ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 41 (2015), No. 3, pp. 677–697

Title:

On meromorphically multivalent functions defined by multiplier transformation

Author(s):

M. P. Jeyaraman and T. K. Suresh

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 41 (2015), No. 3, pp. 677–697
Online ISSN: 1735-8515

ON MEROMORPHICALLY MULTIVALENT FUNCTIONS DEFINED BY MULTIPLIER TRANSFORMATION

M. P. JEYARAMAN* AND T. K. SURESH

(Communicated by Ali Abkar)

Abstract. The purpose of this paper is to derive various useful subordination properties and characteristics for certain subclass of multivalent meromorphic functions, which are defined here by the multiplier transformation. Also, we obtained inclusion relationship for this subclass.
Keywords: Analytic functions, multivalent functions, differential subordination, Gauss hypergeometric function, multiplier transformation.
MSC(2010): Primary: 30C80; Secondary: 30C45, 30D30.

1. Introduction and definitions

Let Σ_p be the class of meromorphic functions f of the form:

$$f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, \dots\}), \tag{1.1}$$

which are analytic and *p-valent* in the punctured unit disk

$$\mathbb{D} = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = \mathbb{U} \setminus \{0\},$$

having a pole of order p at the origin.

For a function $f \in \Sigma_p$ given by (1.1) and $g \in \Sigma_p$ defined by

$$g(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} b_k z^k,$$

the *Hadamard product* (or *convolution*) of f and g is given by

$$(f * g)(z) := \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k b_k z^k =: (g * f)(z).$$

Article electronically published on June 15, 2015.
Received: 11 January 2014, Accepted: 8 April 2014.
*Corresponding author.

Let the functions f and g be analytic in the open unit disk $\mathbb{U} := \mathbb{D} \cup \{0\}$. We say that the function f is said to be *subordinate* to g, or (equivalently) g is said to be *superordinate* to f, written symbolically as

$$f \prec g \quad \text{in} \quad \mathbb{U} \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a *Schwarz function* w analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{U}$, such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

In particular, if the function g is univalent in $\mathbb{U}$, then we have the equivalence (cf., [7,10])

$$f \prec g \quad \Leftrightarrow \quad f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Recently, Ali et al. [1] introduced and investigated the *multiplier transformation* $I_p(n, \lambda)$ on the class Σ_p of meromorphically multivalent analytic functions defined by the infinite series

$$I_p(n,\lambda)f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} \left(\frac{k+\lambda}{\lambda - p}\right)^n a_k z^k$$

$$(\lambda > p; n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathbb{D}).$$

Obviously, we have

$$I_p(m,\lambda)\left(I_p(n,\lambda)f(z)\right) = I_p(m+n,\lambda)f(z) \quad (m, n \in \mathbb{N}_0).$$

We now define the function $f_{n,p}^{\lambda}$ by

$$f_{n,p}^{\lambda}(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} \left(\frac{k-p+\lambda}{\lambda - p}\right)^n z^k \quad (\lambda > p; n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathbb{D}),$$

and let the associated function $f_{n,p}^{\lambda,\mu}$ be defined by the *Hadamard product (or convolution)*:

$$f_{n,p}^{\lambda}(z) * f_{n,p}^{\lambda,\mu}(z) = \frac{1}{z^p(1-z)^{\mu}} \quad (\mu > 0, z \in \mathbb{D}).$$

Then, analogous to $I_p(n, \lambda)$, we here define a new multiplier transformation

$$\mathcal{I}_p^{\mu}(n,\lambda) : \Sigma_p \to \Sigma_p$$

as follows:

$$\mathcal{I}_p^{\mu}(n,\lambda)f(z) = f_{n,p}^{\lambda,\mu}(z) * f(z). \tag{1.2}$$

We note that

$$\mathcal{I}_p^{1}(0,\lambda)f(z) = f(z) \quad \text{and} \quad \mathcal{I}_p^{2}(0,\lambda)f(z) = \frac{\left(z^{p+1}f(z)\right)'}{z^p} = zf'(z) + (p+1)f(z).$$

It is easily verified from (1.2) that

$$z\left(\mathcal{I}_p^{\mu}(n,\lambda)f\right)'(z) = \mu\mathcal{I}_p^{\mu+1}(n,\lambda)f(z) - (\mu+p)\mathcal{I}_p^{\mu}(n,\lambda)f(z) \tag{1.3}$$

and

$$z\left(\mathcal{I}_p^{\mu}(n+1,\lambda)f\right)'(z) = (\lambda-p)\mathcal{I}_p^{\mu}(n,\lambda)f(z) - \lambda\mathcal{I}_p^{\mu}(n+1,\lambda)f(z). \tag{1.4}$$

The definition (1.2) of the multiplier transformation $\mathcal{I}_p^{\mu}(n,\lambda)$ is motivated essentially by the Liu-Srivastava operator [5, 6], which has been used widely on the space of meromorphic functions in $\mathbb{D}$. The multiplier transformation $\mathcal{I}_p^{\mu}(n,\lambda)$ gets reduce to the familiar operators by specializing the parameters λ,μ,n and p. In particular, for $\lambda=2$ and $\mu=p=1$, the operator $\mathcal{I}_p^{\mu}(n,\lambda)f(z)$ reduces the operator $I^n f(z)$, introduced by Flett [2] and investigated by Uralegaddi and Somanatha [16, 17].

Now, we introduce a new subclass of functions in Σ_p, by making use of the multiplier transformation $\mathcal{I}_p^{\mu}(n,\lambda)$ as follows.

Definition 1.1. *A function $f \in \Sigma_p$ is said to be in the class $\Sigma_{p,n}^{\lambda,\mu}(\alpha;A,B)$ if it satisfies*

$$-\frac{1}{p-\alpha}\left(\frac{z\left(\mathcal{I}_p^{\mu}(n,\lambda)f\right)'(z)}{\mathcal{I}_p^{\mu}(n,\lambda)f(z)} + \alpha\right) \prec \frac{1+Az}{1+Bz},$$

$$(n\in\mathbb{N}_0; p\in\mathbb{N}; \lambda>p; 0\le\alpha<p; -1\le B<A\le 1; z\in\mathbb{U}).$$

In particular, for $A=1$ and $B=-1$ we write $\Sigma_{p,n}^{\lambda,\mu}(\alpha;1,-1)=\Sigma_{p,n}^{\lambda,\mu}(\alpha)$, where

$$\Sigma_p^{n,\lambda}(\alpha) = \left\{f\in\Sigma_p : -\mathrm{Re}\left(\frac{z\left(\mathcal{I}_p^{\mu}(n,\lambda)f\right)'(z)}{\mathcal{I}_p^{\mu}(n,\lambda)f(z)}\right) > \alpha, z\in\mathbb{U}\right\}.$$

For $A=\mu=1, B=-1$ and $n=0$, $\Sigma_{p,0}^{\lambda,1}(\alpha;1,-1)$ is the class of p-valent meromorphic starlike functions of order α.

Recently Srivastava et al. [14] and Patel et al. [12] obtained certain subordination properties for certain subclass of multivalent meromorphic functions defined by a linear operator. Some subordination properties of the subclass of multivalent functions associated with the generalized multiplier transformation have been obtained recently by the authors in [3]. Motivated by the aforementioned work, we investigate the subordination properties of the multiplier transformation $\mathcal{I}_p^{\mu}(n,\lambda)$ defined by (1.2) and we derive a number of sufficient conditions for the functions belonging to the subclass $\Sigma_{p,n}^{\lambda,\mu}(\alpha)$. We also obtain a sharp inclusion relationship for the class $\Sigma_{p,n}^{\lambda,\mu}(\alpha;A,B)$.

2. Preliminary lemmas

To establish our main results, we need the following lemmas.

Lemma 2.1. *[7, 10] Let a fucntion h be analytic and convex (univalent) in $\mathbb{U}$, with $h(0) = 1$. Suppose also that the function φ given by*

$$\varphi(z) = 1 + b_1 z + b_2 z^2 + \cdots \tag{2.1}$$

is analytic in $\mathbb{U}$. If

$$\varphi(z) + \frac{z\varphi'(z)}{c} \prec h(z) \quad (\mathrm{Re}\, c \geq 0, c \neq 0), \tag{2.2}$$

then

$$\varphi(z) \prec \psi(z) = \frac{c}{z^c} \int_0^z t^{c-1} h(t) dt \prec h(z),$$

where ψ is the best dominant of (2.2).

We denote by $P(\gamma)$ the class of functions φ given by (2.1) which are analytic in $\mathbb{U}$ and satisfy the following inequality:

$$\mathrm{Re}\, \varphi(z) > \gamma, \quad (0 \leq \gamma < 1, z \in \mathbb{U}).$$

Lemma 2.2. *[11] Let the function φ given by* (2.1) *be in the class $P(\gamma)$. Then*

$$\mathrm{Re}\, \varphi(z) \geq 2\gamma - 1 + \frac{2(1-\gamma)}{1+|z|} \quad (0 \leq \gamma < 1, z \in \mathbb{U}).$$

Lemma 2.3. *[15] For $0 \leq \gamma_1 < \gamma_2 < 1$,*

$$P(\gamma_1) * P(\gamma_2) \subset P(\gamma_3), \quad \textit{where} \quad \gamma_3 = 1 - 2(1-\gamma_1)(1-\gamma_2).$$

The result is the best possible.

For any complex numbers $a, b,$ and c ($c \notin \mathbb{Z}_0^- := \{0, -1, -2, \dots\}$), the *Gaussian hypergeometric function* is defined by

$${}_2F_1(a, b; c; z) = 1 + \frac{ab}{c} \frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{z^2}{2!} + \cdots .$$

Lemma 2.4. *[18] For any complex numbers a, b, c ($c \notin \mathbb{Z}_0^-$), we have*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} \ {}_2F_1(a, b; c; z),$$
$$\mathrm{Re}\, c > \mathrm{Re}\, b > 0,$$

$${}_2F_1(a, b; c; z) = (1-z)^{-a} \ {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right), \quad z \notin (1, \infty),$$
$${}_2F_1(a, b; c; z) = \ {}_2F_1(b, a; c; z),$$
$$(b+1) \ {}_2F_1(1, b; b+1; z) = (b+1) + bz \ {}_2F_1(1, b+1; b+2; z),$$
$$z \ {}_2F_1(1, 1; 2; -z) = \log(1+z).$$

Lemma 2.5. *[13] The function $(1-z)^\varrho \equiv e^{\varrho log(1-z)}, \varrho \neq 0$ is univalent in $\mathbb{U}$ if and only if ϱ is either in the closed disk $|\varrho - 1| \leq 1$ or in the closed disk $|\varrho + 1| \leq 1$.*

Lemma 2.6. *[9] Let $A, B, \beta, \gamma \in \mathbb{C}$ with $\beta \neq 0, |B| \leq 1, A \neq B$, and suppose that these constants satisfy*

$$\operatorname{Re}\left[\beta(1-A)(1-\overline{B})+\gamma|1-B|^2\right]>0$$

and

$$\operatorname{Re}\left[\beta(1-A)(1-\overline{B})+\gamma|1-B|^2\right]\cdot\operatorname{Re}\left[\beta(1+A)(1+\overline{B})+\gamma|1+B|^2\right] - \left[\operatorname{Im}[\beta(\overline{B}-A)+\gamma(\overline{B}-B)]\right]^2 \geq 0,$$

or

$$\operatorname{Re}\left[\beta(1+A)(1+\overline{B})+\gamma|1+B|^2\right]\geq 0$$

and

$$\operatorname{Re}\left[\beta(1-A)(1-\overline{B})+\gamma|1-B|^2\right]=\operatorname{Im}[\beta(\overline{B}-A)+\gamma(\overline{B}-B)]=0.$$

Then the differential equation

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}=\frac{1+Az}{1+Bz}$$

has a univalent solution in $\mathbb{U}$ given by

$$q(z)=\begin{cases}\dfrac{z^{\beta+\gamma}(1+Bz)^{\beta(A-B)/B}}{\beta\int_0^z t^{\beta+\gamma-1}(1+Bt)^{\beta(A-B)/B}dt}-\dfrac{\gamma}{\beta}, & \text{if } B\neq 0\\ \dfrac{z^{\beta+\gamma}e^{\beta Az}}{\beta\int_0^z t^{\beta+\gamma-1}e^{\beta At}dt}-\dfrac{\gamma}{\beta}, & \text{if } B=0.\end{cases}$$

If $\varphi(z)$ is regular in $\mathbb{U}$ and satisfies

$$\varphi(z)+\frac{z\varphi'(z)}{\beta\varphi(z)+\gamma}\prec\frac{1+Az}{1+Bz},$$

then

$$\varphi(z)\prec q(z)\prec\frac{1+Az}{1+Bz}$$

and $q(z)$ is the best dominant.

Lemma 2.7. *[8] Let q be univalent in the unit disk $\mathbb{U}$ and θ and Φ be analytic in a domain D containing $q(\mathbb{U})$, with $\Phi(w)\neq 0$, when $w\in q(\mathbb{U})$. Set $Q(z)=zq'(z)\Phi(q(z))$, $h(z)=\theta(q(z))+Q(z)$ and suppose that either h is convex, or Q is starlike univalent in $\mathbb{U}$. In addition, assume that*

$$\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right)=\operatorname{Re}\left(\frac{\theta'(q(z))}{\Phi(q(z))}+\frac{zQ'(z)}{Q(z)}\right)>0.$$

If p is analytic in $\mathbb{U}$, with $p(0)=q(0), p(\mathbb{U})\subset D$ and

$$\theta(p(z))+zp'(z)\Phi(p(z))\prec\theta(q(z))+zq'(z)\Phi(q(z))=h(z), \tag{2.3}$$

then $p\prec q$, and q is the best dominant.

Lemma 2.8. *[19] Let v be a positive measure on the interval $[0,1]$. Let $h(z,t)$ be a complex-valued function defined on $\mathbb{U}\times[0,1]$ such that $h(.,t)$ is analytic in $\mathbb{U}$ for each $t\in[0,1]$ and $h(z,.)$ is v-integrable on $[0,1]$ for each $z\in\mathbb{U}$. In addition, suppose that $\operatorname{Re}(h(z,t))>0, h(-r,t)$ is real and*

$$\operatorname{Re}\left(\frac{1}{h(z,t)}\right)\geq\frac{1}{h(-r,t)}\quad(|z|\leq r<1;t\in[0,1]).$$

If the function $H(z)$ is defined by

$$H(z)=\int_0^1 h(z,t)dv(t),$$

then

$$\operatorname{Re}\left(\frac{1}{H(z)}\right)\geq\frac{1}{H(-r)}\quad(|z|\leq r<1).$$

Lemma 2.9. *[4] Let $\lambda\neq 0$ be a real number, $\dfrac{a}{\lambda}>0$ and $0\leq\beta<1$. Let $g(z)=1+c_nz^n+c_{n+1}z^{n+1}+\cdots$, be analytic in $\mathbb{U}$ and*

$$g(z)\prec 1+\frac{aMz}{n\lambda+a}\quad(n\in\mathbb{N}),$$

where

$$M=\frac{(1-\beta)|\lambda|\left(1+\frac{n\lambda}{a}\right)}{|1-\lambda+\lambda\beta|+\sqrt{1+\left(1+\frac{n\lambda}{a}\right)^2}}.$$

If $P(z)=1+d_nz^n+d_{n+1}z^{n+1}+\cdots$ is analytic in $\mathbb{U}$ and satisfies the subordination relation

$$g(z)\left\{1-\lambda+\lambda[(1-\beta)P(z)+\beta]\right\}\prec 1+Mz,$$

then $\operatorname{Re}P(z)>0$ for $z\in\mathbb{U}$.

3. Subordination properties of $\mathcal{I}_p^\mu(n,\lambda)$

Unless otherwise mentioned, we assume throughout this paper that

$$-1\leq B<A\leq 1,\alpha\geq 0,\lambda>p,n\in\mathbb{N}_0,\ \text{and}\ p\in\mathbb{N}.$$

Theorem 3.1. *Let $\eta>0$ and $-1\leq B_j<A_j\leq 1,\ j=1,2$. If the functions $f_j\in\Sigma_p$ satisfy the following subordination condition:*

$$z^p\left\{(1-\eta)\mathcal{I}_p^\mu(n+1,\lambda)f_j(z)+\eta\mathcal{I}_p^\mu(n,\lambda)f_j(z)\right\}\prec\frac{1+A_jz}{1+B_jz},\ j=1,2,\tag{3.1}$$

then

$$z^p\left\{(1-\eta)\mathcal{I}_p^\mu(n+1,\lambda)F(z)+\eta\mathcal{I}_p^\mu(n,\lambda)F(z)\right\}\prec\frac{1+(1-2\delta)z}{1-z},$$

where $F = \mathcal{I}_p^\mu(n+1,\lambda)(f_1 * f_2)$ *and*

$$\delta = 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left(1 - \frac{1}{2}\ {}_2F_1\left(1, 1; \frac{\lambda - p}{\eta} + 1; \frac{1}{2}\right)\right). \tag{3.2}$$

The result is the best possible when $B_1 = B_2 = -1$.

Proof. Let the functions $f_j \in \Sigma_p,\ j = 1, 2$, satisfy the subordination condition (3.1). Then, by setting

$$\varphi_j(z) = z^p \left\{(1-\eta)\mathcal{I}_p^\mu(n+1,\lambda)f_j(z) + \eta\mathcal{I}_p^\mu(n,\lambda)f_j(z)\right\} \prec \frac{1 + A_j z}{1 + B_j z}, \quad j = 1, 2, \tag{3.3}$$

we have

$$\varphi_j \in P(\gamma_j), \quad \gamma_j = \frac{1 - A_j}{1 - B_j}, \quad j = 1, 2.$$

By making use of (1.4) and (3.3), we obtain

$$\mathcal{I}_p^\mu(n+1,\lambda)f_j(z) = \frac{\lambda - p}{\eta} z^{-p-\frac{\lambda-p}{\eta}} \int_0^z t^{\frac{\lambda-p}{\eta}-1}\varphi_j(t)dt,\ j = 1, 2. \tag{3.4}$$

Now, if we let $F = \mathcal{I}_p^\mu(n+1,\lambda)(f_1 * f_2)$, then by using (3.4) and the fact that

$$\begin{aligned} \mathcal{I}_p^\mu(n+1,\lambda)F(z) &= \mathcal{I}_p^\mu(n+1,\lambda)\mathcal{I}_p^\mu(n+1,\lambda)(f_1 * f_2)(z)) \\ &= \mathcal{I}_p^\mu(n+1,\lambda)f_1(z) * \mathcal{I}_p^\mu(n+1,\lambda)f_2(z), \end{aligned}$$

a simple computation shows that

$$\mathcal{I}_p^\mu(n+1,\lambda)F(z) = \frac{\lambda - p}{\eta} z^{-p-\frac{\lambda-p}{\eta}} \int_0^z t^{\frac{\lambda-p}{\eta}-1}\varphi_0(t)dt,$$

where

$$\begin{aligned} \varphi_0(z) &= z^p \left\{(1-\eta)\mathcal{I}_p^\mu(n+1,\lambda)F(z) + \eta\mathcal{I}_p^\mu(n,\lambda)F(z)\right\} \\ &= \frac{\lambda - p}{\eta} z^{-\frac{\lambda-p}{\eta}} \int_0^z t^{\frac{\lambda-p}{\eta}-1}(\varphi_1 * \varphi_2)(t)dt. \end{aligned} \tag{3.5}$$

Since $\varphi_j \in P(\gamma_j),\ j = 1, 2$, it follows from Lemma 2.3 that

$$\varphi_1 * \varphi_2 \in P(\gamma_3), \quad \textit{where} \quad \gamma_3 = 1 - 2(1 - \gamma_1)(1 - \gamma_2),$$

and the bound γ_3 is the best possible. Hence, by using Lemma 2.2 in (3.5), we deduce that

$$\begin{aligned}
\operatorname{Re} \varphi_0(z) &= \frac{\lambda-p}{\eta}\int_0^1 u^{\frac{\lambda-p}{\eta}-1}\operatorname{Re}(\varphi_1 * \varphi_2)(uz)du \\
&\geq \frac{\lambda-p}{\eta}\int_0^1 u^{\frac{\lambda-p}{\eta}-1}\left(2\gamma_3 - 1 + \frac{2(1-\gamma_3)}{1+u|z|}\right)du \\
&> \frac{\lambda-p}{\eta}\int_0^1 u^{\frac{\lambda-p}{\eta}-1}\left(2\gamma_3 - 1 + \frac{2(1-\gamma_3)}{1+u}\right)du \\
&= 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left(1 - \frac{\lambda-p}{\eta}\int_0^1 \frac{u^{\frac{\lambda-p}{\eta}-1}}{1+u}du\right) = \delta,
\end{aligned}$$

where δ is given by (3.2).

When $B_1 = B_2 = -1$, we consider the functions $f_j \in \Sigma_p$ $(j = 1, 2)$ which satisfy the hypothesis (3.1) and are given by

$$\mathcal{I}_p^{\mu}(n+1,\lambda)f_j(z) = \frac{\lambda-p}{\eta} z^{-p-\frac{\lambda-p}{\eta}} \int_0^z t^{\frac{\lambda-p}{\eta}-1}\left(\frac{1+A_j t}{1-t}\right)dt, \ j = 1, 2.$$

Since

$$\left(\frac{1+A_1 z}{1-z}\right) * \left(\frac{1+A_2 z}{1-z}\right) = 1 - (1+A_1)(1+A_2) + \frac{(1+A_1)(1+A_2)}{1-z},$$

it follows from (3.5) that

$$\begin{aligned}
\varphi_0(z) &= \frac{\lambda-p}{\eta}\int_0^1 u^{\frac{\lambda-p}{\eta}-1}\left(1 - (1+A_1)(1+A_2) + \frac{(1+A_1)(1+A_2)}{1-uz}\right)du \\
&= 1 - (1+A_1)(1+A_2) + \frac{(1+A_1)(1+A_2)}{(1-z)} \\
&\qquad\qquad \times \ {}_2F_1\left(1, 1; \frac{\lambda-p}{\eta}+1; \frac{z}{z-1}\right).
\end{aligned}$$

Therefore

$$\varphi_0(z) \to 1 - (1+A_1)(1+A_2) + \frac{1}{2}(1+A_1)(1+A_2)\ {}_2F_1\left(1, 1; \frac{\lambda-p}{\eta}+1; \frac{1}{2}\right)$$

as $z \to -1$, which evidently completes our proof of Theorem 3.1. □

By setting $\eta = 1, B_j = -1, A_j = 1 - 2\delta_j$ $j = 1, 2$, in Theorem 3.1, we have the following corollary.

Corollary 3.2. *If the functions $f_j \in \Sigma_p$ satisfy the following subordination condition:*

$$z^p \mathcal{I}_p^{\mu}(n,\lambda)f_j(z) \prec \frac{1+(1-2\delta_j)z}{1-z}, \ j = 1, 2, \tag{3.6}$$

then

$$\operatorname{Re}\left(z^p\mathcal{I}_p^{\mu}(n,\lambda)F(z)\right) > 1-2(1-\delta_1)(1-\delta_2) \times \left[2-\ {}_2F_1\left(1,1;\lambda-p+1;\frac{1}{2}\right)\right]\ (z\in\mathbb{U}),$$

where $F = I_p^{\mu}(n+1,\lambda)(f_1 * f_2)$.

The next theorem gives subordination property of the multiplier transformation $\mathcal{I}_p^{\mu}(n,\lambda)$ with respect to variation of the parameter μ.

Theorem 3.3. *Let* $\eta > 0$ *and* $-1 \le B_j < A_j \le 1,\ j = 1,2$. *If the functions* $f_j \in \Sigma_p$ *satisfy the following subordination condition:*

$$z^p\left\{(1-\eta)\mathcal{I}_p^{\mu}(n,\lambda)f_j(z)+\eta\mathcal{I}_p^{\mu+1}(n,\lambda)f_j(z)\right\} \prec \frac{1+A_jz}{1+B_jz},\ j=1,2,$$

then

$$z^p\left\{(1-\eta)\mathcal{I}_p^{\mu}(n,\lambda)F(z)+\eta\mathcal{I}_p^{\mu+1}(n,\lambda)F(z)\right\} \prec \frac{1+(1-2\delta_1)z}{1-z},$$

where $F = \mathcal{I}_p^{\mu}(n,\lambda)(f_1 * f_2)$ *and*

$$\delta_1 = 1-\frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left(1-\frac{1}{2}\ {}_2F_1\left(1,1;\frac{\mu}{\eta}+1;\frac{1}{2}\right)\right).$$

The result is the best possible when $B_1 = B_2 = -1$.

Proof. The proof is similar to that of Theorem 3.1, and so it is being omitted here. □

In the following Theorem 3.4, we have determined the sufficient condition for the functions $z^p\mathcal{I}_p^{\mu}(n,\lambda)f(z)$ to be a member of the class $P(\rho)$.

Theorem 3.4. *If* $f \in \Sigma_p$ *satisfy the following subordination condition:*

$$z^p\left\{(1-\eta)\mathcal{I}_p^{\mu}(n,\lambda)f(z)+\eta\mathcal{I}_p^{\mu+1}(n,\lambda)f(z)\right\} \prec \frac{1+Az}{1+Bz}, \tag{3.7}$$

then

$$\operatorname{Re}\left(z^p\mathcal{I}_p^{\mu}(n,\lambda)f(z)\right) > \rho \quad (z\in\mathbb{U}),$$

where

$$\rho = \begin{cases} \dfrac{A}{B}+\left(1-\dfrac{A}{B}\right)(1-B)^{-1}\ {}_2F_1\left(1,1;\dfrac{\mu}{\eta}+1;\dfrac{B}{B-1}\right), & \text{if } B\neq 0\\ 1-\dfrac{\mu}{\mu+\eta}A, & \text{if } B=0. \end{cases} \tag{3.8}$$

The result is the best possible.

Proof. Let

$$g(z) = z^p \mathcal{I}_p^{\mu}(n,\lambda) f(z) \quad for \quad f \in \Sigma_p. \tag{3.9}$$

Then the function g is of the form (2.1). Differentiating (3.9) with respect to z and using the identity (1.3), we obtain

$$z^p \mathcal{I}_p^{\mu+1}(n,\lambda) f(z) = g(z) + \frac{1}{\mu} z g'(z). \tag{3.10}$$

By using (3.7), (3.9), and (3.10), we get

$$g(z) + \frac{\eta}{\mu} z g'(z) \prec \frac{1+Az}{1+Bz}.$$

Now, by applying Lemma 2.1, we have

$$g(z) \prec Q(z) = \frac{\mu}{\eta} z^{-\frac{\mu}{\eta}} \int_0^z t^{\frac{\mu}{\eta}-1} \left(\frac{1+At}{1+Bt} \right) dt.$$

By applying Lemma 2.4, we get

$$Q(z) = \begin{cases} \dfrac{A}{B} + \left(\dfrac{1-\frac{A}{B}}{1+Bz} \right) {}_2F_1\left(1,1;\dfrac{\mu}{\eta}+1;\dfrac{Bz}{Bz+1}\right), & \text{if } B \neq 0 \\ 1 + \dfrac{\mu}{\mu+\eta} Az, & \text{if } B = 0. \end{cases} \tag{3.11}$$

Now, we will show that

$$\inf \ \{\mathrm{Re}\, Q(z) : |z| < 1\} = Q(-1). \tag{3.12}$$

We have

$$\mathrm{Re}\, \frac{1+Az}{1+Bz} \geq \frac{1-Ar}{1-Br} \quad |z| = r < 1,$$

and setting

$$h(s,z) = \frac{1+Azs}{1+Bzs} \quad (0 \leq s \leq 1) \quad and \quad d\mu(s) = \frac{\mu}{\eta} s^{\frac{\mu}{\eta}} ds,$$

which is a positive measure on the closed interval $[0,1]$, we get

$$Q(z) = \int_0^1 h(s,z) d\mu(s),$$

so that

$$\mathrm{Re}\, Q(z) \geq \int_0^1 \frac{1-Asr}{1-Bsr} d\mu(s) = Q(-r), \quad |z| = r < 1. \tag{3.13}$$

As $r \to 1^-$ in (3.13), we obtain the assertion (3.12). Now, by using (3.11) and (3.12), we get

$$\mathrm{Re}\ \left(z^p \mathcal{I}_p^{\mu}(n,\lambda) f(z) \right) > \rho$$

where ρ is given by (3.8).

To show the estimate (3.8) is the best possible, we consider the function $f \in \Sigma_p$ defined by

$$z^p \mathcal{I}_p^{\mu}(n,\lambda)f(z) = \frac{\mu}{\eta}\int_0^1 u^{\frac{\mu}{\eta}-1}\left(\frac{1+Auz}{1+Buz}\right)du.$$

For the above function, we find that

$$z^p\left\{(1-\eta)\mathcal{I}_p^{\mu}(n,\lambda)f(z)+\eta\mathcal{I}_p^{\mu+1}(n,\lambda)f(z)\right\} = \frac{1+Az}{1+Bz}$$

and

$$z^p \mathcal{I}_p^{\mu}(n,\lambda)f(z) \to \frac{\mu}{\eta}\int_0^1 u^{\frac{\mu}{\eta}-1}\left(\frac{1-Au}{1-Bu}\right)du$$

$$= \begin{cases} \dfrac{A}{B}+\left(1-\dfrac{A}{B}\right)(1-B)^{-1}\ {}_2F_1\left(1,1;\dfrac{\mu}{\eta}+1;\dfrac{B}{B-1}\right), & \text{if } B \neq 0 \\ 1-\dfrac{\mu}{\mu+\eta}A, & \text{if } B = 0. \end{cases}$$

as $z \to -1$, and the proof of the Theorem 3.4 is completed. □

In its special case when $A = 1-2\gamma, B = -1$ and $\eta = 1$, Theorem 3.4 yields the following corollary.

Corollary 3.5. *If $f \in \Sigma_p$ satisfy the following condition:*

$$z^p \mathcal{I}_p^{\mu+1}(n,\lambda)f(z) \prec \frac{1+(1-2\gamma)z}{1-z} \quad (z \in \mathbb{U}),$$

then

$$\operatorname{Re}\left(z^p \mathcal{I}_p^{\mu}(n,\lambda)f(z)\right) > \gamma + (1-\gamma)\left[\ {}_2F_1\left(1,1;\mu+1;\frac{1}{2}\right)-1\right] \quad (z \in \mathbb{U}).$$

The result is the best possible.

Theorem 3.6. *If $f \in \Sigma_p$ satisfy the following subordination condition:*

$$z^p\left\{(1-\eta)\mathcal{I}_p^{\mu}(n+1,\lambda)f(z)+\eta\mathcal{I}_p^{\mu}(n,\lambda)f(z)\right\} \prec \frac{1+Az}{1+Bz}, \tag{3.14}$$

then

$$\operatorname{Re}\left(z^p \mathcal{I}_p^{\mu}(n,\lambda)f(z)\right) > \rho_0 \quad (z \in \mathbb{U}),$$

where

$$\rho_0 = \begin{cases} \dfrac{A}{B}+\left(\dfrac{1-\frac{A}{B}}{1-B}\right)\ {}_2F_1\left(1,1;\dfrac{\lambda-p}{\eta}+1;\dfrac{B}{B-1}\right), & \textit{if } B \neq 0 \\ 1-\dfrac{\lambda-p}{\lambda-p+\eta}A, & \textit{if } B = 0. \end{cases}$$

The result is the best possible.

Proof. Let

$$h(z) = z^p \mathcal{I}_p^{\mu}(n+1,\lambda)f(z) \quad for \quad f \in \Sigma_p. \tag{3.15}$$

Then by using the hypothesis (3.14) together with (1.4) and (3.15), we obtain

$$h(z) + \frac{\eta}{\lambda - p} zh'(z) = (1-\eta)z^p \mathcal{I}_p^{\mu}(n+1,\lambda)f(z) + \eta \mathcal{I}_p^{\mu}(n,\lambda)f(z) \prec \frac{1+Az}{1+Bz}.$$

The remaining part of the proof of Theorem 3.6 is similar to that of Theorem 3.4 and hence, we omit the details. □

For a function $f \in \Sigma_p$, the integral operator $F_{c,p}$ is defined by

$$\begin{aligned} F_{c,p}f(z) &= \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t)dt \\ &= \left(\frac{{}_2F_1(1,c;c+1;z)}{z^p} \right) * f(z) \quad (c>0, z \in \mathbb{D}). \end{aligned} \tag{3.16}$$

Also, it is easily verified from (3.16) that

$$z\left(\mathcal{I}_p^{\mu}(n,\lambda)F_{c,p}f\right)'(z) = c\mathcal{I}_p^{\mu}(n,\lambda)f(z) - (c+p)\mathcal{I}_p^{\mu}(n,\lambda)F_{c,p}f(z). \tag{3.17}$$

In the next Theorem 3.7, by using the integral operator defined by (3.16), we establish sufficient condition for the functions $z^p\mathcal{I}_p^{\mu}(n,\lambda)F_{c,p}f(z)$ to belong to $P(\rho_1)$.

Theorem 3.7. *If $f \in \Sigma_p$ and $F_{c,p}f$ given by* (3.16)*, satisfies the subordination condition:*

$$z^p \left\{(1-\eta)\mathcal{I}_p^{\mu}(n,\lambda)F_{c,p}f(z) + \eta\mathcal{I}_p^{\mu}(n,\lambda)f(z)\right\} \prec \frac{1+Az}{1+Bz}, \tag{3.18}$$

then

$$\operatorname{Re}\left(z^p\mathcal{I}_p^{\mu}(n,\lambda)F_{c,p}f(z)\right) > \rho_1 \quad (z \in \mathbb{U}),$$

where

$$\rho_1 = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1-B)^{-1}\ {}_2F_1\left(1,1;\dfrac{c}{\eta}+1;\dfrac{B}{B-1}\right), & if\ B \neq 0 \\ 1 - \dfrac{c}{c+\eta}A, & if\ B = 0. \end{cases}$$

The result is the best possible.

Proof. Let

$$h(z) = z^p\mathcal{I}_p^{\mu}(n,\lambda)F_{c,p}f(z). \tag{3.19}$$

Then by using the hypothesis (3.18) together with (3.17) and (3.19), we obtain

$$h(z) + \frac{c}{\eta} zh'(z) = z^p \left\{(1-\eta)\mathcal{I}_p^{\mu}(n,\lambda)F_{c,p}f(z) + \eta I_p(n,\lambda)f(z)\right\} \prec \frac{1+Az}{1+Bz}.$$

The remaining part of the proof of Theorem 3.7 is similar to that of Theorem 3.4 and hence, we omit the details. □

Theorem 3.8. *If $f \in \Sigma_p$ and the function $F_{\mu,p}f$ defined by* (3.16), *satisfies*

$$-\frac{z^{p+1}}{p}\left\{(1-\eta)\left(\mathcal{I}_p^{\mu}(n,\lambda)F_{\mu,p}f\right)'(z)+\eta\left(\mathcal{I}_p^{\mu}(n,\lambda)f\right)'(z)\right\} \prec \frac{1+Az}{1+Bz},$$

then

$$-\operatorname{Re}\left(\frac{z^{p+1}}{p}\left(\mathcal{I}_p^{\mu}(n,\lambda)F_{\mu,p}f\right)'(z)\right) > \rho_1 \quad (z \in \mathbb{U}), \tag{3.20}$$

where ρ_1 is given as in Theorem 3.7. The result is the best possible.

Proof. Upon replacing $h(z)$ by $-\frac{z^{p+1}}{p}\left(\mathcal{I}_p^{\mu}(n,\lambda)F_{\mu,p}f\right)'(z)$ in (3.19) and using the same technique as in the proof of the Theorem 3.4, we can prove the assertion (3.20) of Theorem 3.8. □

Theorem 3.9. *Let $0 \neq \delta \in \mathbb{C}$ and $0 < \gamma \leq p$ such that either $|1+2\gamma\delta| \leq 1$ or $|1-2\gamma\delta| \leq 1$. If $f \in \Sigma_p$ satisfies*

$$\operatorname{Re}\left(\frac{\mathcal{I}_p^{\mu+1}(n,\lambda)f(z)}{\mathcal{I}_p^{\mu}(n,\lambda)f(z)}\right) < 1+\frac{\gamma}{\mu} \quad (z \in \mathbb{U}), \tag{3.21}$$

then

$$\left(z^p\mathcal{I}_p^{\mu}(n,\lambda)f(z)\right)^{\delta} \prec q(z) = (1-z)^{2\gamma\delta} \quad (z \in \mathbb{U})$$

and q is the best dominant.

Proof. Let

$$\phi(z) = \left(z^p\mathcal{I}_p^{\mu}(n,\lambda)f(z)\right)^{\delta} \quad (z \in \mathbb{U}) \tag{3.22}$$

and choose the principal branch in (3.22). We note that ϕ is analytic in $\mathbb{U}$ and $\phi(0) = 1$. Differentiating (3.22), we deduce that

$$-p+\frac{z\phi'(z)}{\delta\phi(z)} = \frac{z\mathcal{I}_p^{\mu+1}(n,\lambda)f(z)}{\mathcal{I}_p^{\mu}(n,\lambda)f(z)}. \tag{3.23}$$

Using (1.3) and (3.21) in (3.23), we get

$$-p+\frac{z\phi'(z)}{\delta\phi(z)} \prec \frac{-p+(p-2\gamma)z}{1-z}. \tag{3.24}$$

Define the functions θ and Φ by $\theta(z) = -p$, and $\Phi(z) = 1/\delta z$. Then θ and Φ are analytic in $\mathbb{C}\setminus\{0\}$ and $\Phi(z) \neq 0$. Letting $q(z) = (1-z)^{2\gamma\delta}$, by Lemma 2.5, q is univalent in $\mathbb{U}$ with q(0)=1. Since

$$Q(z) = zq'(z)\Phi(q(z)) = -\frac{2\gamma z}{1-z}$$

is starlike univalent in $\mathbb{U}$ with $Q(0) = 0$ and $Q'(0) \neq 0$,

$$h(z) = \frac{-p+(p-2\gamma)z}{1-z} \quad \text{and} \quad \operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \operatorname{Re}\left((1-z)^{-1}\right) > 0,$$

the functions Q and h satisfy the conditions of Lemma 2.7. Thus, the assertion of the Theorem 3.9 follows from (3.24) and Lemma 2.7. $\square$

4. Inclusion properties of the class $\Sigma_{p,n}^{\lambda,\mu}(\alpha; A, B)$

Theorem 4.1. *If* $f(z) \in \Sigma_{p,n}^{\lambda,\mu+1}(\alpha; A, B)$ *and*

$$(\mu + p - \alpha)(1 - B) - (p - \alpha)(1 - A) \geq 0, \tag{4.1}$$

then

$$-\frac{1}{p-\alpha}\left(\frac{z\left(\mathcal{I}_p^{\mu}(n,\lambda)f\right)'(z)}{\mathcal{I}_p^{\mu}(n,\lambda)f(z)} + \alpha\right) \prec q(z) \prec \frac{1+Az}{1+Bz}, \quad (z \in \mathbb{U}), \tag{4.2}$$

where

$$q(z) = \frac{1}{p-\alpha}\left((\mu + p - \alpha) - \frac{1}{Q(z)}\right),$$

$$Q(z) = \begin{cases} \displaystyle\int_0^1 t^{p+l-1}\left(\frac{1+Btz}{1+Bz}\right)^{-(p-\alpha)(A-B)/B} dt, & \text{if } B \neq 0 \\ \displaystyle\int_0^1 t^{p+l-1} exp(-A(p-\alpha)(t-1)z)dt, & \text{if } B = 0. \end{cases} \tag{4.3}$$

and $q(z)$ *is the best dominant of* (4.2). *If, in addition to* (4.1),

$$A < \frac{B(p+\mu+1-\alpha)}{p-\alpha} \quad \textit{with} \quad 0 < B < 1,$$

then

$$\Sigma_{p,n}^{\lambda,\mu+1}(\alpha; A, B) \subset \Sigma_{p,n}^{\lambda,\mu}(\alpha; 1-2\rho, -1) \tag{4.4}$$

where

$$\rho = \frac{1}{p-\alpha}\left((\mu + p - \alpha) - \mu\left[{}_2F_1\left(1, \frac{(p-\alpha)(A-B)}{B}; \mu+1; \frac{B}{B-1}\right)\right]^{-1}\right).$$

The bound on ρ *is the best possible.*

Proof. Let $f(z) \in \Sigma_{p,n}^{\lambda,\mu+1}(\alpha; A, B)$. Define the function g by

$$g(z) = z\left(z^p \mathcal{I}_p^{\mu}(n,\lambda)f(z)\right)^{-\frac{1}{p-\alpha}} \tag{4.5}$$

and $r_1 = \sup\{r : g(z) \neq 0, 0 < |z| < r < 1\}$. Then $g(z)$ is an analytic function in $|z| < r_1$. By logarithmic differentiation in (4.5), it follows that the function $\phi(z)$ given by

$$\phi(z) = \frac{zg'(z)}{g(z)} = -\frac{1}{p-\alpha}\left(\frac{z\left(\mathcal{I}_p^{\mu}(n,\lambda)f\right)'(z)}{\mathcal{I}_p^{\mu}(n,\lambda)f(z)} + \alpha\right) \prec \frac{1+Az}{1+Bz} \tag{4.6}$$

is analytic in $|z| < r_1$ and $\phi(0) = 1$. Using the identity (1.3) in (4.6) and logarithmic differentiation of the resulting equation yields the following:

$$-\frac{1}{p-\alpha}\left(\frac{z\left(\mathcal{I}_p^{\mu+1}(n,\lambda)f\right)'(z)}{\mathcal{I}_p^{\mu+1}(n,\lambda)f(z)}+\alpha\right) = \phi(z) + \frac{z\phi'(z)}{-(p-\alpha)\phi(z)+\mu+p-\alpha}$$
$$\prec \frac{1+Az}{1+Bz} \quad (|z| < r_1).$$

Hence, by using Lemma 2.6 with $\beta = \alpha - p$ and $\gamma = \mu + p - \alpha$, we find that

$$\phi(z) \prec \frac{1}{p-\alpha}\left((\mu+p-\alpha) - \frac{1}{Q(z)}\right) = q(z) \prec \frac{1+Az}{1+Bz} \quad (|z| < r_1), \tag{4.7}$$

where $q(z)$ is the best dominant of (4.7) and $Q(z)$ is given by (4.3). Since

$$\mathrm{Re}\left(\frac{1+Az}{1+Bz}\right) > 0 \quad (-1 \le B < A \le 1; z \in \mathbb{U}),$$

by (4.7), we have $\mathrm{Re}(\phi(z)) > 0$ $(|z| < r_1)$. Now (4.6) shows that $g(z)$ is starlike (univalent) in $|z| < r_1$. Thus it is not possible that $g(z)$ vanishes on $|z| = r_1$ if $r_1 < 1$. So we conclude that $r_1 = 1$, and therefore $\phi(z)$ is analytic in $\mathbb{U}$. Hence (4.7) implies that

$$\phi(z) \prec q(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}).$$

This proves the assertion (4.2) of Theorem 4.1.

In order to establish (4.4), we have to find the least upper bound of ρ $(0 < \rho < 1)$ such that

$$\phi(z) \prec \frac{1+(1-2\rho)z}{1-z} \quad (z \in \mathbb{U}).$$

By (4.7), we have to show that

$$\rho = \sup_{z\in\mathbb{U}} \mathrm{Re}(q(z)) = q(-1). \tag{4.8}$$

To prove (4.8), we need to show that

$$\inf_{z\in\mathbb{U}} \mathrm{Re}\left(\frac{1}{Q(z)}\right) = \frac{1}{Q(-1)}.$$

From (4.3), we see that, for $B \neq 0$,

$$Q(z) = (1+Bz)^a \int_0^1 t^{b-1}(1-t)^{c-b-1}(1+Bzt)^{-a}dt \quad (z \in \mathbb{U}),$$

where

$$a = \frac{(p-\alpha)(A-B)}{B}, \quad b = \mu, \quad c = \mu + 1.$$

Since $c > b > 0$, by using Lemma 2.4, we get the following:

$$\begin{aligned} Q(z) &= (1+Bz)^a \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} \; {}_2F_1(a,b;c;-Bz) \\ &= \frac{\Gamma(b)}{\Gamma(c)} \; {}_2F_1\left(a, c-b; c; \frac{Bz}{Bz+1}\right) \\ &= \frac{\Gamma(b)}{\Gamma(c)} \; {}_2F_1\left(1, a; c; \frac{Bz}{Bz+1}\right). \end{aligned} \tag{4.9}$$

Since

$$A < \frac{B(p+\mu+1-\alpha)}{p-\alpha} \quad with \quad 0 < B < 1,$$

implies that $c > a > 0$, by using Lemma 2.4, we find from (4.9) that

$$Q(z) = \int_0^1 h(z,t)dv(t),$$

where

$$h(z,t) = \frac{1+Bz}{1+(1-t)Bz} \; (0 \le t \le 1) \text{ and } dv(t) = \frac{\Gamma(b)t^{a-1}(1-t)^{c-a-1}}{\Gamma(a)\Gamma(c-a)}dt,$$

which is a positive measure on $[0,1]$. For $0 < B < 1$, it may be noted that $\operatorname{Re}(h(z,t)) > 0$ and $h(-r,t)$ is real for $0 \le |z| \le r < 1$ and $t \in [0,1]$. Hence, by using Lemma 2.8, we have

$$\operatorname{Re}\left(\frac{1}{Q(z)}\right) \ge \frac{1}{Q(-r)} \quad (|z| \le r < 1),$$

and

$$\inf_{z\in\mathbb{U}} \operatorname{Re}\left(\frac{1}{Q(z)}\right) = \inf_{-1<r<1} \frac{1}{Q(-r)} = \frac{1}{\int_0^1 h(-1,t)dv(t)} = \frac{1}{Q(-1)}.$$

We note that $Q(-1) \neq 0$. Thus, by using (4.8) and (4.9), we have

$$\rho = \frac{1}{p-\alpha}\left((\mu+p-\alpha) - \mu\left[{}_2F_1\left(1, \frac{(p-\alpha)(A-B)}{B}; \mu+1; \frac{B}{B-1}\right)\right]^{-1}\right),$$

when $A < \dfrac{B(p+\mu+1-\alpha)}{p-\alpha}$. Further by taking

$$A \to \left(\frac{B(p+\mu+1-\alpha)}{p-\alpha}\right)^+ \text{ for the case } A = \frac{B(p+\mu+1-\alpha)}{p-\alpha},$$

and using (4.2), we get (4.4). The result is the best possible as the function $q(z)$ is the best dominant of (4.2). This completes the proof of the Theorem 4.1. $\square$

5. Sufficient conditions for the class $\Sigma_{p,n}^{\lambda,\mu}(\alpha)$

Theorem 5.1. *Let $\eta > 0$ and if $f \in \Sigma_p$ such that $z^p \mathcal{I}_p^\mu(n,\lambda) f(z) \neq 0$, $z \in \mathbb{U}$, and satisfies the following differential subordination:*

$$(1-\eta)\left(z^p \mathcal{I}_p^\mu(n,\lambda) f(z)\right)^{-\sigma} + \frac{\eta}{p} z^{p+1} \left(-\mathcal{I}_p^\mu(n,\lambda) f(z)\right)' \left(z^p \mathcal{I}_p^\mu(n,\lambda) f(z)\right)^{-\sigma-1} \prec 1 + M_1 z, \tag{5.1}$$

where the powers are understood as the principle value, and

$$M_1 = \begin{cases} \dfrac{(p-\alpha)\eta\left(1+\frac{\eta}{\sigma p}\right)}{|p-(p-\alpha)\eta| + \sqrt{p^2 + \left(p+\frac{\eta}{\sigma}\right)^2}}, & \text{if } \sigma \neq 0 \\ \dfrac{(p-\alpha)\eta}{p}, & \text{if } \sigma = 0, \end{cases}$$

then $f \in \Sigma_{p,n}^{\lambda,\mu}(\alpha)$.

Proof. If $\sigma = 0$, then the condition (5.1) is equivalent to

$$\left| \frac{z\left(\mathcal{I}_p^\mu(n,\lambda) f\right)'(z)}{\mathcal{I}_p^\mu(n,\lambda) f(z)} + p \right| < p - \alpha \quad (z \in \mathbb{U}),$$

which implies that $f \in \Sigma_{p,n}^{\lambda,\mu}(\alpha)$.

Now we consider $\sigma > 0$ and suppose that

$$g(z) = \left(z^p \mathcal{I}_p^\mu(n,\lambda) f(z)\right)^{-\sigma} \quad (z \in \mathbb{D}). \tag{5.2}$$

Choosing the principal value in (5.2), we note that g is of the form (2.1) and is analytic in $\mathbb{U}$. Differentiating (5.2) with respect to z, we obtain

$$\begin{aligned} g(z) + \frac{\eta}{\sigma p} z g'(z) &= (1-\eta)\left(z^p \mathcal{I}_p^\mu(n,\lambda) f(z)\right)^{-\sigma} + \frac{\eta}{p} z^{p+1}\left(-\mathcal{I}_p^\mu(n,\lambda) f(z)\right)' \\ &\qquad \times \left(z^p \mathcal{I}_p^\mu(n,\lambda) f(z)\right)^{-\sigma-1} \end{aligned}$$

which, in view of Lemma 2.1 with $c = \sigma p/\eta$, yields

$$g(z) \prec 1 + \frac{\sigma p}{\sigma p + \eta} M_1 z.$$

Also, with the aid of (5.2), (5.1) can be written as follows:

$$g(z)\left\{1 - \eta + \eta\left[\left(1 - \frac{\alpha}{p}\right) P(z) + \frac{\alpha}{p}\right]\right\} \prec 1 + M_1 z$$

where P is given by

$$P(z) = -\frac{1}{p-\alpha}\left(\frac{z\left(\mathcal{I}_p^{\mu}(n,\lambda)f\right)'(z)}{\mathcal{I}_p^{\mu}(n,\lambda)f(z)} + \alpha\right) \quad (0 \leq \alpha < p), \tag{5.3}$$

Therefore, by Lemma 2.9, we find that

$$\mathrm{Re}\, P(z) > 0 \quad (z \in \mathbb{U}),$$

that is

$$-\mathrm{Re}\, \frac{z\left(\mathcal{I}_p^{\mu}(n,\lambda)f\right)'(z)}{\mathcal{I}_p^{\mu}(n,\lambda)f(z)} > \alpha \quad (0 \leq \alpha < p, z \in \mathbb{U}),$$

which completes the proof of Theorem 5.1. □

Theorem 5.2. *If $f \in \Sigma_p$ satisfies the following subordination condition:*

$$z^p\left\{(1-\eta)\mathcal{I}_p^{\mu}(n,\lambda)f(z) + \eta\mathcal{I}_p^{\mu+1}(n,\lambda)f(z)\right\} \prec 1 + M_2 z, \tag{5.4}$$

where

$$M_2 = \frac{\eta(\alpha-p)\left(1+\frac{\eta}{\mu}\right)}{|\mu-\eta(\alpha-p)| + \sqrt{\mu^2+(\mu+\eta)^2}},$$

then $f \in \Sigma_{p,n}^{\lambda;\mu}(\alpha)$.

Proof. Let

$$g(z) = z^p\mathcal{I}_p^{\mu}(n,\lambda)f(z). \tag{5.5}$$

Then the function g is of the form (2.1) and is analytic in $\mathbb{U}$. From Theorem 3.4 with $A = M_1$, and $B = 0$, we have

$$g(z) \prec 1 + \frac{\mu}{\mu+\eta}M_2 z,$$

which is equivalent to

$$|g(z) - 1| < \frac{\mu}{\mu+\eta}M_2 = N < 1 \quad (z \in \mathbb{U}). \tag{5.6}$$

By using the identity (1.3) followed by (5.5), we obtain

$$z^p\mathcal{I}_p^{\mu+1}(n,\lambda)f(z) = \left(1 - \frac{(\alpha-p)}{\mu} + \frac{(\alpha-p)}{\mu}P(z)\right)g(z), \tag{5.7}$$

where $P(z)$ is given by (5.3). In view of (5.7), the hypothesis (5.4) can be written as follows:

$$\left|\left(1 - \frac{\eta(\alpha-p)}{\mu}\right)g(z) + \frac{\eta(\alpha-p)}{\mu}P(z)g(z) - 1\right| < M_2 \quad (z \in \mathbb{U}). \tag{5.8}$$

We need to show that (5.8) yields

$$\mathrm{Re}\, P(z) > 0 \quad (z \in \mathbb{U}). \tag{5.9}$$

Suppose that this is false. Since $P(0)=1$, there exists a point $z_0 \in \mathbb{U}$ such that $P(z_0)=ix$ for some $x \in \mathbb{R}$. Therefore, in order to show that (5.9), it is sufficient to obtain the contradiction from the inequality

$$E=\left|\left(1-\frac{\eta(\alpha-p)}{\mu}\right)g(z_0)+\frac{\eta(\alpha-p)}{\mu}P(z_0)g(z_0)-1\right|\geq M_2. \tag{5.10}$$

If we let $g(z_0)=u+iv$, then by using (5.6) and the triangle inequality, we obtain that

$$\begin{aligned}
E^2 &= \left|\left(1-\frac{\eta(\alpha-p)}{\mu}\right)g(z_0)+\frac{\eta(\alpha-p)}{\mu}P(z_0)g(z_0)-1\right|^2\\
&=(u^2+v^2)\left(\frac{\eta x(\alpha-p)}{\mu}\right)^2+\frac{2v\eta x(\alpha-p)}{\mu}+\left|\left(1-\frac{\eta(\alpha-p)}{\mu}\right)g(z_0)-1\right|^2\\
&\geq (u^2+v^2)\left(\frac{\eta x(\alpha-p)}{\mu}\right)^2+\frac{2v\eta x(\alpha-p)}{\mu}\\
&\qquad\qquad +\left(\frac{\eta(\alpha-p)}{\mu}-\left(1-\frac{\eta(\alpha-p)}{\mu}\right)N\right)^2....
\end{aligned}$$

If we let

$$\begin{aligned}
\Psi(x) &= E^2-M_1^2\\
&\geq (u^2+v^2)\left(\frac{\eta x(\alpha-p)}{\mu}\right)^2+\frac{2v\eta x(\alpha-p)}{\mu}\\
&\qquad +\left(\frac{\eta(\alpha-p)}{\mu}-\left|1-\frac{\eta(\alpha-p)}{\mu}\right|N\right)^2-N^2\left(\frac{\mu+\eta}{\mu}\right)^2,
\end{aligned}$$

then (5.10) holds true if $\Psi(x)\geq 0$, for any $x\in\mathbb{U}$. Since $(u^2+v^2)\left(\frac{\eta(\alpha-p)}{\mu}\right)^2>0$, the inequality $\Psi(x)\geq 0$ holds true if the discriminant $\Delta\leq 0$, that is,

$$\begin{aligned}
\Delta = 4\left(\frac{\eta(\alpha-p)}{\mu}\right)^2 &\left\{v^2-(u^2+v^2)\left[\left(\frac{\eta(\alpha-p)}{\mu}-\left|1-\frac{\eta(\alpha-p)}{\mu}\right|N\right)^2\right.\right.\\
&\qquad\left.\left.-N^2\left(\frac{\mu+\eta}{\mu}\right)^2\right]\right\}\leq 0,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&v^2\left\{1-\left[\left(\frac{\eta(\alpha-p)}{\mu}-\left|1-\frac{\eta(\alpha-p)}{\mu}\right|N\right)^2+N^2\left(\frac{\mu+\eta}{\mu}\right)^2\right]\right\}\\
&\qquad\leq u^2\left[\left(\frac{\eta(\alpha-p)}{\mu}-\left|1-\frac{\eta(\alpha-p)}{\mu}\right|N\right)^2-N^2\left(\frac{\mu+\eta}{\mu}\right)^2\right].
\end{aligned}$$

After a simple computation, by using (5.6) we obtain the inequality

$$\frac{v^2}{u^2} \le \frac{\rho^2}{1-\rho^2} \le \frac{N^2}{1-N^2} \le \frac{\left(\frac{\eta(\alpha-p)}{\mu} - \left|1-\frac{\eta(\alpha-p)}{\mu}\right| N\right)^2 - N^2\left(\frac{\mu+\eta}{\mu}\right)^2}{1-\left[\left(\frac{\eta(\alpha-p)}{\mu} - \left|1-\frac{\eta(\alpha-p)}{\mu}\right| N\right)^2 + N^2\left(\frac{\mu+\eta}{\mu}\right)^2\right]},$$

which yields $\Delta \le 0$. Therefore $E \ge M_1$, which contradicts (5.8). It follows that Re $P(z) > 0$, and $f \in \Sigma_{p,n}^{\lambda;\mu}(\alpha)$. □

Acknowledgments

The authors thank the referees for their insightful comments. The first author work in this present investigation was supported by UGC New Delhi, project No: (MRP-5591/15 (SERO/UGC)).

References

[1] R. M. Ali, V. Ravichandran and N. Seenivasagan, On subordination and superordination of the multiplier transformation for meromorphic functions, *Bull. Malays. Math. Sci. Soc. (2)* **33** (2010), no. 2, 311–324.
[2] T. M. Flett, The dual of an inequality of Hardy and Littlewood and some related inequalities, *J. Math. Anal. Appl.* **38** (1972) 746–765.
[3] M. P. Jeyaraman and T. K. Suresh, *J. Complex Anal.* (2014), Article ID: 187482, 12 pages.
[4] M. S. Liu, On certain sufficient condition for starlike functions, *Soochow J. Math.* **29** (2003), no. 4, 407–412.
[5] M. S. Liu and H. M. Srivastava, A linear operator and associated families of meromorphically multivalent functions, *J. Math. Anal. Appl.* **259** (2001), no. 2, 566–581.
[6] J. L. Liu and H. M. Srivastava, Classes of meromorphically multivalent functions associated with the generalized hypergeometric function, *Math. Comput. Modelling* **39** (2004), no. 1, 21–34.
[7] S. S. Miller and P. T. Mocanu, Differential subordinations and univalent functions, *Michigan Math. J.* **28** (1981), no. 2, 157–171.
[8] S. S. Miller and P. T. Mocanu, On some classes of first-order differential subordinations, *Michigan Math. J.* **32** (1985), no. 2, 185–195.
[9] S. S. Miller and P. T. Mocanu, Univalent solution of Briot-Bouquet differential subordinations, *J. Differential Equations* **56** (1985), no. 3, 297–309.
[10] S. S. Miller and P. T. Mocanu, Differential Subordination: Theory and Applications, Marcel Dekker, Inc., New York, 2000.
[11] D. Z. Pashkouleva, The starlikeness and spiral-convexity of certain subclass of analytic, Current Topics in Analytic Function Theory, 266–273, World Sci. Publ., River Edge, 1992.
[12] J. Patel and Ashis Ku. Palit, On certain subclasses of meromorphically multivalent functions associated with the generalized hypergeometric function, *JIPAM. J. Inequal. Pure Appl. Math.* **10** (2009), no. 1, 1–17.
[13] M. S. Robertson, Certain classes of starlike functions, *Michigan Math. J.* **32** (1985), no. 2, 134–140.

[14] H. M. Srivastava and J. Patel, Applications of differential subordination to certain subclasses of meromorphically multivalent functions, *JIPAM. J. Inequal. Pure Appl. Math.* **6** (2005), no. 3, 1–15.

[15] J. Stankiewicz and Z. Stankiewicz, Some applications of the Hadamard convolution in the theory of functions, *Ann. Univ. Mariae Curie-Sklodwska Sect. A* **40** (1986) 251–265.

[16] B. A. Uralegaddi and C. Somanatha, New criteria for meromorphic starlike univalent functions, *Bull. Austral. Math. Soc.* **43** (1991), no. 1, 137–140.

[17] B. A. Uralegaddi and C. Somanatha, Certain differential operators for meromorphic functions, *Houston Journal of Mathematics*, **17** (1991) 279–284.

[18] E. T. Whittaker and G. N. Watson, A course on Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions with an Account of the Principal Transcendental Functions, Fourth Edition, Cambridge Univ. Press, Cambridge, 1996.

[19] D. R. Wilken and J. Feng, A remark on convex and starlike functions, *J. London Math. Soc. (2)* **21** (1980), no. 2, 287–290.

(M. P. Jeyaraman) Department of Mathematics, L. N. Government College, Ponneri, Chennai, 601- 204, Tamilnadu, India
E-mail address: `jeyaraman_mp@yahoo.co.in`

(T. K. Suresh) Department of Mathematics, Easwari Engineering College, Ramapuram, Chennai, 600-089, Tamilnadu, India
E-mail address: `tksuresh73@yahoo.com`