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ON WEAKLY $\mathfrak{F}_s$-QUASINORMAL SUBGROUPS OF FINITE GROUPS

Y. MAO, X. CHEN* AND W. GUO

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Abstract. Let $\mathfrak{F}$ be a formation and G a finite group. A subgroup H of G is said to be weakly $\mathfrak{F}_s$-quasinormal in G if G has an S-quasinormal subgroup T such that HT is S-quasinormal in G and $(H \cap T)H_G/H_G \leq Z_{\mathfrak{F}}(G/H_G)$, where $Z_{\mathfrak{F}}(G/H_G)$ denotes the $\mathfrak{F}$-hypercenter of G/H_G. In this paper, we study the structure of finite groups by using the concept of weakly $\mathfrak{F}_s$-quasinormal subgroup.
Keywords: $\mathfrak{F}$-hypercenter, weakly $\mathfrak{F}_s$-quasinormal subgroups, Sylow subgroups, p-nilpotence, supersolubility.
MSC(2010): Primary: 20D20; Secondary: 20D10.

1. Introduction

Throughout this paper, all groups considered are finite. G always denotes a group, π denotes a set of primes and p denotes a prime. Let $|G|_p$ denote the order of Sylow p-subgroups of G. For any subgroup H of G, we use H_G and H^G to denote the largest normal subgroup of G contained in H and the smallest normal subgroup of G containing H, respectively.

A class of groups $\mathfrak{F}$ is called a formation if it is closed under taking homomorphic images and subdirect products. A formation $\mathfrak{F}$ is called saturated if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$. Also, a formation $\mathfrak{F}$ is said to be S-closed if every subgroup of G belongs to $\mathfrak{F}$ whenever $G \in \mathfrak{F}$. The $\mathfrak{F}$-residual of G, denoted by $G^{\mathfrak{F}}$, is the smallest normal subgroup of G with quotient in $\mathfrak{F}$. We use $\mathfrak{U}$, $\mathfrak{U}_p$ and $\mathfrak{N}_p$ to denote the formations of all supersoluble groups, p-supersoluble groups and p-nilpotent groups, respectively.

For a class of groups $\mathfrak{F}$, a chief factor L/K of G is said to be $\mathfrak{F}$-central in G if $L/K \rtimes G/C_G(L/K) \in \mathfrak{F}$. A normal subgroup N of G is called $\mathfrak{F}$-hypercentral in G if either $N = 1$ or every chief factor of G below N is $\mathfrak{F}$-central in G. Let $Z_{\mathfrak{F}}(G)$ denote the $\mathfrak{F}$-hypercentre of G, that is, the product

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of all $\mathfrak{F}$-hypercentral normal subgroups of G. All unexplained notation and terminology are standard, as in [3, 6, 14].

Recall that a subgroup H of G is said to be quasinormal (resp. S-quasinormal) in G if H permutes with every subgroup (resp. Sylow subgroup) of G. Let $\mathfrak{F}$ be a formation. Recently, Huang [12] introduced the concept of $\mathfrak{F}_s$-quasinormal subgroup: a subgroup H of G is said to be $\mathfrak{F}_s$-quasinormal in G if G has a normal subgroup T such that HT is S-quasinormal in G and $(H \cap T)H_G/H_G \leq Z_{\mathfrak{F}}(G/H_G)$. Also, Miao and Li [15] introduced the concept of $\mathfrak{F}$-quasinormal subgroup: a subgroup H of G is said to be $\mathfrak{F}$-quasinormal in G if G has a quasinormal subgroup T such that HT is quasinormal in G and $(H \cap T)H_G/H_G \leq Z_{\mathfrak{F}}(G/H_G)$. By using these two concepts, the authors obtained some interesting results on the structure of finite groups. As a continuation of the above ideas, we introduce the following weaker concept.

Definition 1.1. *Let $\mathfrak{F}$ be a formation. A subgroup H of G is said to be weakly $\mathfrak{F}_s$-quasinormal in G if G has an S-quasinormal subgroup T such that HT is S-quasinormal in G and $(H \cap T)H_G/H_G \leq Z_{\mathfrak{F}}(G/H_G)$.*

Note that not only the concepts of $\mathfrak{F}_s$-quasinormal subgroup and $\mathfrak{F}$-quasinormal subgroup, but also many other subgroup embedding properties are generalized by our concept (see Section 4 below). In this paper, we study the properties of weakly $\mathfrak{F}_s$-quasinormal subgroups, and derive some criteria for a finite group to be p-nilpotent or supersoluble in terms of weakly $\mathfrak{F}_s$-quasinormal subgroups.

2. **Preliminaries**

Lemma 2.1. [7, Lemma 2.1] *Let $\mathfrak{F}$ be a non-empty saturated formation, $H \leq G$ and $N \trianglelefteq G$. Then:*

(1) $Z_{\mathfrak{F}}(G)N/N \leq Z_{\mathfrak{F}}(G/N)$.

(2) *If $\mathfrak{F}$ is S-closed, then $Z_{\mathfrak{F}}(G) \cap H \leq Z_{\mathfrak{F}}(H)$.*

Lemma 2.2. *Let $H, K \leq G$ and $N \trianglelefteq G$.*

(1) *If H is S-quasinormal in G, then H is subnormal in G.*

(2) *If H is S-quasinormal in G, then HN/N is S-quasinormal in G/N.*

(3) *If $N \leq H$, then H/N is S-quasinormal in G/N if and only if H is S-quasinormal in G.*

(4) *If H is S-quasinormal in G, then $H \cap K$ is S-quasinormal in K.*

(5) *If H is S-quasinormal in G, then H/H_G is nilpotent.*

(6) *If H is a p-group, then H is S-quasinormal in G if and only if $O^p(G) \leq N_G(H)$.*

(7) *If H and K are S-quasinormal in G, then $H \cap K$ is S-quasinormal in G.*

Proof. See [2, Lemma 1.2.7, Theorem 1.2.14, Lemma 1.2.16 and Theorem 1.2.19]. □

Lemma 2.3. *Let $H \leq K \leq G$ and $N \trianglelefteq G$. Then:*

(1) *If H is weakly $\mathfrak{F}_s$-quasinormal in G such that $(|H|, |N|) = 1$, then HN/N is weakly $\mathfrak{F}_s$-quasinormal in G/N.*

(2) *H/N is weakly $\mathfrak{F}_s$-quasinormal in G/N if and only if H is weakly $\mathfrak{F}_s$-quasinormal in G.*

(3) *If $\mathfrak{F}$ is S-closed and H is weakly $\mathfrak{F}_s$-quasinormal in G, then H is weakly $\mathfrak{F}_s$-quasinormal in K.*

Proof. (1) Since H is weakly $\mathfrak{F}_s$-quasinormal in G, G has an S-quasinormal subgroup T such that HT is S-quasinormal in G and $(H\cap T)H_G/H_G \leq Z_{\mathfrak{F}}(G/H_G)$. It is easy to see that $HN \cap TN = (H \cap T)N$ for $(|H|, |N|) = 1$. By Lemma 2.2(2), TN/N and HTN/N are S-quasinormal in G/N. Since $(H\cap T)H_G/H_G \leq Z_{\mathfrak{F}}(G/H_G)$, $(H \cap T)(HN)_G/(HN)_G \leq Z_{\mathfrak{F}}(G/(HN)_G)$ by Lemma 2.1(1). This implies that $(HN/N \cap TN/N)(HN/N)_{G/N}/$
$(HN/N)_{G/N} \leq Z_{\mathfrak{F}}((G/N)/(HN/N)_{G/N})$. Hence HN/N is weakly $\mathfrak{F}_s$-quasinormal in G/N.

(2) First suppose that H/N is weakly $\mathfrak{F}_s$-quasinormal in G/N. Then G/N has an S-quasinormal subgroup T/N such that $(H/N)(T/N)$ is S-quasinormal in G/N and $((H/N)\cap(T/N))(H/N)_{G/N}/(H/N)_{G/N} \leq Z_{\mathfrak{F}}((G/N)/(H/N)_{G/N})$. It follows that T and HT are S-quasinormal in G by Lemma 2.2(3) and $(H \cap T)H_G/H_G \leq Z_{\mathfrak{F}}(G/H_G)$. Hence H is weakly $\mathfrak{F}_s$-quasinormal in G. Now assume that H is weakly $\mathfrak{F}_s$-quasinormal in G. Then a similar argument as in (1) shows that H/N is weakly $\mathfrak{F}_s$-quasinormal in G/N.

(3) As H is weakly $\mathfrak{F}_s$-quasinormal in G, G has an S-quasinormal subgroup T such that HT is S-quasinormal in G and $(H \cap T)H_G/H_G \leq Z_{\mathfrak{F}}(G/H_G)$. Then $T \cap K$ and $H(T \cap K)$ are S-quasinormal in K by Lemma 2.2(4). By Lemma 2.1(2), $(H \cap T)H_G/H_G \leq Z_{\mathfrak{F}}(G/H_G) \cap (K/H_G) \leq Z_{\mathfrak{F}}(K/H_G)$, and so $(H \cap T)H_K/H_K \leq Z_{\mathfrak{F}}(K/H_K)$ by Lemma 2.1(1). Therefore, H is weakly $\mathfrak{F}_s$-quasinormal in K. □

Lemma 2.4. [5, Main Theorem] *Suppose that G has a Hall π-subgroup and $2 \notin \pi$. Then all the Hall π-subgroups are conjugate in G.*

Recall that a group G is called π-closed if G has a normal Hall π-subgroup. Moreover, a group G is said to be a C_π-group if G has a Hall π-subgroup and any two Hall π-subgroups of G are conjugate in G.

Lemma 2.5. [8, Corollary 3.7] *Let P be a p-subgroup of G. Suppose that G is a C_π-group with $p \notin \pi$. If every maximal subgroup of P has a π-closed supplement in G, then G is π-closed.*

The next lemma is clear.

Lemma 2.6. *Let p be a prime divisor of $|G|$ with $(|G|, p-1) = 1$.*

(1) *If G has cyclic Sylow p-subgroups, then $G \in \mathfrak{N}_p$.*

(2) *If N is a normal subgroup of G such that $|N|_p \leq p$ and $G/N \in \mathfrak{N}_p$, then $G \in \mathfrak{N}_p$.*

Lemma 2.7. [17, Lemma 2.16] *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Suppose that $N \trianglelefteq G$ such that $G/N \in \mathfrak{F}$. If N is cyclic, then $G \in \mathfrak{F}$.*

3. Main results

Lemma 3.1. *Let P be a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P either is weakly $(\mathfrak{U}_p)_s$-quasinormal or has a p-nilpotent supplement in G, then $G \in \mathfrak{N}_p$.*

Proof. Suppose that the result is false and let G be a counterexample of minimal order. Then:

(1) $O_{p'}(G) = 1$.

If $O_{p'}(G) > 1$, then by Lemma 2.3 (1), $G/O_{p'}(G)$ satisfies the hypothesis of the lemma. The choice of G implies that $G/O_{p'}(G) \in \mathfrak{N}_p$, and so $G \in \mathfrak{N}_p$, a contradiction.

(2) *G is soluble.*

Assume that G is not soluble. Then $p = 2$ by the Feit-Thompson theorem. If $O_2(G) > 1$, then $G/O_2(G)$ satisfies the hypothesis of the lemma by Lemma 2.3 (2). The choice of G implies that $G/O_2(G) \in \mathfrak{N}_2$. Thus G is soluble. This contradiction shows that $O_2(G) = 1$. If every maximal subgroup of P has a 2-nilpotent supplement in G, then G has a Hall $2'$-subgroup. By Lemma 2.4, G is a $C_{2'}$-group, and so $G \in \mathfrak{N}_2$ by Lemma 2.5, which is impossible. Therefore, P has a maximal subgroup P_1 that is weakly $(\mathfrak{U}_2)_s$-quasinormal in G. Then G has an S-quasinormal subgroup T such that P_1T is S-quasinormal in G and $(P_1 \cap T)(P_1)_G/(P_1)_G \leq Z_{\mathfrak{U}_2}(G/(P_1)_G)$. Clearly, $(P_1)_G \leq O_2(G) = 1$. Then we have that $P_1 \cap T \leq Z_{\mathfrak{U}_2}(G)$. Since $O_2(G) = O_{2'}(G) = 1$ by (1), $Z_{\mathfrak{U}_2}(G) = 1$, and so $P_1 \cap T = 1$. This implies that $|T|_2 \leq 2$. Then by Lemma 2.6(1), $T \in \mathfrak{N}_2$, and consequently $T \leq O_{2',2}(G) = 1$ by Lemma 2.2(1). Thus P_1 is S-quasinormal in G. By Lemma 2.2(1) again, $P_1 \leq O_2(G) = 1$, and so $|G|_2 \leq 2$. It follows that G is soluble, a contradiction.

(3) *G has a unique minimal normal subgroup N, $G/N \in \mathfrak{N}_p$ and $G = N \rtimes M$, where M is a maximal subgroup of G. Moreover, $N = O_p(G)$ and $|N| > p$.*

Let N be a minimal normal subgroup of G. Then by (1) and (2), N is an elementary abelian p-group. By Lemma 2.3(2), the hypothesis of the lemma still holds for G/N. By the choice of G, $G/N \in \mathfrak{N}_p$. Evidently, N is the unique minimal normal subgroup of G and $\Phi(G) = 1$. Thus there exists a maximal subgroup M of G such that $G = N \rtimes M$. Since $C_G(N) \cap M = 1$, $N = C_G(N)$, and thereby $N = O_p(G)$. If $|N| = p$, then by Lemma 2.6(2), $G \in \mathfrak{N}_p$, a contradiction. Hence $|N| > p$.

(4) *Final contradiction.*

Let P_1 be any maximal subgroup of P such that $N \nleq P_1$. Then $P = P_1N$, $(P_1)_G = 1$ and $P_1 > 1$ by (3). Suppose that P_1 is weakly $(\mathfrak{U}_p)_s$-quasinormal in

G. Then G has an S-quasinormal subgroup T such that P_1T is S-quasinormal in G and $P_1 \cap T \leq Z_{\mathfrak{U}_p}(G)$. If $Z_{\mathfrak{U}_p}(G) > 1$, then $N \leq Z_{\mathfrak{U}_p}(G)$ by (3), and so $|N| = p$, which is impossible. Thus $Z_{\mathfrak{U}_p}(G) = 1$. Then $P_1 \cap T = 1$, and we can conclude that $|T|_p \leq p$. If $T = 1$, then P_1 is S-quasinormal in G. By (3) and Lemma 2.2(6), $N \leq (P_1)^G = (P_1)^P = P_1$. This contradiction shows that $T > 1$. By Lemma 2.6(1), $T \in \mathfrak{N}_p$. Let $T_{p'}$ be the normal p-complement of T. Then $T_{p'}$ is subnormal in G by Lemma 2.2(1), and so $T_{p'} \leq O_{p'}(G) = 1$ by (1). This implies that T is a group of order p. Then P_1T is a Sylow p-subgroup of G. By (3) and Lemma 2.2(1), $P = P_1T = O_p(G) = N$. Consequently, $N \leq T^G = T^P = T$ by (3) and Lemma 2.2(6), and so $|N| = p$, which contradicts (3). Therefore, P_1 has a p-nilpotent supplement in G. Since $G = N \rtimes M$ and $M \in \mathfrak{N}_p$ by (3), every maximal subgroup of P has a p-nilpotent supplement in G. Note that G is a $C_{p'}$-group because G is p-soluble. Then by Lemma 2.5, $G \in \mathfrak{N}_p$. The final contradiction ends the proof. □

Theorem 3.2. *Let p be a prime divisor of $|G|$ with $(|G|, p-1) = 1$ and E be a normal subgroup of G such that $G/E \in \mathfrak{N}_p$. If E has a Sylow p-subgroup P such that every maximal subgroup of P either is weakly $(\mathfrak{U}_p)_s$-quasinormal or has a p-nilpotent supplement in G, then $G \in \mathfrak{N}_p$.*

Proof. By Lemmas 2.3(3) and 3.1, $E \in \mathfrak{N}_p$. Let $E_{p'}$ be the normal p-complement of E. Then $E_{p'} \trianglelefteq G$. Suppose that $E_{p'} > 1$. Then by Lemma 2.3(1), we see that $G/E_{p'}$ satisfies the hypothesis of the theorem. Hence $G/E_{p'} \in \mathfrak{N}_p$ by induction on $|G|$, and so $G \in \mathfrak{N}_p$. We may, therefore, assume that $E_{p'} = 1$. Then $E = P$ is a p-group. Let V/P be the normal p-complement of G/P. By Schur-Zassenhaus Theorem, there exists a Hall p'-subgroup $V_{p'}$ of V such that $V = P \rtimes V_{p'}$. Since $V \in \mathfrak{N}_p$ by Lemmas 2.3(3) and 3.1, $V = P \times V_{p'}$. This induces that $V_{p'}$ is the normal p-complement of G. Consequently, $G \in \mathfrak{N}_p$. □

Lemma 3.3. *Let P be a Sylow p-subgroup of G, where p is a prime divisor of $|G|$. If $N_G(P) \in \mathfrak{N}_p$ and every maximal subgroup of P either is weakly $(\mathfrak{U}_p)_s$-quasinormal or has a p-nilpotent supplement in G, then $G \in \mathfrak{N}_p$.*

Proof. If $p = 2$, then obviously, $G \in \mathfrak{N}_2$ by Lemma 3.1. So we only need to prove the lemma in the case that $p > 2$. Suppose that the result is false and let G be a counterexample of minimal order. Then:

(1) $O_{p'}(G) = 1$.

Suppose that $O_{p'}(G) > 1$. Since $N_{G/O_{p'}(G)}(PO_{p'}(G)/O_{p'}(G)) = N_G(P)O_{p'}(G)/O_{p'}(G) \in \mathfrak{N}_p$, $G/O_{p'}(G)$ satisfies the hypothesis of the lemma by Lemma 2.3(1). The choice of G implies that $G/O_{p'}(G) \in \mathfrak{N}_p$, and thereby $G \in \mathfrak{N}_p$, a contradiction.

(2) *If $P \leq H < G$, then $H \in \mathfrak{N}_p$.*

By Lemma 2.3(3), H satisfies the hypothesis of the lemma, and so $H \in \mathfrak{N}_p$ by the choice of G.

(3) *G is p-soluble.*

Since $G \notin \mathfrak{N}_p$, then there exists a non-trivial characteristic subgroup L of P such that $N_G(L) \notin \mathfrak{N}_p$ by [4, Chap. 8, Theorem 3.1]. If $L \ntrianglelefteq G$, then $P \leq N_G(L) < G$, and so $N_G(L) \in \mathfrak{N}_p$ by (2), which is impossible. Thus $L \trianglelefteq G$. This implies that $O_p(G) > 1$. Since $N_{G/O_p(G)}(P/O_p(G)) = N_G(P)/O_p(G) \in \mathfrak{N}_p$, $G/O_p(G)$ satisfies the hypothesis of the lemma by Lemma 2.3(2). The choice of G induces that $G/O_p(G) \in \mathfrak{N}_p$, and thereby G is p-soluble.

(4) *G has a unique minimal normal subgroup N, $G/N \in \mathfrak{N}_p$ and $G = N \rtimes M$, where M is a maximal subgroup of G. Moreover, $N = O_p(G)$ and $|N| > p$.*

Let N be a minimal normal subgroup of G. Then by (1) and (3), $N \leq O_p(G)$. Since $N_{G/N}(P/N) = N_G(P)/N \in \mathfrak{N}_p$, the hypothesis of the lemma holds for G/N by Lemma 2.3(2), and so $G/N \in \mathfrak{N}_p$ by the choice of G. It is easy to see that $N = O_p(G)$ is the unique minimal normal subgroup of G and G has a maximal subgroup M such that $G = N \rtimes M$. If $|N| = p$, then by Lemma 2.7, $G \in \mathfrak{U}_p$. As $O_{p'}(G) = 1$, $P \trianglelefteq G$ by [2, Lemma 2.1.6], and thus $G = N_G(P) \in \mathfrak{N}_p$, a contradiction. Hence $|N| > p$.

(5) *Final contradiction.*

Let P_1 be any maximal subgroup of P such that $N \nleq P_1$. Then by (4), we have that $P = P_1N$, $(P_1)_G = 1$ and $P_1 > 1$. Assume that P_1 is weakly $(\mathfrak{U}_p)_s$-quasinormal in G. Then G has an S-quasinormal subgroup T such that P_1T is S-quasinormal in G and $P_1 \cap T \leq Z_{\mathfrak{U}_p}(G)$. It follows from (4) that $Z_{\mathfrak{U}_p}(G) = 1$. Otherwise $|N| = p$, a contradiction. Then $P_1 \cap T = 1$, and so $|T|_p \leq p$. If $T = 1$, then P_1 is S-quasinormal in G. By (4) and Lemma 2.2(6), $N \leq (P_1)^G = (P_1)^P = P_1$, which is impossible. Thus $T > 1$. If $T_G > 1$, then $N \leq T$ by (4), and so $|N| = p$, a contradiction. Hence $T_G = 1$. By Lemma 2.2(5), T is nilpotent. Since T is subnormal in G by Lemma 2.2(1), T is a group of order p, because $O_{p'}(G) = 1$ by (1). Then P_1T is a Sylow p-subgroup of G. By (4) and Lemma 2.2(1), $P = P_1T = O_p(G) = N$. Thus $N \leq T^G = T^P = T$ by (4) and Lemma 2.2(6), and so $|N| = p$, which contradicts (4). Therefore, P_1 has a p-nilpotent supplement in G. Since $G = N \rtimes M$ and $M \in \mathfrak{N}_p$ by (4), every maximal subgroup of P has a p-nilpotent supplement in G. Then by (3) and Lemma 2.5, $G \in \mathfrak{N}_p$. This is the final contradiction. □

Theorem 3.4. *Let p be a prime divisor of $|G|$ and E be a normal subgroup of G such that $G/E \in \mathfrak{N}_p$. If E has a Sylow p-subgroup P such that $N_G(P) \in \mathfrak{N}_p$ and every maximal subgroup of P either is weakly $(\mathfrak{U}_p)_s$-quasinormal or has a p-nilpotent supplement in G, then $G \in \mathfrak{N}_p$.*

Proof. By Lemmas 2.3(3) and 3.3, $E \in \mathfrak{N}_p$. Let $E_{p'}$ be the normal p-complement of E. Clearly, $E_{p'} \trianglelefteq G$. Suppose that $E_{p'} > 1$. Then by Lemma 2.3(1), $G/E_{p'}$ satisfies the hypothesis of the theorem. By induction on $|G|$, we have that $G/E_{p'} \in \mathfrak{N}_p$, and so $G \in \mathfrak{N}_p$. Hence we may assume that $E_{p'} = 1$. Then $E = P$ is a p-group. Therefore, $G = N_G(P) \in \mathfrak{N}_p$. □

Theorem 3.5. *Suppose that for every prime divisor p of $|G|$ and every non-cyclic Sylow p-subgroup P of G, every maximal subgroup of P either is weakly $(\mathfrak{U}_p)_s$-quasinormal or has a p-supersoluble supplement in G. Then $G \in \mathfrak{U}$.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. Then:

(1) *G is a Sylow tower group of supersoluble type.*

Let q be the smallest prime dividing $|G|$ and Q a Sylow q-subgroup of G. If Q is cyclic, then $G \in \mathfrak{N}_q$ by Lemma 2.6(1). Now suppose that Q is non-cyclic. Since G satisfies the hypothesis of Lemma 3.1, $G \in \mathfrak{N}_q$ too. Then by Lemma 2.3(1), we can deduce that G is a Sylow tower group of supersoluble type by analogy.

(2) *Let r be the largest prime dividing $|G|$ and R a Sylow r-subgroup of G. Then R is the unique minimal normal subgroup of G, $G/R \in \mathfrak{U}$ and $G = R \rtimes M$, where M is a maximal subgroup of G. Moreover, $R = F(G)$ and R is non-cyclic.*

By (1), G is soluble and $R \trianglelefteq G$. Let N be any minimal normal subgroup of G. Then N is elementary abelian. By Lemmas 2.3(1) and 2.3(2), the hypothesis of the theorem holds for G/N, and so the choice of G implies that $G/N \in \mathfrak{U}$. Clearly, N is the unique minimal normal subgroup of G and $\Phi(G) = 1$. It follows that $N \leq R$ and G has a maximal subgroup M such that $G = N \rtimes M$. Since $C_G(N) \cap M = 1$, $N = C_G(N)$, and thereby $N = F(G)$. This induces that $R = N$. If R is cyclic, then by Lemma 2.7, $G \in \mathfrak{U}$, which is impossible. Thus R is non-cyclic.

(3) *Final contradiction.*

Let R_1 be any maximal subgroup of R. Then by (2), $(R_1)_G = 1$ and $R_1 > 1$. Suppose that R_1 is weakly $(\mathfrak{U}_r)_s$-quasinormal in G. Then we can derive a contradiction as in step (5) of the proof of Lemma 3.3. Hence R_1 has a r-supersoluble supplement in G, say K. Since $R \cap K \trianglelefteq G$, by (2), either $R \cap K = 1$ or $R \leq K$. In the former case, $R_1 \cap K = 1$, and so $R = R_1$, a contradiction. In the latter case, $G = K \in \mathfrak{U}_r$. Then $|R| = r$, a contradiction too. The proof is thus completed. □

Lemma 3.6. *Let P be a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every cyclic subgroup of P of order p or 4 (when P is a non-abelian 2-group) either is weakly $(\mathfrak{U}_p)_s$-quasinormal or has a p-nilpotent supplement in G, then $G \in \mathfrak{N}_p$.*

Proof. Suppose that the result is false and let G be a counterexample of minimal order. Let M be any maximal subgroup of G. By Lemma 2.3(3), it is easy to see that the hypothesis of the lemma still holds on M. Hence $M \in \mathfrak{N}_p$ by the choice of G, and so G is a minimal non-p-nilpotent group. In view of [14, Chap. IV, Satz 5.4] and [3, Chap. VII, Theorem 6.18], G is a minimal non-nilpotent group; $G = P \rtimes Q$, where Q is a Sylow q-subgroup of G with $q \neq p$; $P/\Phi(P)$ is a chief factor of G; the exponent of P is p or 4 (when P is a non-abelian 2-group). If $P/\Phi(P) \leq Z_{\mathfrak{U}_p}(G/\Phi(P))$, then $G/\Phi(P) \in \mathfrak{U}_p$, and thereby $G \in \mathfrak{U}_p$. Since $(|G|, p-1) = 1$, $G \in \mathfrak{N}_p$, which is impossible. Thus $P/\Phi(P) \nleq Z_{\mathfrak{U}_p}(G/\Phi(P))$, and so $|P/\Phi(P)| > p$.

Let $x \in P\backslash\Phi(P)$, $H = \langle x \rangle$ and $V = H\Phi(P)$. Then $|H| = p$ or 4 (when P is a non-abelian 2-group) and $H < P$. Since $P/\Phi(P)$ is a chief factor of G, $H_G \leq \Phi(P)$. First suppose that H is weakly $(\mathfrak{U}_p)_s$-quasinormal in G. Then G has an S-quasinormal subgroup T such that HT is S-quasinormal in G and $(H \cap T)H_G/H_G \leq Z_{\mathfrak{U}_p}(G/H_G)$. By Lemma 2.2(7), we may assume that $T \leq P$. Also, by Lemma 2.1(1), $(H \cap T)\Phi(P)/\Phi(P) \leq P/\Phi(P) \cap Z_{\mathfrak{U}_p}(G/\Phi(P)) = 1$, and so $T < P$. It follows from Lemma 2.2(6) that $T^G = T^P < P$. Since $P/\Phi(P)$ is a chief factor of G, $T \leq T^G \leq \Phi(P)$. Thus $V = HT\Phi(P)$ is S-quasinormal in G. By Lemma 2.2(6) again, we have that $P = V^G = V^P = V$. Hence $P = H$, a contradiction. Now suppose that H has a p-nilpotent supplement K in G. Then $(P \cap K)\Phi(P) \trianglelefteq G$. Since $P/\Phi(P)$ is a chief factor of G, $(P \cap K)\Phi(P) = P$ or $\Phi(P)$. If $P \leq K$, then $K = G$, and so $G \in \mathfrak{N}_p$, which is impossible. Thus $P \cap K \leq \Phi(P)$. This implies that $P = H(P \cap K) = H$, which is also impossible. The proof is thus finished. □

Theorem 3.7. *Let p be a prime divisor of $|G|$ with $(|G|, p-1) = 1$ and E be a normal subgroup of G such that $G/E \in \mathfrak{N}_p$. If E has a Sylow p-subgroup P such that every cyclic subgroup of P of order p or* 4 (*when P is a non-abelian* 2*-group*) *either is weakly $(\mathfrak{U}_p)_s$-quasinormal or has a p-nilpotent supplement in G, then $G \in \mathfrak{N}_p$.*

Proof. The conclusion follows by a similar argument as in Theorem 3.2 and using Lemma 3.6 instead of Lemma 3.1. □

Theorem 3.8. *Let E be a normal subgroup of G such that $G/E \in \mathfrak{U}$. Suppose that for every prime p dividing $|E|$ and every non-cyclic Sylow p-subgroup P of E, every cyclic subgroup of P of order p or* 4 (*when P is a non-abelian* 2*-group*) *either is weakly $(\mathfrak{U}_p)_s$-quasinormal or has a p-supersoluble supplement in G, then $G \in \mathfrak{U}$.*

Proof. Suppose that the result is false and let G be a counterexample of minimal order. A similar discussion as in the proof of Lemma 3.6 shows that G is a minimal non-supersoluble group. In view of [1, Theorem 12] and [3, Chap. VII, Theorem 6.18], G is a soluble group that has a normal Sylow p-subgroup, say G_p; $G_p = G^{\mathfrak{U}}$; $G_p/\Phi(G_p)$ is a chief factor of G; the exponent of G_p is p or 4 (when G_p is a non-abelian 2-group). Since $G/E \in \mathfrak{U}$, we have that $G_p \leq E$. If $|G_p/\Phi(G_p)| = p$, then by Lemma 2.7, $G/\Phi(G_p) \in \mathfrak{U}$, and so $G \in \mathfrak{U}$, which is impossible. Thus $|G_p/\Phi(G_p)| > p$. This implies that $G_p/\Phi(G_p) \nleq Z_{\mathfrak{U}_p}(G/\Phi(G_p))$.

Let $x \in G_p\backslash\Phi(G_p)$ and $H = \langle x \rangle$. Then $|H| = p$ or 4 (when G_p is a non-abelian 2-group) and $H < G_p$. Suppose that H is weakly $(\mathfrak{U}_p)_s$-quasinormal in G. Then we can get a contradiction similarly as in the proof of Lemma 3.6. Now consider that H has a p-supersoluble supplement K in G. Then $(G_p \cap K)\Phi(G_p) \trianglelefteq G$. Since $G_p/\Phi(G_p)$ is a chief factor of G, $(G_p \cap K)\Phi(G_p) = G_p$ or $\Phi(G_p)$. If $G_p \leq K$, then $K = G$, and so $G \in \mathfrak{U}_p$. This induces that $G_p \leq Z_{\mathfrak{U}}(G)$, and therefore $G \in \mathfrak{U}$, a contradiction. Thus $G_p \cap K \leq \Phi(G_p)$. Then $G_p = H(G_p \cap K) = H$, a contradiction too. The theorem is proved. □

4. **Some applications**

Let $\mathfrak{F}$ be a formation. In Section 1, we observe that all $\mathfrak{F}_s$-quasinormal and $\mathfrak{F}$-quasinormal subgroups of G are weakly $\mathfrak{F}_s$-quasinormal in G. Besides, recall that a subgroup H of G is said to be c-normal [18] in G if G has a normal subgroup T such that $G = HT$ and $H \cap T \leq H_G$. A subgroup H of G is called $\mathfrak{F}_n$-supplemented [19] in G if G has a normal subgroup T such that $G = HT$ and $(H \cap T)H_G/H_G \leq Z_{\mathfrak{F}}(G/H_G)$. A subgroup H of G is said to be $\mathfrak{F}_h$-normal [9] in G if G has a normal subgroup T such that HT is a normal Hall subgroup of G and $(H \cap T)H_G/H_G \leq Z_{\mathfrak{F}}(G/H_G)$. A subgroup H of G is called $\mathfrak{F}_n$-normal [10] in G if G has a normal subgroup T such that HT is normal in G and $(H \cap T)H_G/H_G \leq Z_{\mathfrak{F}}(G/H_G)$. It is easy to see that all above-mentioned subgroups of G are also weakly $\mathfrak{F}_s$-quasinormal in G.

Therefore, many results in former literatures can be viewed as special cases of our theorems in Section 3, and we list some of them below:

Corollary 4.1. [11, Theorem 3.4] *Let p be the smallest prime dividing $|G|$ and P be a Sylow p-subgroup of G. If every maximal subgroup of P is c-normal in G, then $G \in \mathfrak{N}_p$.*

Corollary 4.2. [9, Theorem 5.1] *Let p be a prime divisor of $|G|$ with $(|G|, p-1) = 1$ and P be a Sylow p-subgroup of G. Then $G \in \mathfrak{N}_p$ if and only if every maximal subgroup of P is $\mathfrak{U}_h$-normal in G.*

Corollary 4.3. [10, Theorem 4.2] *Let p be a prime divisor of $|G|$ with $(|G|, p-1) = 1$ and P be a Sylow p-subgroup of G. Then $G \in \mathfrak{N}_p$ if and only if every maximal subgroup of P not having a p-nilpotent supplement in G is $\mathfrak{U}_n$-normal in G.*

Corollary 4.4. [13, Theorem 3.2] *Let p be a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Assume that G has a normal subgroup N such that $G/N \in \mathfrak{N}_p$ and for every maximal subgroup M of each Sylow p-subgroup of N which is not $(\mathfrak{N}_p)_s$-quasinormal in G, M has a p-nilpotent supplement in G. Then $G \in \mathfrak{N}_p$.*

Corollary 4.5. [15, Lemma 2.7] *Let p be the smallest prime divisor of $|G|$ and P be a Sylow p-subgroup of G. Then $G \in \mathfrak{N}_p$ if and only if every maximal subgroup of P having no p-nilpotent supplement in G is $\mathfrak{N}_p$-quasinormal in G.*

Corollary 4.6. [11, Theorem 3.1] *Let p be an odd prime dividing $|G|$ and P be a Sylow p-subgroup of G. If $N_G(P) \in \mathfrak{N}_p$ and every maximal subgroup of P is c-normal in G, then $G \in \mathfrak{N}_p$.*

Corollary 4.7. [9, Theorem 5.2] *Let p be a prime divisor of $|G|$ and P be a Sylow p-subgroup of G. Then $G \in \mathfrak{N}_p$ if and only if $N_G(P) \in \mathfrak{N}_p$ and every maximal subgroup of P is $\mathfrak{U}_h$-normal in G.*

Corollary 4.8. [10, Theorem 4.3] *Let p be a prime divisor of $|G|$ and P be a Sylow p-subgroup of G. Then $G \in \mathfrak{N}_p$ if and only if $N_G(P) \in \mathfrak{N}_p$ and every maximal subgroup of P not having a p-nilpotent supplement in G is $\mathfrak{U}_n$-normal in G.*

Corollary 4.9. [18, Theorem 4.1] *Suppose that P_1 is c-normal in G for every Sylow subgroup P of G and every maximal subgroup P_1 of P. Then $G \in \mathfrak{U}$.*

Corollary 4.10. [19, Corollary 3.8] *$G \in \mathfrak{U}$ if and only if every maximal subgroup of every non-cyclic Sylow subgroup of G is $\mathfrak{U}_n$-supplemented in G.*

Corollary 4.11. [16, Lemma 3.8] *Let p be the smallest prime dividing $|G|$ and P be a Sylow p-subgroup of G. If the subgroups of P of order p or order 4 are c-normal in G, then $G \in \mathfrak{N}_p$.*

Corollary 4.12. [13, Theorem 3.3] *Let p be a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Assume that G has a normal subgroup N such that $G/N \in \mathfrak{N}_p$ and for every cyclic subgroup L of order p or 4 of N which is not $(\mathfrak{N}_p)_s$-quasinormal in G, L has a p-nilpotent supplement in G. Then $G \in \mathfrak{N}_p$.*

Corollary 4.13. [18, Theorem 4.2] *Suppose that $\langle x \rangle$ is c-normal in G for every element x of G with prime order or order 4. Then $G \in \mathfrak{U}$.*

Corollary 4.14. [9, Corollary 3.6] *$G \in \mathfrak{U}$ if and only if every cyclic subgroup of G of prime order or order 4 is $\mathfrak{U}_h$-normal in G.*

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References

[1] A. Ballester-Bolinches and R. Esteban-Romero, On minimal non-supersoluble groups, *Rev. Mat. Iberoam.* **23** (2007), no. 1, 127–142.

[2] A. Ballester-Bolinches, R. Esteban-Romero and M. Asaad, Products of Finite Groups, Walter de Gruyter & Co., Berlin, 2010.

[3] K. Doerk and T. Hawkes, Finite Soluble Groups, Walter de Gruyter & Co., Berlin, 1992.
[4] D. Gorenstein, Finite Groups, Harper & Row Publishers, New York-London, 1968.
[5] F. Gross, Conjugacy of odd order Hall subgroups, *Bull. London Math. Soc.* **19** (1987), no. 4, 311–319.
[6] W. Guo, The Theory of Classes of Groups, Science Press-Kluwer Academic Publishers, Beijing, 2000.
[7] W. Guo, On $\mathfrak{F}$-supplemented subgroups of finite groups, *Manu. Math.* **127** (2008), no. 2, 139–150.
[8] W. Guo and A. N. Skiba, On $\mathfrak{F}\Phi^*$-hypercentral subgroups of finite groups, *J. Algebra* **372** (2012) 275–292.
[9] W. Guo, X. Feng and J. Huang, New characterizations of some classes of finite groups, *Bull. Malays. Math. Sci. Soc. (2)* **34** (2011), no. 3, 575–589.
[10] W. Guo and X. Yu, On $\mathfrak{F}_n$-normal subgroups of finite groups, *Sib. Math. J.* **52** (2011), no. 2, 197–206.
[11] X. Guo and K. P. Shum, On c-normal maximal and minimal subgroups of Sylow p-subgroup of finite groups, *Arch. Math. (Basel)* **80** (2003), no. 6, 561–569.
[12] J. Huang, On $\mathfrak{F}_s$-quasinormal subgroups of finite groups, *Comm. Algebra* **38** (2010), no. 11, 4063–4076.
[13] J. Huang, N. Yang, B. Hu and X. Yu, A system of covering subgroups for the class of p-nilpotent groups, *Sib. Math. J.* **53** (2012), no. 2, 352–360
[14] B. Huppert, Endliche Gruppen I, Springer-Verlag, Berlin-New York, 1967.
[15] L. Miao and B. Li, On $\mathfrak{F}$-quasinormal primary subgroups of finite groups, *Comm. Algebra* **39** (2011), no. 10, 3515–3525.
[16] M. Ramadan, M. Ezzat Mohamed and A. A. Heliel, On c-normality of certain subgroups of prime power order of finite groups, *Arch. Math. (Basel)* **85** (2005), no. 3, 203–210.
[17] A. N. Skiba, On weakly s-permutable subgroups of finite groups, *J. Algebra* **315** (2007), no. 1, 192–209.
[18] Y. Wang, C-normality of groups and its properties, *J. Algebra* **180** (1996), no. 3, 954–965.
[19] N. Yang and W. Guo, On $\mathfrak{F}_n$-supplemented subgroups of finite groups, *Asian-Eur. J. Math.* **1** (2008), no. 4, 619–629.

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