Title:

On uniqueness of meromorphic functions sharing five small functions on annuli

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ON UNIQUENESS OF MEROMORPHIC FUNCTIONS
SHARING FIVE SMALL FUNCTIONS ON ANNULI

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Abstract. The purpose of this article is to investigate the uniqueness of meromorphic functions sharing five small functions on annuli.

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1. Introduction and results

We assume that the reader is familiar with Nevanlinna’s theory of meromorphic functions (for references, see [5,15]). The uniqueness of meromorphic functions in the complex plane \( \mathbb{C} \) is an important subject in the value distribution theory (see [16]). We say that two meromorphic functions \( f \) and \( g \) share the value \( a \) \( ( a \in \mathbb{C} = \mathbb{C} \cup \{ \infty \} ) \) in \( X \subset \mathbb{C} \) provided that in \( X \), we have \( f(z) = a \) if and only if \( g(z) = a \). We will state whether a shared value is by \( CM \) (counting multiplicities) or by \( IM \) (ignoring multiplicities).

In 1926, Nevanlinna [14] proved the following well-known five value theorem.

**Theorem A** ([14]). Let \( f \) and \( g \) be two nonconstant meromorphic functions in the complex plane \( \mathbb{C} \), \( a_j \in \mathbb{C} \) \( (j=1,2,3,4,5) \) be five distinct values. If \( f \) and \( g \) share the values \( a_j \) \( (j=1,2,3,4,5) \) \( IM \) in \( \mathbb{C} \), then \( f \equiv g \).

After his very work, the uniqueness theory of meromorphic functions in \( \mathbb{C} \) attracted many investigations (for references, see [16]). For the uniqueness of meromorphic functions in the unit disc, refer to [4]. In [19], J.H. Zheng suggested first time to investigate the uniqueness of meromorphic functions in a precise subset of \( \mathbb{C} \) and posed the following question.

**Question 1.1.** Under what conditions, must two meromorphic functions on \( X(\neq \mathbb{C}) \) be identical?
It is an interesting topic how to extend some important uniqueness results in the complex plane to an angular domain. In 2003, J. H. Zheng firstly took into account the uniqueness of meromorphic functions sharing values in an angular domain and extended five value theorem in the complex plane to an angular domain (see [19, 20]).

**Theorem B** ([21]). Let $f$ and $g$ be two nonconstant meromorphic functions in an angular domain $\Omega(\alpha, \beta) = \{z : \alpha < \arg z < \beta\}$ ($0 < \beta - \alpha < 2\pi$), and

$$\limsup_{r \to \infty} \frac{T_{\alpha, \beta}(r, f)}{\log r} = \infty.$$

If $f$ and $g$ share five distinct values $a_j$ ($j = 1, 2, 3, 4, 5$) IM in $\Omega(\alpha, \beta)$, then $f \equiv g$.

The Nevanlinna five distinct value theorem has been extended in [11, 17] to the case of five IM shared small functions; see the following result.

**Theorem C** ([11, 17]). Let $f$ and $g$ be two nonconstant meromorphic functions in complex plane $\mathbb{C}$, and $\alpha_j$ ($j = 1, 2, 3, 4, 5$) be five distinct small functions with respect to $f$ and $g$. If $f$ and $g$ share $\alpha_j$ ($j = 1, 2, 3, 4, 5$) IM in $\mathbb{C}$, then $f(z) \equiv g(z)$.

Also, we may raise the following natural question:

**Question 1.2.** What is the analogous result for Theorem C in one angular domain?

In [12], H. F. Liu and Z. Q. Mao firstly extended five small functions theorem in the complex plane to an angular domain.

**Theorem D** ([12]). Let $f$ and $g$ be two nonconstant meromorphic functions in an angular domain $\Omega(\alpha, \beta)(0 < \beta - \alpha < 2\pi)$ such that

$$\limsup_{r \to \infty} \frac{T_{\alpha, \beta}(r, f)}{\log r} = \infty$$

and let $\alpha_j$ ($j = 1, 2, 3, 4, 5$) be five distinct small functions with respect to $f$ and $g$ in $\Omega(\alpha, \beta)$. If $f$ and $g$ share $\alpha_j$ ($j = 1, 2, 3, 4, 5$) IM in $\Omega(\alpha, \beta)$, then $f \equiv g$.

However, all the above cases take place in simply connected domains. Thus it is very interesting to consider the uniqueness theory of meromorphic functions in doubly connected domains.

Here we shall mainly study the uniqueness of meromorphic functions in doubly connected domains of the complex plane $\mathbb{C}$. By the Doubly Connected Mapping Theorem [1] each doubly connected domain is conformally equivalent to an annulus $\{z : r < |z| < R\}$, $0 \leq r < R \leq +\infty$. We consider only two cases: $r = 0$, $R = +\infty$ simultaneously and $0 < r < R < +\infty$. In the latter case, the homothety $z \mapsto \frac{z}{\sqrt{rR}}$ reduces the given domain to the annulus $\{z : \frac{1}{R_0} < |z| < R_0\}$, where $R_0 = \sqrt{\frac{r}{R}}$. Thus, in both cases every annulus is invariant with respect to the inversion $z \mapsto \frac{1}{z}$. Hence in this paper, we consider the uniqueness of meromorphic functions on the annulus $\Lambda = \{z : \frac{1}{R_0} < |z| < R_0\}$.
\[ R_0 \}, \text{ where } 1 < R_0 \leq +\infty. \] We denote by \( S \) a subset of distinct elements in \( \mathbb{C} = \mathbb{C} \cup \{\infty\} \). For a function \( f \) meromorphic in \( \Lambda \), we define
\[
E(S, f) = \bigcup_{a \in S} \{z \in \Lambda : f(z) - a = 0, \ \text{ counting multiplicity}\},
\]
\[
\overline{E}(S, f) = \bigcup_{a \in S} \{z \in \Lambda : f(z) - a = 0, \ \text{ ignoring multiplicity}\}.
\]

The Nevanlinna characteristic \( T_0(r, f) \) of a meromorphic function \( f \) on the annulus \( \Lambda \) shall be introduced in the next section.

Let \( f \) be a nonconstant meromorphic function on the annulus \( \Lambda = \{z : \frac{1}{R_0} < |z| < R_0\} \), where \( 1 < R_0 \leq +\infty \). The function \( f \) is called a transcendental or admissible meromorphic function on the annulus \( \Lambda \) provided that
\[
\limsup_{r \to \infty} \frac{T_0(r, f)}{\log r} = \infty, \quad 1 \leq r < R_0 = +\infty
\]
or
\[
\limsup_{r \to R_0} \frac{T_0(r, f)}{-\log(R_0 - r)} = \infty, \quad 1 \leq r < R_0 < +\infty.
\]

In 2009, T. B. Cao, H. X. Yi and H. Y. Xu \[2\] proved a general theorem on the multiple values and uniqueness of meromorphic functions in the annulus \( \Lambda \).

**Theorem E** (\[2\]). Let \( f \) and \( g \) be two transcendental or admissible meromorphic functions on the annulus \( \Lambda = \{z : \frac{1}{R_0} < |z| < R_0\} \), where \( 1 < R_0 \leq +\infty \).

Let \( a_j (j = 1, 2, 3, 4, 5) \) be five distinct complex numbers in \( \mathbb{C} \). If \( \overline{E}(a_j, f) = \overline{E}(a_j, g) \) for \( j = 1, 2, 3, 4, 5 \), then \( f(z) \equiv g(z) \).

Let \( f \) and \( g \) be two meromorphic functions on the annulus \( \Lambda \) satisfying: if \( T_0(r, \alpha) = o(T_0(r, f)) \) as \( r \to \infty \), possibly outside a set \( E \) of finite linear measure for \( R_0 = +\infty \) or \( T_0(r, \alpha) = o(T_0(r, f)) \), as \( r \to R_0 \) possibly outside a set \( E \) with \( \int_E dr/(R_0 - r) < \infty \) for \( R_0 < +\infty \).

The following is the question we consider in this paper.

**Question 1.3.** Do \( f \) and \( g \) coincide if \( \overline{E}(a_j, f) = \overline{E}(a_j, g) \) for \( j = 1, 2, 3, 4, 5 \) on the annulus \( \Lambda \), where \( a_j (j = 1, 2, 3, 4, 5) \) are five distinct small functions with respect to \( f \) and \( g \).

Dealing with the above question, we obtain the following results which give an affirmative answer to Question 1.3.

**Theorem 1.1.** Let \( f \) and \( g \) be two transcendental or admissible meromorphic functions on the annulus \( \Lambda = \{z : \frac{1}{R_0} < |z| < R_0\} \), where \( 1 < R_0 \leq +\infty \). Let \( \alpha_j (j = 1, 2, 3, 4, 5) \) be five distinct small functions with respect to \( f \) and \( g \) on the annulus \( \Lambda \). If \( \overline{E}(\alpha_j, f) = \overline{E}(\alpha_j, g) \) for \( j = 1, 2, 3, 4, 5 \), then \( f(z) \equiv g(z) \).
We complete the proof of Theorem 1.1 with the help of the method in Yi [17, 18] and Liu and Mao [12, 13]. These papers investigate the uniqueness of meromorphic functions sharing small functions in the complex plane or in the angular domains.

2. Preliminaries and Some Lemmas

Let \( f \) be a meromorphic function on the annulus \( \Lambda = \{ z : \frac{1}{R_0} < |z| < R_0 \} \), where \( 1 < R_0 \leq +\infty \). We recall the classical notations of Nevanlinna theory as follows:

\[
N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} \, dt + n(0, f) \log r,
\]

\[
m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta,
\]

\[
T(r, f) = N(r, f) + m(r, f),
\]

where \( \log^+ x = \max\{\log x, 0\} \), and \( n(t, f) \) is the counting function of poles of function \( f \) in \( \{ z : |z| \leq t \} \). Here we give the notations of the Nevanlinna theory on annuli. Let

\[
N_1(r, f) = \int_{\frac{1}{R_0}}^1 n_1(t, f) \, dt,
\]

\[
N_2(r, f) = \int_1^r n_2(t, f) \, dt,
\]

\[
m_0(r, f) = m(r, f) + m\left(\frac{1}{r}, f\right) - 2m(1, f),
\]

\[
N_0(r, f) = N_1(r, f) + N_2(r, f),
\]

where \( n_1(t, f) \) and \( n_2(t, f) \) are the counting functions of poles of function \( f \) in \( \{ z : t < |z| \leq 1 \} \) and \( \{ z : 1 < |z| \leq t \} \), respectively. Set

\[
N_0(r, \frac{1}{1-a}) = N_1(r, \frac{1}{1-a}) + N_2(r, \frac{1}{1-a})
\]

\[
= \int_{\frac{1}{1-a}}^{\frac{1}{R_0}} \frac{n_1(t, \frac{1}{1-a})}{t} \, dt + \int_{1}^{r} \frac{n_2(t, \frac{1}{1-a})}{t} \, dt.
\]

in which each zero of the function \( f - a \) is counted only once. The Nevanlinna characteristic of \( f \) on the annulus \( \Lambda \) is defined by

\[
T_0(r, f) = m_0(r, f) + N_0(r, f).
\]

Throughout, we denote by \( S(r, \ast) \) quantities satisfying

(i) in the case \( R_0 = \infty \),

\[
S(r, \ast) = O(\log(rT_0(r, \ast)))
\]

for \( r \in (1, +\infty) \) except for the set \( \Delta_r \) such that \( \int_{\Delta_r} r^{\lambda-1} < +\infty \);

(ii) if \( R_0 < \infty \), then

\[
S(r, \ast) = O(\log\left(\frac{T_0(r, \ast)}{R_0-r}\right))
\]

for \( r \in (1, R_0) \) except for the set \( \Delta'_r \) such that \( \int_{\Delta'_r} \frac{dr}{(R_0-r)^{\ast}} < +\infty \);
Thus for an admissible meromorphic function on the annulus, $S(r, f) = o(T_0(r, f))$ holds for all $1 \leq r < R_0$ except for the set $\Delta_r$ or the set $\Delta_r'$ mentioned above, respectively.

**Lemma 2.1** ([7,9]). Let $f$ be a nonconstant meromorphic function on the annulus $\Lambda = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 \leq r < R_0 \leq +\infty$. Then

(i) $T_0(r, f) = T_0(r, \frac{1}{f-a})$,

(ii) $\max\{T_0(r, f_1), T_0(r, f_2), T_0(r, f_1 f_2), T_0(r, f_1 + f_2)\} \leq T_0(r, f_1) + T_0(r, f_2) + O(1)$.

By Lemma 2.1, the first fundamental theorem on the annulus is immediately obtained.

**Lemma 2.2** ([7,9] The first fundamental theorem). Let $f$ be a nonconstant meromorphic function on the annulus $\Lambda = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 \leq r < R_0 \leq +\infty$. Then

$$T_0(r, \frac{1}{f-a}) = T_0(r, f) + O(1)$$

for every fixed $a \in \mathbb{C}$.

**Lemma 2.3** ([8,9] The lemma of the logarithmic derivative). Let $f$ be a nonconstant meromorphic function on the annulus $\Lambda = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 \leq r < R_0 \leq +\infty$. Then

$$m_0(r, F^{(k)}) \leq S(r, f)$$

for every $k \in \mathbb{N}$.

Khrystiyanyn and Kondratyuk also obtained the second fundamental theorem on the annulus $\Lambda$. We show here the reduced form due to Cao, Yi and Xu.

**Lemma 2.4** ([2] The second fundamental theorem). Let $f$ be a nonconstant meromorphic function on the annulus $\Lambda = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 \leq r < R_0 \leq +\infty$. Let $a_1, a_2, \ldots, a_q$ be $q$ distinct complex numbers in $\bar{\mathbb{C}}$. Then

$$(q - 2)T_0(r, f) < \sum_{j=1}^q N_0(r, \frac{1}{f-a_j}) + S(r, f).$$

**Lemma 2.5** ([3]). Let $f$ be a nonconstant meromorphic function on the annulus $\Lambda$, $P_1(f)$ and $P_2(f)$ be two mutually prime polynomials in $f$ with degrees $m$ and $n$ respectively. Then

$$T_0(r, \frac{P_1(f)}{P_2(f)}) = \max\{m, n\} T_0(r, f) + S(r, f).$$
Lemma 2.6. Let $f$ be a nonconstant meromorphic function on the annulus \( \Lambda = \{ z : \frac{1}{r_1} < |z| < r_0 \} \), where \( 1 \leq r < R_0 \leq +\infty \). Let \( \beta_j (j = 1, 2, 3) \) be small functions with respect to $f$ on the annulus $\Lambda$. Then

\[
T_0(r, f) \leq \sum_{j=1}^{3} N_0(r, \frac{1}{f - \beta_j}) + S(r, f) + o(T_0(r, f)).
\]

Proof. Set

\[ F(z) = \frac{f(z) - \beta_1(z)}{f(z) - \beta_2(z)} \frac{\beta_3(z) - \beta_2(z)}{\beta_3(z) - \beta_1(z)}. \]

Then combining Lemma 2.4 and Lemma 2.5, we obtain the result. \( \square \)

Lemma 2.7. \( \{10\} \). Let $g : (0, \infty) \to R$ and $h : (0, \infty) \to R$ be monotone nondecreasing functions such that $g(r) \leq h(r)$ outside an exceptional set $E$ of finite linear measure. Then for any $\alpha > 1$, there exists $r_0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.

Lemma 2.8. \( \{6\} \). Let $h_1(r)$ and $h_2(r)$ be monotonically increasing and real valued functions on $[0, R_0)$ such that $h_1(r) \leq h_2(r)$ possibly outside an exceptional set $E \subset [0, R_0)$, for which $\int_E \frac{dr}{R_0 - r} < \infty$. Then there exists a constant $b \in (0, R_0)$ such that if $s(r) = 1 - b(1 - r)$, then $h_1(r) \leq h_2(s(r))$ for all $r \in [0, R_0)$.

3. Proof of Theorem 1.1

The idea of the proof is from \( \{17, 18\} \) and \( \{12, 13\} \).

Suppose that $f \neq g$. Set

\[ L(w) = \frac{\omega - \alpha_1}{\omega - \alpha_2} \frac{\alpha_3 - \alpha_2}{\alpha_3 - \alpha_1}. \]

Let $F(z) = L(f(z)), G(z) = L(g(z)), \beta_j = L(\alpha_j), (j = 1, \ldots, 5)$. By (3.1) and Lemma 2.5, we get $\beta_1 = 0, \beta_2 = \infty, \beta_3 = 1, \beta_4 = 0, \infty, 1, \beta_5 = 0, \infty, 1, \beta_4 \neq \beta_5$, and $\beta_1, \ldots, \beta_5$ are small functions with respect to $F$ and $G$. By the assumption of Theorem 1.1 and (3.1), we know that $F$ and $G$ share $0, 1, \infty$ IM.

Then by the second fundamental theorem on $\Lambda$, we get

\[
T_0(r, F) \leq N_0(r, \frac{1}{F}) + N_0(r, \frac{1}{F - 1}) + N_0(r, F) + S(r, F)
\]

(3.2)

\[
\leq N_0(r, \frac{1}{G}) + N_0(r, \frac{1}{G - 1}) + N_0(r, G) + S(r, F)
\]

\[
\leq 3T_0(r, G) + S(r, F).
\]

Similarly, we have

(3.3)

\[ T_0(r, G) \leq 3T_0(r, F) + S(r, G). \]

Hence by (3.2) and (3.3), we get

(3.4)

\[ S(r, F) = S(r, G). \]
We claim that at least three among $\mathcal{N}_0(r, \frac{1}{F - \beta}) (j = 1, 2, 3, 4, 5)$ are not equal to $S(r, F) + o(T_0(r, F))$. Otherwise, by Lemma 2.6, we get
\begin{equation}
T_0(r, F) \leq S(r, F) + o(T_0(r, F)).
\end{equation}
By (3.5) and Lemma 2.7 and Lemma 2.8, we get
\[ \limsup_{r \to \infty} \frac{T_0(r, f)}{\log r} < \frac{1}{1 - r}, \quad 1 \leq r < R_0 = +\infty, \]
or
\[ \limsup_{r \to R_0} \frac{T_0(r, f)}{-\log(R_0 - r)} < \frac{1}{1 - r}, \quad 1 \leq r < R_0 < +\infty. \]
which contradicts $f$ is transcendental or admissible.
Without loss of generality, we assume that
\begin{equation}
N_0(r, \frac{1}{F - \beta}) \neq S(r, F) + o(T_0(r, F)).
\end{equation}
Set
\begin{equation}
H = \frac{F'(\beta_4 G - \beta_4 G')(F - G)}{F(F - 1)G(G - 1) - G'(\beta_4 F - \beta_4 F')(F - G)}.
\end{equation}
Then by (3.7), we get
\begin{equation}
H = \frac{(F - G)H_1}{F(F - 1)(F - \beta_4)G(G - 1)(G - \beta_4)},
\end{equation}
where
\begin{equation}
H_1 = \frac{F'(\beta_4 G - \beta_4 G')(G - 1)(F - \beta_4) - G'(\beta_4 F - \beta_4 F')(F - 1)(G - \beta_4)}{G(G - 1)F(F - \beta_4)}.
\end{equation}
Noting that $f \neq g$, by (3.1), we have
\begin{equation}
F \neq G.
\end{equation}
We discuss the following two cases.
\textbf{Case 1.} $H \equiv 0$. By (3.7) and (3.10), we get
\begin{equation}
\frac{F'(\beta_4 G - \beta_4 G')}{(F - 1)(G - \beta_4)} = \frac{G'(\beta_4 F - \beta_4 F')}{(G - 1)(F - \beta_4)}.
\end{equation}
If $\beta_4$ is a constant, then by $\beta_4 \neq 1$ and (3.11), we get $F \equiv G$, which contradicts (3.10). So $\beta_4$ is not a constant. By (3.11), we get
\[ \frac{F'(\beta_4 G - \beta_4 G')}{G'(\beta_4 F - \beta_4 F')} - 1 = \frac{(F - 1)(G - \beta_4)}{(G - 1)(F - \beta_4)} - 1. \]
Hence we get

\[(3.12) \quad \frac{F' - G'}{F - G} = \frac{(1 - \beta_4)G'(\beta'_4 F - \beta_4 F')}{\beta_4 G(G - 1)(F - \beta_4)} + \frac{G'}{G} \]

By (3.6), we know that there is a point \( z_0 \) such that \( z_0 \) is a common zero of \( F - \beta_5 \) and \( G - \beta_5 \), but is not a zero or a pole of \( \beta_4, \beta'_4, \beta_5, \beta_5 - 1, \beta_5 - \beta_4 \). It is obvious that \( z_0 \) is a pole of the left side of (3.12), and not a pole of the right side of (3.12), which is a contradiction.

**Case 2.** \( H \not\equiv 0 \). By (3.7), we get

\[(3.13) \quad H = \frac{F'}{F - 1} \cdot \frac{\beta'_4 G - \beta_4 G'}{G(G - \beta_4)} - \frac{(\frac{F'}{F - 1} \cdot \frac{F'}{F})}{F - \beta_4} \cdot \frac{\beta'_4 G - \beta_4 G'}{G - \beta_4} \]
\[-\left(\frac{G'}{G - 1} - \frac{G'}{G}\right) \cdot \frac{\beta'_4 F - \beta_4 F'}{F - \beta_4} + \frac{G'}{G - 1} \cdot \frac{\beta'_4 F - \beta_4 F'}{F - \beta_4}. \]

Since
\[(3.14) \quad \frac{\beta'_4 G - \beta_4 G'}{G(G - \beta_4)} = \frac{G'}{G(G - \beta_4)} - \frac{\beta'_4 G}{G(G - \beta_4)} = \frac{\beta'_4 G - \beta_4 G'}{G - \beta_4}, \]
then by Lemma 2.3 and (3.4), we get
\[(3.15) \quad m_0(r, \frac{\beta'_4 G - \beta_4 G'}{G(G - \beta_4)}) \leq m_0(r, \frac{G'}{G}) + m_0(r, \frac{G'}{G - \beta_4}) = S(r, F) + o(T_0(r, F)), \]
\[(3.16) \quad m_0(r, \frac{\beta'_4 G - \beta_4 G'}{G - \beta_4}) \leq m_0(r, \frac{\beta'_4 G - \beta_4 G'}{G - \beta_4}) = S(r, F) + o(T_0(r, F)). \]

Combining (3.13), (3.15) and (3.16), we get
\[(3.17) \quad m_0(r, H) = S(r, F) + o(T_0(r, F)). \]

Next we estimate \( N_0(r, H) \). By (3.7), we know that the poles of \( H \) only possibly occur from the zeros of \( F, G, F - 1, G - 1, F - \beta_4 \) and \( G - \beta_4 \), the poles of \( F, G \) and \( \beta_4 \). Let \( E_0 \) be the set of all zeros, 1-points and poles of \( \beta_4 \). We discuss the following four subcases.

**Subcase 1.** Suppose that \( z_1 \) is a zero of \( F \) with multiplicity \( m_1 \) and \( G \) with multiplicity \( n_1 \), but \( z_1 \not\in E_0 \). Then by (3.9), we know that \( z_1 \) is a zero of \( H_1 \) with multiplicity at least \( m_1 + n_1 - 1 \). Noting that \( z_1 \) is a zero of \( F - G \) with multiplicity \( \min\{m_1, n_1\} \), by (3.8), we deduce that \( z_1 \) is not a pole of \( H \).

**Subcase 2.** Suppose that \( z_2 \) is a pole of \( F \) with multiplicity \( m_2 \) and \( G \) with multiplicity \( n_2 \), but \( z_2 \not\in E_0 \). Then by (3.9), we know that \( z_2 \) is a pole of \( H_1 \) with multiplicity at most \( 2m_2 + 2n_2 - 1 \). Noting that \( z_2 \) is a pole of \( F - G \) with multiplicity at most \( \max\{m_2, n_2\} \), by (3.8), we deduce that \( z_2 \) is not a pole of \( H \).
Subcase 3. Suppose that $z_3$ is a zero of $F - 1$ with multiplicity $m_3$ and $G - 1$ with multiplicity $n_3$, but $z_3 \notin E_0$. Noting that $z_3$ is a zero of $F - G$ with multiplicity $\min\{m_3, n_3\}$, a simple pole of $F_e'/(F - 1)$ and $G_e'/G - 1$, by (3.7), we deduce that $z_3$ is not a pole of $H$.

Subcase 4. Suppose that $z_4$ is a zero of $F - \beta_4$ with multiplicity $m_4$ and $G - \beta_4$ with multiplicity $n_4$, but $z_4 \notin E_0$. By (3.14), we know that $z_4$ is a simple pole of $F'(G - \beta_4)e/(G - \beta_4)$ and $G'(F - \beta_4)e/(F - \beta_4)$. Noting that $z_4$ is a zero of $F - G$, by (3.7), we deduce that $z_4$ is not a pole of $H$.

From the above, we get

$$N_0(r, H) = o(T_0(r, F)).$$

Thus by (3.17) and (3.18), we get

$$T_0(r, H) = S(r, F) + o(T_0(r, F)).$$

Since $F$ and $G$ share $\beta_5$ IM, by (3.7) and (3.19), we get

$$N_0(r, \frac{1}{F - \beta_5}) \leq T_0(r, \frac{1}{H}) \leq S(r, F) + o(T_0(r, F)),$$

which contradicts (3.6). Theorem 1.1 is now completely proved.

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References

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