_h} + \delta(\mathbf{g}_{h,t} + A\mathbf{g}_h))_n.$$

Here δ is the SD parameter (usually $\delta \sim h$). We use finite element approximation on a space-time slab with the trial functions being piecewise polynomials in space and piecewise linear in time; that is, for $(x,t) \in S_n$. We seek the approximate solution

$$(5.2) \qquad \mathbf{w}_h^n(x,t) = \sum_{i=1}^{M} \left\{ \varphi_i(x)(\theta_1(t)\tilde{\mathbf{w}}_i^n + \theta_2(t)\mathbf{w}_i^{n+1}) \right\}$$
$$= \begin{pmatrix} u_h^n(x,t) = \sum_{i=1}^M \varphi_i(x)(\theta_1(t)\tilde{u}_i^n + \theta_2(t)u_i^{n+1}) \\ v_h^n(x,t) = \sum_{i=1}^M \varphi_i(x)(\theta_1(t)\tilde{v}_i^n + \theta_2(t)v_i^{n+1}) \end{pmatrix},$$

such that $\varphi_i(x_j) = \delta_{ij}, (j = 1, ..., M)$ are the spatial shape functions at node i, $\theta_1(t)$ and $\theta_2(t)$ are the time interpolation functions defined for Bézier curve. Also, the nodal value of $\mathbf{w}$ for node i at $(t_n)_+$ and $(t_{n+1})_-$ are denoted by $\tilde{\mathbf{w}}_i^n$ and $\mathbf{w}_i^{n+1}$. Then, for each time slab, the test functions $\mathbf{g}_h^n$ are defined as $\varphi_j(x)\theta_1(t)$ and $\varphi_j(x)\theta_2(t)$, for $j = 1, .., M$. We choose φ_i as the *hat-functions*

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_{i+1} - x_i} = \frac{x - x_{i-1}}{h}, & x \in [x_{i-1}, x_i], \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} = \frac{x_{i+1} - x}{h}, & x \in [x_i, x_{i+1}], \\ 0, & \text{elsewhere} , \end{cases}$$

defined on a uniform partition of $\Omega = [\overline{a}, \overline{b}]$.

5.1. **Numerical experimenters.** We carry out experimental computations to solve (5.1), by an AMD Opteron computer with 15 Gigabytes RAM memory with 2.2 GHz CPU. For each slab S_n, we choose a partition of the spatial interval into the subinterval $J_i^n = (x_{i-1}^n, x_i^n)$, with $h_i^n = x_i^n - x_{i-1}^n$. For $h > 0$, small, we let T_h^n be a triangulation of the slab S_n into, a time-space triangular elements K (cf. Fig. 1. below), satisfying a minimum angle condition (see, e.g. [17,23]). The triangulation for S_n may be chosen independently from that of S_{n-1}, but for the sake of simplicity we assume the quasi-uniformity condition. We shall use finite element approximation on a space time slab with the trial functions being piecewise polynomials in space and piecewise linear in time; that is, for $(x,t) \in S_n$. In the following, we show some numerical examples:

5.1.1. *Test Problem 1 (when we have the exact solution).* Streamline diffusion method is computed by given δ, $h = 0.01$, $k = 0.0005$ and in (1.1) we assume that $\Omega = [\pi/2, 3\pi/2]$,

$$\begin{cases} u_{tt} - u_{xx} = pe^{qt}, & (x,t) \in (-\pi/2, \pi/2) \times (0,T), \\ u(x,0) = \alpha \sin(mx), & -\pi/2 \leq x \leq \pi/2, \\ u_t(x,0) = \beta \cos(nx), & -\pi/2 \leq x \leq \pi/2, \end{cases}$$

the exact solution of the above equation is

$$u(x,t) = \frac{p}{q^2}(e^{qt} - qt - 1) + \frac{\beta}{n}\cos(nx)\sin(nt) + \alpha\cos(mt)\sin(mx).$$

We assume $p = 0$ and $\alpha = 0$ then in the Figures 2-4, we verify numerically the rate of convergence for Error in L_2-norm of u, i.e. $\|u - u_h\|_{L_2}$. The results are given after $0, 2, 4, \ldots 200$ time steps . Also, the error in time or $t = k \times$time step for example we observe that time is between $t = 0$ until $t = 0.0001 \times 100 = 10^{-1}$. Therefore, we compare the absolute error of $u(x,0) - u_h(x,0), ..., u(x, 0.0005 \times \text{ time step}) - u_h(x, 0.0005 \times \text{ time step}), ...,$ and $u(x,10^{-1}) - u_h(x,10^{-1})$ also, we assume $x = -1.00 \times \pi/2, -0.99 \times \pi/2, ..., 0, 0.01 \times \pi/2, ..., 1.00 \times \pi/2$ (see Figs 2, 3 and 4). Finally we show the error of SD approximation solution in Table 1. The order of error is calculated using the following formula:

$$\text{Order of error for } u \approx \ln \frac{E_{h_i}(u)}{E_{h_{i+1}}(u)},$$

where $E_{h_i}(u) = \|u(x,t) - u_{h_i}(x,t)\|_{L_2}$ and $i = 1,2,3,4,5$. The agreement of the error estimates between theoretical analysis and numerical results show that our method is efficient (see Table 1). The reason maybe due to that the error order of SD method is close to 4.

5.1.2. *Test Problem 2 (when we haven't the exact solution).* In (1.1), we consider

$$u_0(x) = \begin{cases} 1+x, & x \in [0,1], \\ 1-x, & x \in [1,2], \\ 0, & \text{elsewhere} \ , \end{cases}$$

and

$$u_1(x) = \begin{cases} 1+x^2, & x \in [0,1], \\ 1-x^2, & x \in [1,2], \\ 0, & \text{elsewhere} \ . \end{cases}$$

Also, we consider $f(x) = sinx$ and $\Omega = [0,2]$. We give some results in Figures 5-8 for different $a = a(x)$ based on DG method (see [6]) and streamline diffusion method.

6. Conclusion

To this end, a spatial linear second order hyperbolic initial-boundary value problem is investigated. We use the new streamline diffusion method for generalizing wave equation and obtain a posteriori error estimates in $L^2(L^2)$-norm. The posteriori error estimate is a very powerful mathematical tool in this problem by SD method. Also, the obtained results reveal that the proposed technique is very effective, convenient and quite accurate to such considered problems. Therefore, we try to obtain optimal bounds and *hp*-version of this method remind challenges that deserves special attention and will be consideration elsewhere.

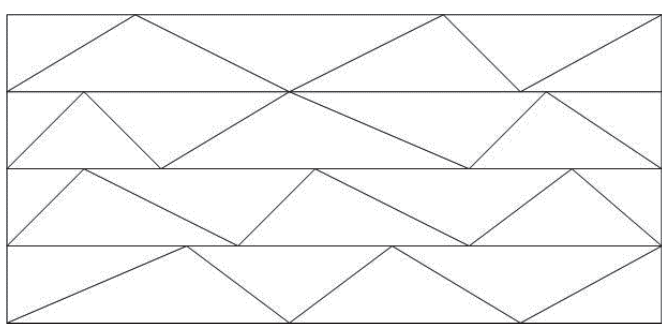

Fig. 1: The slabs on Rectangle

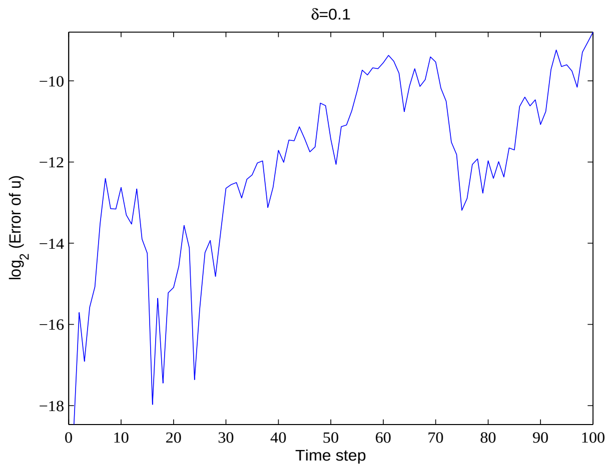

Figure 2: The behavior of error in time step for u

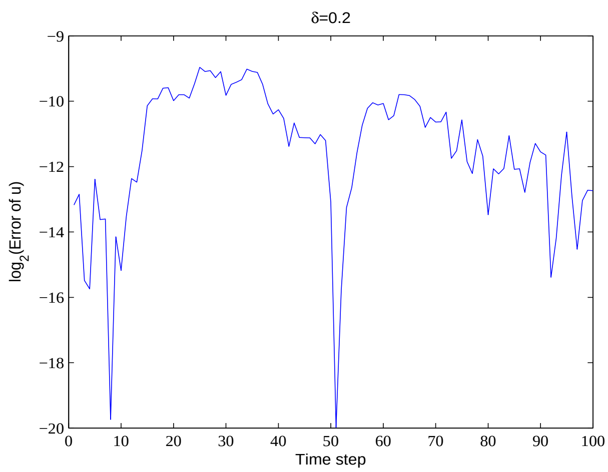

Fig. 3: The behavior of error in time step for u

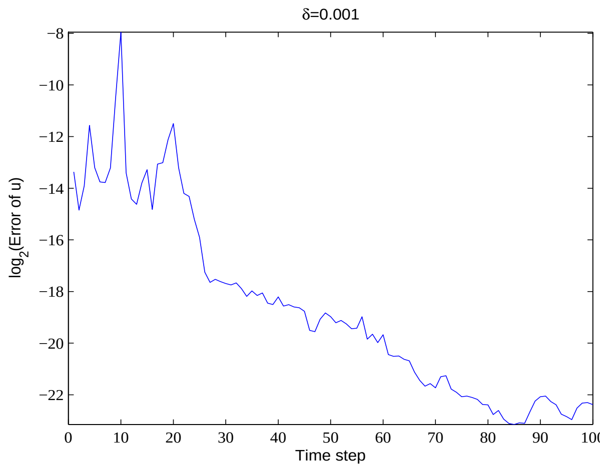

Fig. 4: The behavior of error in time step for u.

Table 1. $E_{h_i}(u)$ and order of error for u by SD method at $\delta = 0.1$ and $k = 0.01$

x	$h_1 = 0.15$	$h_2 = 0.10$	$h_3 = 0.05$	$h_4 = 0.01$	$h_5 = 0.005$
$-\pi/2$	0.231e-6	0.212e-6	0.431e-8	0.751e-8	0.321e-9
0.0	0.231e-5	0.761e-7	0.454e-7	0.983e-9	0.522e-12
$+\pi/2$	0.514e-6	0.234e-7	0.713e-10	0.761e-9	0.510e-9
order	-	2.6	2.9	2.9	3.9

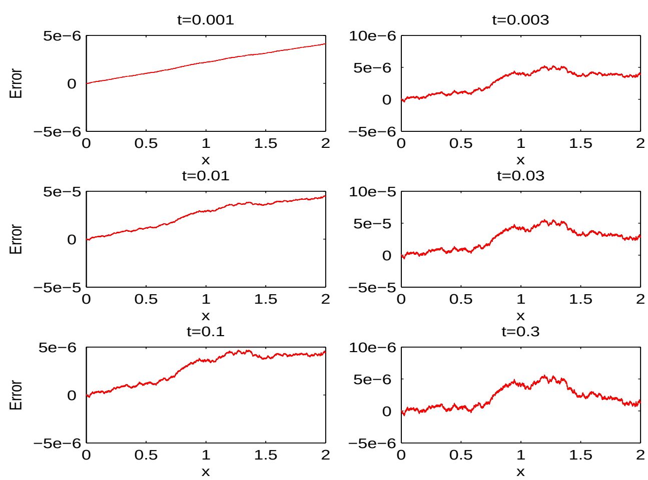

Fig. 5: Error of DG method and streamline diffusion method for $\delta = h = 0.001$, and $k = 0.0005$. The results are given, first row: after 2 times (left), after 24 times (right), second row: after 76

times (left), after 137 times (right) and three row: after 287 times (left), after 424 times (right)

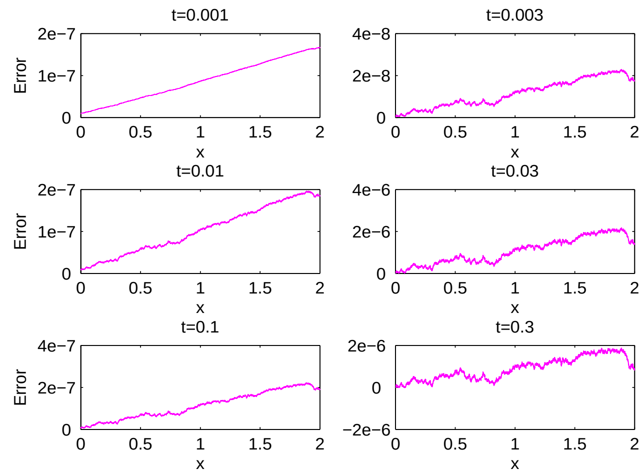

Fig. 6: Error of DG method and streamline diffusion method for $\delta = h = 0.001$, and $k = 0.0005$. The results are given, first row: after 2 times (left), after 24 times (right), second row: after 76 times (left), after 137 times (right) and three row: after 287 times (left), after 424 times (right) when $a(x) = \sqrt{x}$.

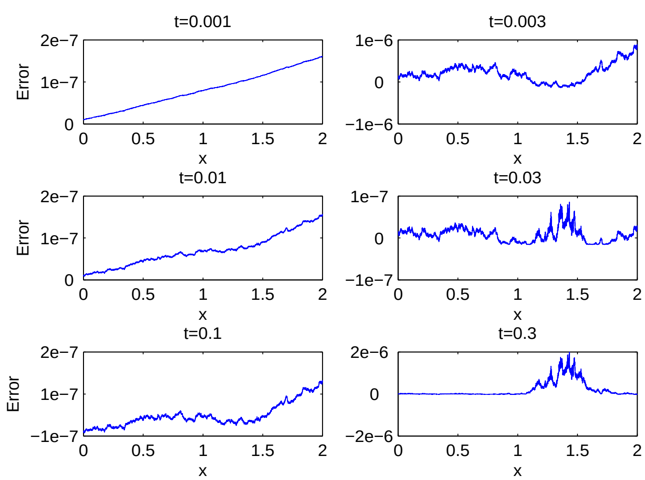

Fig. 7: Error of DG method and streamline diffusion method for $\delta = h = 0.001$, and $k = 0.0005$. The results are given, first row: after 2 times (left), after 24 times (right), second row: after 76 times (left), after 137 times (right) and three row: after 287 times (left), after 424 times (right)when $a(x) = x$.

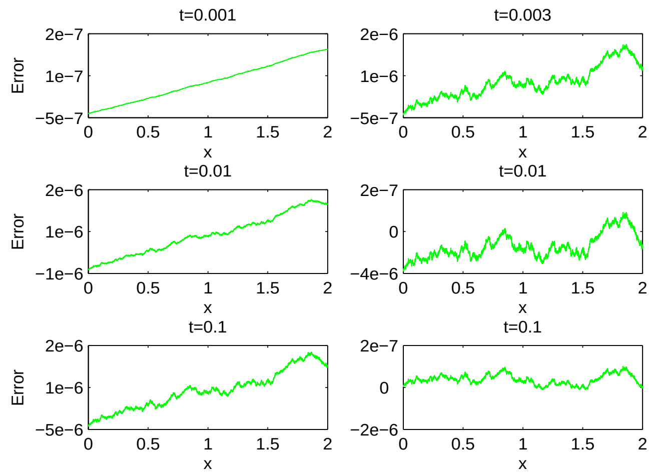

Fig. 8: Error of DG method and streamline diffusion method for $\delta = h = 0.001$, and $k = 0.0005$. The results are given, first row: after 2 times (left), after 24 times (right), second row: after 76 times (left), after 137 times (right) and three row: after 287 times (left), after 424 times (right) $a(x) = x^2$.

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