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STOCHASTIC FUNCTIONAL POPULATION DYNAMICS WITH JUMPS

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Abstract. In this paper we use a class of stochastic functional Kolmogorov-type model with jumps to describe the evolutions of population dynamics. By constructing a special Lyapunov function, we show that the stochastic functional differential equation associated with our model admits a unique global solution in the positive orthant, and, by the exponential martingale inequality with jumps, we discuss the asymptotic pathwise estimation of such a model.
Keywords: Kolmogorov-type population dynamics, jumps, exponential martingale inequality with jumps, asymptotic pathwise estimation.
MSC(2010): Primary: 60H15; Secondary: 60H30.

1. Introduction

Predator-prey behavior is a very common biological interaction in nature. To describe such behavior, one of the most general mathematical models is the n-dimensional Kolmogorov-type system on $\mathbb{R}^n$

$$\mathrm{d}X(t) = \mathrm{diag}(X_1(t), \cdots, X_n(t))f(X(t))\mathrm{d}t,$$

where $X = (X_1, \cdots, X_n)^T, f = (f_1, \cdots, f_n)^T$, $X_i(t)$ represents the population size of species i at time t and $f_i(X(t))$ denotes the inherent net growth rate of the i-th species, depending on the population size of each species.

From the practical point of view, the most realistic model should include some hereditary characteristics such as after-effect, time-lag and time-delay appearing in the variables, where functional differential equations give a mathematical formulation for such systems. Tang and Kuang [13] discussed the permanence of n-species Kolmogorov-type functional differential system

$$\mathrm{d}X_i(t) = X_i(t)f_i(X_t)\mathrm{d}t, \quad i = 1, \cdots, n. \tag{1.1}$$

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For more details on the asymptotical stability, periodic solution, extinction and permanence of (1.1), we refer the reader to, e.g., [6,12,13].

However, the deterministic models assume that parameters in the systems are all deterministic and irrespective of environmental fluctuations, while, from the biological point of view, there are some limitations in mathematical modeling of ecological systems. Moreover, population dynamics in the real world is affected inevitably by environmental noise. Wu and Hu [16] stochastically perturbed system (1.1) and discussed the following stochastic functional Kolmogorov-type population dynamics

$$\mathrm{d}X(t) = \mathrm{diag}(X_1(t), \cdots, X_n(t))[F(X_t)\mathrm{d}t + G(X_t)\mathrm{d}W(t)].$$

We should also point out that Mao, Marion, and Renshaw [9] formulated a proper starting point to investigate stochastic n-dimensional Lotka-Volterra system, and revealed that the environmental noise can suppress a potential population explosion. For some recent progress with respect to stochastic delay population dynamics and hybrid Lotka-Volterra ecosystems, we refer to, e.g., [2,11,15,17,18].

Furthermore, the population may suffer sudden environmental shocks, e.g., earthquakes, hurricanes, epidemics, etc. However, model (1.1) cannot explain the phenomena above. To explain these phenomena, introducing a jump process into underlying population dynamics is one of the important methods. So in this paper we introduce the following *stochastic functional Kolmogorov-type population dynamics with jumps*

(1.2)
$$\begin{cases} \mathrm{d}X(t) = \mathrm{diag}(X_1(t), \cdots, X_n(t))\Big[F(X_t)\mathrm{d}t + \int_{\mathbb{Y}} H(X_{t^-}, u)\tilde{N}(\mathrm{d}t, \mathrm{d}u)\Big] \\ X_0 = \xi \in D^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n_+) \end{cases}$$

to describe the evolutions of population dynamics.

We will first introduce some definitions and notation. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a probability space, where $\mathcal{F}$ is a σ-algebra on the outcome space $\Omega, \mathbb{P}$ is a probability measure on the measurable space $(\Omega, \mathcal{F})$, and $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration of sub-σ-algebra of $\mathcal{F}$, where the usual conditions are satisfied, i.e., $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets and $\mathcal{F}_t = \mathcal{F}_{t_+} := \bigcap_{s>t} \mathcal{F}_s$. Recall that a path $X : [-\tau, 0] \to \mathbb{R}^n$ is called càdlàg if it is right-continuous having finite left-hand limits, where $\tau \in (0, \infty)$ is referred to as the delay or memory. Let $\mathscr{D} := D([-\tau, 0]; \mathbb{R}^n)$ stand for the family of all $\mathbb{R}^n$-valued càdlàg paths on $[-\tau, 0]$, endowed with the Skorokhod topology. For a càdlàg function $X : [-\tau, \infty) \to \mathbb{R}^n$ and $t \geq 0$, let $X_t \in \mathscr{D}$ be such that $X_t(\theta) = X(t + \theta), \theta \in [-\tau, 0]$. $D^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n_+)$ means the family of all bounded, $\mathcal{F}_0$-measurable, $D([-\tau, 0]; \mathbb{R}^n_+)$-valued random variables, where $\mathbb{R}^n_+ := \{x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n | x_i > 0\}$. For (1.2), $N(\mathrm{d}t, \mathrm{d}u)$ is a real-valued Poisson counting measure with characteristic measure λ on measurable subset

$\mathbb{Y}$ of $[0,\infty)$ such that $\int_{\mathbb{Y}}(1\wedge u^2)\lambda(\mathrm{d}u)<\infty$, $\tilde{N}(\mathrm{d}t,\mathrm{d}u):=N(\mathrm{d}t,\mathrm{d}u)-\lambda(\mathrm{d}u)\mathrm{d}t$, $F:D([-\tau,0];\mathbb{R}^n)\to\mathbb{R}^n$ and $H:D([-\tau,0];\mathbb{R}^n)\times\mathbb{Y}\to\mathbb{R}^n$.

For population dynamics with jumps, we refer to Bao et al. [3] and Bao and Yuan [4], where both F and H are independent of the past history, and, in particular, are linear in [3].

In reference to the existing results in the literature, our contributions are as follows:

- We use stochastic functional Kolmogorov-type model with jumps to describe the evolutions of population dynamics, which suffer sudden environmental shocks;
- We show by the classical Lyapunov function argument that stochastic functional differential equation associated with our model admits a unique global solution in the positive orthant. In particular, our established theory also demonstrates that jump processes can also be applied to suppress solution explosion for functional differential equations.
- We verify by the exponential martingale inequality with jumps that population size is at most of polynomial growth almost surely.

2. **Global positive solution**

Since the vector $X=(X_1,X_2,\cdots,X_n)^T$ denotes the population sizes of the n interacting species, it is natural to require the solution of (1.2) not only to be positive but also not to explode in finite time. Therefore, in this section we intend to claim that (1.2) has a unique global solution in the positive orthant.

Throughout the paper, we impose the following assumptions.

(H1) There exists $L_k>0$ such that

$$|F(\varphi)-F(\phi)|^2+\int_{\mathbb{Y}}|H(\varphi,u)-H(\phi,u)|^2\lambda(\mathrm{d}u)\le L_k\|\varphi-\phi\|_\infty^2,$$

where $\varphi,\phi\in D([-\tau,0];\mathbb{R}^n)$ with $\|\varphi\|_\infty\vee\|\phi\|_\infty\le k$.

(H2) $H_i(\varphi,u)>-1$ for $\varphi\in D([-\tau,0];\mathbb{R}_+^n)$, $u\in\mathbb{Y}$ and $i=1,\cdots,n$.

Since the coefficients don't satisfy linear growth condition or weak coercivity condition, e.g., [14, Theorem 2.3], though they satisfy local Lipschitz condition, the solutions of (1.2) may explode in finite time. Khasminskii [5, Theorem 4.1, p. 85] gave a Lyapunov function argument, which is a powerful test for nonexplosion of solutions and generalized to delay SDEs in, e.g., [7,8] . In the sequel, we shall follow similar arguments to those of [5,7,8] to show that (1.2) has a unique global positive solution $\{X(t)\}_{t\ge0}$ under appropriate conditions.

To this end, we further assume that, for some $p\in(0,1)$ and any $\varphi\in D([-\tau,0];\mathbb{R}_+^n)$, there exist $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2,\delta_j>0, j=1,\cdots,5$, where $\alpha_1\ge\alpha_2,\beta_1>\beta_2$, $\gamma_1\ge\gamma_2$ and $\beta_1>\max\{p,\alpha_1,\gamma_1\}$, and probability measures ρ_1,ρ_2,ρ_3 on $[-\tau,0]$ such that

(H3) $|F(\varphi)| \le \delta_1\Big(1 + |\varphi(0)|^{\alpha_1} + \int_{-\tau}^0 |\varphi(\theta)|^{\alpha_2}\rho_1(\mathrm{d}\theta)\Big)$;

(H4) $J_i(\varphi,p) := \int_{\mathbb{Y}}[(1+H_i(\varphi,u))^p - 1 - pH_i(\varphi,u)]\lambda(\mathrm{d}u) \le \delta_2 - \delta_3|\varphi(0)|^{\beta_1} + \delta_4\int_{-\tau}^0 |\varphi(\theta)|^{\beta_2}\rho_2(\mathrm{d}\theta)$;

(H5) $\bar{J}_i(\varphi,p) := \int_{\mathbb{Y}}[H_i(\varphi,u) - \ln(1+H_i(\varphi,u))]\,\lambda(\mathrm{d}u) \le \delta_5\Big(1 + |\varphi(0)|^{\gamma_1} + \int_{-\tau}^0 |\varphi(\theta)|^{\gamma_2}\rho_3(\mathrm{d}\theta)\Big)$.

Theorem 2.1. Under (**H1**)-(**H5**), for any initial condition $\xi \in D^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n_+)$, (1.2) has a unique global solution $\{X(t)\}_{t\ge 0} \in \mathbb{R}^n_+$ a.s.

Proof. Since both the drift term and the jump-diffusion term associated with (1.2) are locally Lipschitzian, for any initial condition $\xi \in D^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n_+)$, (1.2) admits by a standard truncation argument, e.g., [10, Theorem 3.4, p. 56], a unique local solution $X(t), t \in [0,\tau_e)$, where τ_e is the explosion time defined by $\tau_e := \inf\{t > 0 : |X(t)| = \infty\}$. To show that $X(t)$ is not only positive but also global, for some $k_0 > 0$ such that $\|\xi\|_\infty < k_0$ and each $k > k_0$, define a stopping time

$$\tau_k := \inf\{t \in (0,\tau_e) : X_i(t) \notin (1/k,k), \text{ for some } i = 1,2,\cdots,n\}.$$

Due to the fact that τ_k is increasing as $k \uparrow \infty$, the limit $\tau_\infty := \lim_{k\to\infty}\tau_k$ exists and $\tau_\infty \le \tau_e$ a.s. Thus, to show that the local solution $X(t)$ on the time interval $[0,\tau_e)$ is positive and global, it is sufficient to verify $\tau_\infty = \infty$. Define the standard Lyapunov function

$$V(x) := \sum_{i=1}^n (x_i^p - 1 - p\ln x_i),\ x \in \mathbb{R}^n_+,\ p \in (0,1). \tag{2.1}$$

For arbitrary $T > 0$, applying the Itô formula yields that

$$\mathbb{E}V(X(\tau_k \wedge T)) = V(X(0)) + \mathbb{E}\int_0^{\tau_k\wedge T}\mathcal{L}V(X_t)\mathrm{d}t,$$

where, for $\varphi \in D([-\tau,0];\mathbb{R}^n_+)$,

$$\mathcal{L}V(\varphi) := p\sum_{i=1}^n(\varphi_i^p(0) - 1)F_i(\varphi) + \sum_{i=1}^n J_i(\varphi,p)\varphi_i^p(0) + p\sum_{i=1}^n \bar{J}_i(\varphi,p).$$

Recall the fundamental inequalities

$$n^{(1-\frac{q}{2})\wedge 0}|x|^q \le \sum_{i=1}^n x_i^q \le n^{(1-\frac{q}{2})\vee 0}|x|^q, \quad x \in \mathbb{R}^n_+, \quad \forall q > 0, \tag{2.2}$$

and

$$x^\kappa y^{1-\kappa} \le \kappa x + (1-\kappa)y, \quad x,y \in \mathbb{R}_+, \quad \kappa \in (0,1). \tag{2.3}$$

Thus, (**H3**), (**H4**)and (**H5**), together with (2.2) and (2.3), give that for any $t \le \tau_k \wedge T$

(2.4)

$$\begin{aligned}
\int_0^t \mathcal{L}V(X_s)\mathrm{d}s \le{}& p\int_0^t \sum_{i=1}^n (X_i^p(s)+1)|F(X_s)|\mathrm{d}s \\
&+ \int_0^t \sum_{i=1}^n \Big(\delta_2 - \delta_3|X(s)|^{\beta_1} + \delta_4 \int_{-\tau}^0 |X(s+\theta)|^{\beta_2}\rho_2(\mathrm{d}\theta)\Big)X_i^p(s)\mathrm{d}s \\
&+ n\delta_5 \int_0^t \Big(1+|X(s)|^{\gamma_1} + \int_{-\tau}^0 |X(s+\theta)|^{\gamma_2}\rho_3(\mathrm{d}\theta)\Big)\mathrm{d}s \\
&\le C\int_0^t (1+|X(s)|^p)\Big(1+|X(s)|^{\alpha_1} + \int_{-\tau}^0 |X(s+\theta)|^{\alpha_2}\rho_1(\mathrm{d}\theta)\Big)\mathrm{d}s \\
&+ \int_0^t C|X(s)|^p + \sum_{i=1}^n \Big(-\delta_3|X(s)|^{\beta_1} + \delta_4\int_{-\tau}^0 |X(s+\theta)|^{\beta_2}\rho_2(\mathrm{d}\theta)\Big)X_i^p(s)\mathrm{d}s \\
&+ n\delta_5 \int_0^t \Big(1+|X(s)|^{\gamma_1} + \int_{-\tau}^0 |X(s+\theta)|^{\gamma_2}\rho_3(\mathrm{d}\theta)\Big)\mathrm{d}s \\
&\le \int_0^t \Big\{ -\delta_3|X(s)|^{p+\beta_1} + C(1+|X(s)|^p + |X(s)|^{\gamma_1} \\
&+ |X(s)|^{p+\alpha_1} + |X(s)|^{p+\beta_2})\Big\}\mathrm{d}s =: \int_0^t \tilde{J}(s)\mathrm{d}s,
\end{aligned}$$

where we have also utilized

$$\int_0^t \int_{-\tau}^0 |X(s+\theta)|^\beta \rho(\mathrm{d}\theta)\mathrm{d}s \le \int_{-\tau}^0 |X(\theta)|^\beta \mathrm{d}\theta + \int_0^t |X(s)|^\beta \mathrm{d}s$$

for any constant $\beta > 0$ and some probability measure ρ. On the other hand, since the leading term of polynomial $\tilde{J}(s)$ is negative, there exists $K > 0$ such that $\tilde{J}(s) \le K$. Hence it follows that

$$\mathbb{E}V(X(\tau_k \wedge T)) \le V(X(0)) + KT. \tag{2.5}$$

Define for each $u > 0$

$$\mu(u) := \inf\{V(x) : x_i \ge u \text{ or } x_i \le \frac{1}{u} \text{ for some } i = 1, 2, \cdots, n\}.$$

Note that $\mu(u) \to \infty$ as $u \to \infty$. Then we obtain from (2.5) that

$$\mu(k)\mathbb{P}(\tau_k \le T) \le \mathbb{E}(V(X(\tau_k))I_{\tau_k \le T}) \le \mathbb{E}V(X(\tau_k \wedge T)) \le V(X(0)) + KT.$$

Letting $k \to \infty$ yields

$$\mathbb{P}(\tau_\infty \le T) = 0.$$

Since T is arbitrary, we must have

$$\mathbb{P}(\tau_\infty = \infty) = 1$$

and (1.2) admits a unique global solution $\{X(t)\}_{t\ge 0} \in \mathbb{R}_+^n$. □

Next we construct an example to demonstrate that Theorem 2.1 is applicable.

Example 2.2. Consider a stochastic population dynamics model with jumps on $\mathbb{R}$

$$\begin{aligned} \mathrm{d}X(t) &= X(t)\Big[(a+bX^\alpha(t)+cX^\alpha(t-\tau))\mathrm{d}t \\ &\quad + \int_0^\infty \Big\{|X(t-)|^\beta + \int_{-\tau}^0 |X((t+\theta)-)|^\gamma \mathrm{d}\theta\Big\} u\tilde{N}(\mathrm{d}t,\mathrm{d}u)\Big], \end{aligned} \tag{2.6}$$

where $a,b,c\in\mathbb{R}, \alpha,\gamma>1$, $\beta>\alpha\vee\gamma$, $\int_0^\infty(1\vee u)\lambda(\mathrm{d}u)<\infty$, and $\int_0^\infty u^2\lambda(\mathrm{d}u)<\infty$. For arbitrary $\varphi\in D([-\tau,0];\mathbb{R}_+)$ and $u\in(0,\infty)$, let

$$F(\varphi) := a+b\varphi^\alpha(0)+c\varphi^\alpha(-\tau) \text{ and } H(\varphi,u) := \Big(|\varphi(0)|^\beta + \int_{-\tau}^0 |\varphi(\theta)|^\gamma \mathrm{d}\theta\Big)u.$$

Then (2.6) can be written in the framework of (1.2). By the elementary inequality $(a+b)^2\le 2a^2+2b^2$, for any $\|\varphi\|_\infty\vee\|\phi\|_\infty\le k$, we get that there exists $L_k>0$ such that

$$\begin{aligned} &|F(\varphi)-F(\phi)|^2+\int_0^\infty |H(\varphi,u)-H(\phi,u)|^2\lambda(\mathrm{d}u) \\ &= |b(\varphi^\alpha(0)-\phi^\alpha(0))+c(\varphi^\alpha(-\tau)-\phi^\alpha(-\tau))|^2 \\ &\quad + \int_0^\infty \Big|u(|\varphi(0)|^\beta-|\phi(0)|^\beta)+u\int_{-\tau}^0(|\varphi(\theta)|^\gamma-\phi(\theta)|^\gamma)\mathrm{d}\theta\Big|^2\lambda(\mathrm{d}u) \\ &\le 2b^2|\varphi^\alpha(0)-\phi^\alpha(0)|^2+2c^2|\varphi^\alpha(-\tau)-\phi^\alpha(-\tau)|^2 \\ &\quad + 2\int_0^\infty u^2\lambda(\mathrm{d}u)|\varphi^\beta(0)-\phi^\beta(0)|^2 + 2\int_0^\infty u^2\lambda(\mathrm{d}u)\Big|\int_{-\tau}^0(|\varphi(\theta)|^\gamma-\phi(\theta)|^\gamma)\mathrm{d}\theta\Big|^2 \\ &\le L_k\|\varphi-\phi\|_\infty^2. \end{aligned}$$

So (**H1**) holds. It is trivial to see that (**H2**) is true due to $u\in(0,\infty)$, and (**H3**) holds with $\delta_1=|a|\vee|b|\vee|c|$, $\alpha_1=\alpha_2=\alpha$ and ρ_1 being the Dirac measure on the point $-\tau$. Moreover, for arbitrary $\varphi\in D([-\tau,0];\mathbb{R}_+)$ and $p\in(0,1)$, observe by the elemental inequality $(a+b)^\kappa\le a^\kappa+b^\kappa, a,b\ge0,\kappa\in(0,1)$, that

$$\begin{aligned} J(\varphi,p) &:= \int_0^\infty\{(1+H(\varphi,u))^p-1-pH(\varphi,u)\}\lambda(\mathrm{d}u) \\ &= \int_0^\infty\Big\{\Big(1+\Big(|\varphi(0)|^\beta+\int_{-\tau}^0|\varphi(\theta)|^\gamma\mathrm{d}\theta\Big)u\Big)^p-1-p\Big(|\varphi(0)|^\beta+\int_{-\tau}^0|\varphi(\theta)|^\gamma\mathrm{d}\theta\Big)u\Big\}\lambda(\mathrm{d}u) \\ &\le \int_0^\infty\Big\{\Big(|\varphi(0)|^{p\beta}+\Big(\int_{-\tau}^0|\varphi(\theta)|^\gamma\mathrm{d}\theta\Big)^p\Big)u^p-p\Big(|\varphi(0)|^\beta+\int_{-\tau}^0|\varphi(\theta)|^\gamma\mathrm{d}\theta\Big)u\Big\}\lambda(\mathrm{d}u). \end{aligned}$$

By virtue of $\beta > \gamma > 0$, we choose $\delta \in (1, \beta/\gamma)$. Applying the Young inequality (2.3) and the Hölder inequality leads to

$$\Big(\int_{-\tau}^{0} \varphi^{\gamma}(\theta)d\theta\Big)^{p} = \Big(\Big(\int_{-\tau}^{0} \varphi^{\gamma}(\theta)d\theta\Big)^{\delta}\Big)^{p/\delta} \le \varepsilon\tau^{1/\delta-1}\Big(\int_{-\tau}^{0} \varphi^{\gamma}(\theta)d\theta\Big)^{\delta} + C$$
$$\le \varepsilon\int_{-\tau}^{0} \varphi^{\delta\gamma}(\theta)d\theta + C, \quad \varepsilon \in (0,1).$$

Hence

$$J(\varphi,p) \le -p\int_0^\infty u\lambda(\mathrm{d}u)|\varphi(0)|^{\beta} + \int_0^\infty (1+Cu^p)\lambda(\mathrm{d}u) + \int_0^\infty u^p\lambda(\mathrm{d}u)|\varphi(0)|^{p\beta}$$
$$+ \varepsilon\int_0^\infty u^p\lambda(\mathrm{d}u)\int_{-\tau}^{0}\varphi^{\delta\gamma}(\theta)\mathrm{d}\theta,$$

and therefore (**H4**) follows. Note that for $x > 0$, we have $\ln(1+x) > 0$, thus

$$\begin{aligned}\bar{J}(\varphi,p) &:= \int_0^\infty [H(\varphi,u) - \ln(1+H(\varphi,u))]\,\lambda(\mathrm{d}u)\\ &= \int_0^\infty \Big\{\Big(|\varphi(0)|^{\beta} + \int_{-\tau}^{0}|\varphi(\theta)|^{\gamma}\mathrm{d}\theta\Big)u\\ &\quad - \ln\Big[1 + \Big(|\varphi(0)|^{\beta} + \int_{-\tau}^{0}|\varphi(\theta)|^{\gamma}\mathrm{d}\theta\Big)u\Big]\Big\}\,\lambda(\mathrm{d}u)\\ &\le \int_0^\infty u\lambda(\mathrm{d}u)|\varphi(0)|^{\beta} + \int_0^\infty u\lambda(\mathrm{d}u)\int_{-\tau}^{0}|\varphi(\theta)|^{\gamma}\mathrm{d}\theta.\end{aligned}$$

Then (**H5**) follows due to $\beta > \gamma$.

Note that (**H4**) excludes the case $\beta_2 = \beta_1$. For such case, we replace (**H4**) by the following one: There exist $\beta > \alpha_1 \vee \gamma_1$, $\delta_2', \delta_3', \delta_4' > 0$ with $\delta_3' > \delta_4'$ and probability measure ρ_2' on $[-\tau, 0]$ such that for $\varphi \in D([-\tau,0];\mathbb{R}_+^n)$

(**H**4′) $\int_{\mathbb{Y}}[(1+H_i(\varphi,u))^p - 1 - pH_i(\varphi,u)]\lambda(\mathrm{d}u) \le \delta_2' - \delta_3'\varphi_i^{\beta}(0) + \delta_4'\int_{-\tau}^{0}\varphi_i^{\beta}(\theta)\rho_2'(\mathrm{d}\theta)$.

Theorem 2.3. Under (**H1**)-(**H3**), (**H**4′) and (**H5**), for any initial condition $\xi \in D^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}_+^n)$, (1.2) has a unique global solution $\{X(t)\}_{t\ge 0} \in \mathbb{R}_+^n$ a.s.

Proof. Since the proof is very similar to that of Theorem 2.1, we here only give an outline to point out the corresponding differences. Note from (2.2) and (2.3) that

$$\begin{aligned}
&\sum_{i=1}^n\int_0^t\Big(-\delta_3'X_i^{p+\beta}(s)+\delta_4'X_i^p(s)\int_{-\tau}^0X_i^\beta(s+\theta)\rho_2(\mathrm{d}\theta)\Big)\mathrm{d}s\\
&\le\sum_{i=1}^n\int_0^t\Big(-\delta_3'X_i^{p+\beta}(s)+\frac{p\delta_4'}{p+\beta}X_i^{p+\beta}(s)\Big)\mathrm{d}s+\frac{\beta\delta_4'}{p+\beta}\sum_{i=1}^n\int_0^t\int_{-\tau}^0X_i^{p+\beta}(s+\theta)\rho_2(\mathrm{d}\theta)\mathrm{d}s\\
&\le\sum_{i=1}^n\int_0^t\Big(-\delta_3'X_i^{p+\beta}(s)+\frac{p\delta_4'}{p+\beta}X_i^{p+\beta}(s)\Big)\mathrm{d}s+\frac{\beta\delta_4'}{p+\beta}\sum_{i=1}^n\int_{-\tau}^0\int_{-\tau}^tX_i^{p+\beta}(s)\mathrm{d}s\rho_2(\mathrm{d}\theta)\\
&\le-(\delta_3'-\delta_4')\int_0^t\sum_{i=1}^nX_i^{p+\beta}(s)\mathrm{d}s+\frac{\beta\delta_4'}{p+\beta}\int_{-\tau}^0\sum_{i=1}^nX_i^{p+\beta}(s)\mathrm{d}s\\
&\le-(\delta_3'-\delta_4')\int_0^t|X(s)|^{p+\beta}\mathrm{d}s+C.
\end{aligned}$$

Then, following the argument of (2.4) we derive from (**H3**), (**H**4′) and (**H5**) that

$$\begin{aligned}
&\int_0^t\mathcal{L}V(X_s)\mathrm{d}s\\
&\le p\int_0^t\sum_{i=1}^n(X_i^p(s)+1)|F(X_s)|\mathrm{d}s\\
&+\int_0^t\sum_{i=1}^n\Big(\delta_2'-\delta_3'X_i^\beta(s)+\delta_4'\int_{-\tau}^0X_i^\beta(s+\theta)\rho_2(\mathrm{d}\theta)\Big)X_i^p(s)\mathrm{d}s\\
&+n\delta_5\int_0^t\Big(1+|X(s)|^{\gamma_1}+\int_{-\tau}^0|X(s+\theta)|^{\gamma_2}\rho_3(\mathrm{d}\theta)\Big)\mathrm{d}s\\
&\le\int_0^t\Big\{-(\delta_3'-\delta_4')|X(s)|^{p+\beta}+C(1+|X(s)|^p+|X(s)|^{\gamma_1}+|X(s)|^{p+\alpha_1})\Big\}\mathrm{d}s.
\end{aligned}$$

The desired assertion follows by carrying out a similar argument of the second half part of Theorem 2.1. □

Remark 2.1. Following the argument of Theorem 2.1 and taking the Young inequality (2.3) into consideration, we can also deduce that, for any initial condition $\xi\in D^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n_+)$, (1.2) has a unique global solution $\{X(t)\}_{t\ge0}\in\mathbb{R}^n_+$ a.s. under the assumptions: for $\varphi\in D([-\tau,0];\mathbb{R}^n_+)$

$$|F(\varphi)|\le\delta_1\Big(1+\sum_{i=1}^m|\varphi(0)|^{\alpha_{1i}}+\int_{-\tau}^0\sum_{i=1}^m|\varphi(\theta)|^{\alpha_{2i}}\rho_{1i}(\mathrm{d}\theta)\Big);$$

$$\begin{aligned}
&\int_{\mathbb{Y}}[(1+H_i(\varphi,u))^p-1-pH_i(\varphi,u)]\lambda(\mathrm{d}u)\\
&\le\delta_2-\delta_3|\varphi(0)|^{\beta_1}+\delta_4\int_{-\tau}^0\sum_{i=1}^m|\varphi(\theta)|^{\beta_{2i}}\rho_{2i}(\mathrm{d}\theta);
\end{aligned}$$

$$\int_{\mathbb{Y}} [H_i(\varphi, u) - \ln(1 + H_i(\varphi, u))] \, \lambda(\mathrm{d}u)$$
$$\leq \delta_5 \Big(1 + \sum_{i=1}^{m} |\varphi(0)|^{\gamma_{1i}} + \int_{-\tau}^{0} \sum_{i=1}^{m} |\varphi(\theta)|^{\gamma_{2i}} \rho_{3i}(\mathrm{d}\theta)\Big),$$

where $m \in \mathbb{N}, \alpha_{1i}, \alpha_{2i}, \beta_{2i}, \gamma_{1i}, \alpha_{2i}, \delta_1, \cdots, \delta_5 > 0, \beta_1 > (\max\{\alpha_{1i}\} \vee \max\{\alpha_{2i}\} \vee \max\{\beta_{2i}\} \vee \max\{\gamma_{1i}\} \vee \max\{\gamma_{2i}\})$ and $\rho_{1i}, \rho_{2i}, \rho_{3i}$ are probability measures on $[-\tau, 0]$.

Remark 2.2. By the argument of Theorem 2.1 and that of [4, Theorem 2.2], under some appropriate conditions Theorem 2.1 can also be generalized to population model

$$\mathrm{d}X(t) = \mathrm{diag}(X_1(t), \cdots, X_n(t)) \Big[F(X_t)\mathrm{d}t + G(X_t)\mathrm{d}W(t) + \int_{\mathbb{Y}} H(X_{t-}, u)\tilde{N}(\mathrm{d}t, \mathrm{d}u)\Big],$$

where W is an m-dimensional Brownian motion.

Remark 2.3. Theorem 2.1 and Theorem 2.3 demonstrate that Lévy noise can also be applied to suppress solution explosion for functional differential equations. Therefore we can utilize jump processes to investigate stabilization problems for functional differential equations, which will be reported in the forthcoming paper.

3. **Pathwise analysis**

Theorem 3.1. Let (**H1**)-(**H5**) hold. Assume further that there exist $\delta_6 > 0$ and $\kappa \in (0, \beta_1)$ such that

$$\int_{\mathbb{Y}} [(\ln Q(\varphi, u))^2 + Q(\varphi, u)]\lambda(du) \leq \delta_6 \Big(1 + |\varphi(0)|^{\kappa} + \int_{-\tau}^{0} |\varphi(\theta)|^{\kappa} \mathrm{d}\theta\Big), \tag{3.1}$$

where, for $p \in (0, 1)$ and $\varphi \in D([-\tau, 0]; \mathbb{R}_+^n)$,

$$Q(\varphi, u) := \sum_{i=1}^{n} (1 + H_i(\varphi, u))^p \varphi_i^p(0) \Big/ \sum_{i=1}^{n} \varphi_i^p(0).$$

Then the solution $X(t), t \geq 0$, of (1.2) has the property

$$\limsup_{t \to \infty} \frac{\ln(|X(t)|)}{t} \leq 0, \quad \text{a.s.} \tag{3.2}$$

Proof. Observe from Theorem 2.1 that (1.2) admits a unique global positive solution. Let

$$V(x) := \sum_{i=1}^{n} x_i^p, \; x \in \mathbb{R}_+^n.$$

By the Itô formula

$$
\begin{aligned}
e^t \ln V(X(t)) = \ln V(\xi(0)) &+ \int_0^t e^s\Big\{ \ln V(X(s)) + \frac{p}{V(X(s))}\sum_{i=1}^n X_i^p(s)F_i(X_s) \\
&+ \int_{\mathbb{Y}} \Big(\ln Q(X_s,u) - \frac{p}{V(X(s))}\sum_{i=1}^n X_i^p(s)H_i(X_s,u)\Big)\lambda(\mathrm{d}u)\Big\}\mathrm{d}s \\
&+ \int_0^t \int_{\mathbb{Y}} e^s \ln Q(X_{s_-},u)\tilde N(\mathrm{d}s,\mathrm{d}u).
\end{aligned}
\tag{3.3}
$$

Applying the exponential martingale inequality with jumps, e.g., [1, Theorem 5.2.9, p. 291], for any $\alpha,\ \nu,\ T>0$, we have

$$
\begin{aligned}
\mathbb{P}\Big\{\omega : \sup_{0\le t\le T} &\Big\{ \int_0^t \int_{\mathbb{Y}} e^s \ln Q(X_{s^-},u)\tilde N(\mathrm{d}s,\mathrm{d}u) \\
&- \frac{1}{\alpha}\int_0^t\int_{\mathbb{Y}} \Big(Q^{\alpha e^s}(X_s,u) - 1 - \alpha e^s \ln Q(X_s,u)\Big)\lambda(\mathrm{d}u)\mathrm{d}s\Big\} \ge \nu\Big\} \le e^{-\alpha\nu}.
\end{aligned}
$$

Choose $T=k, \alpha=\varepsilon e^{-k}$ and $\nu = 2\varepsilon^{-1}e^k\ln k$, where $k\in\mathbb{N}, \varepsilon\in(0,1/2)$, in the above equation. Since $\sum_{k=1}^\infty k^{-2}<\infty$, we can deduce from the Borel-Cantelli lemma that there exists a measurable subset $\Omega_0\subseteq\Omega$ with $\mathbb{P}(\Omega_0)=1$ such that for any $\omega\in\Omega_0$ we can find an integer $k_0(\omega)>0$ such that

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{Y}} e^s \ln Q(X_{s^-},u)\tilde N(\mathrm{d}s,\mathrm{d}u) \\
&\le 2\varepsilon^{-1}e^k\ln k + \frac{1}{\varepsilon e^{-k}}\int_0^t\int_{\mathbb{Y}} \Big(Q^{\varepsilon e^{s-k}}(X_s,u) - 1 - \varepsilon e^{s-k}\ln Q(X_s,u)\Big)\lambda(\mathrm{d}u)\mathrm{d}s,
\end{aligned}
\tag{3.4}
$$

where $0\le t\le k$ and $k\ge k_0(\omega)$. Hence, for $\omega\in\Omega_0$, $0\le t\le k$ and $k\ge k_0(\omega)$,

$$
\begin{aligned}
e^t \ln V(X(t)) &\le \ln V(\xi(0)) + 2\varepsilon^{-1}e^k\ln k \\
&\quad + \int_0^t e^s\Big\{ \ln V(X(s)) + \frac{p}{V(X(s))}\sum_{i=1}^n X_i^p(s)F_i(X_s) \\
&\quad + \int_{\mathbb{Y}} \Big(\ln Q(X_s,u) - \frac{p}{V(X(s))}\sum_{i=1}^n X_i^p(s)H_i(X_s,u)\Big)\lambda(\mathrm{d}u)\Big\}\mathrm{d}s \\
&\quad + \frac{1}{\varepsilon e^{-k}}\int_0^t\int_{\mathbb{Y}} \Big(Q^{\varepsilon e^{s-k}}(X_s,u) - 1 - \varepsilon e^{s-k}\ln Q(X_s,u)\Big)\lambda(\mathrm{d}u)\mathrm{d}s \\
&=: \ln V(\xi(0)) + 2\varepsilon^{-1}e^k\ln k + J_1(t) + J_2(t) + J_3(t).
\end{aligned}
\tag{3.5}
$$

Recall the inequalities

$$\ln x \le x-1, \quad x>0,$$

and

$$(a+b)^\theta \le a^\theta + b^\theta, \quad a,b>0,\ \theta\in(0,1). \tag{3.6}$$

Then, by the Hölder inequality, (2.2) and (**H3**) it follows that

(3.7)

$$\begin{aligned} J_1(t) &\leq \int_0^t e^s\Big\{V(X(s)) + \frac{p}{V(X(s))}\Big(\sum_{i=1}^n X_i^{2p}(s)\Big)^{1/2}\Big(\sum_{i=1}^n F_i^2(X_s)\Big)^{1/2}\Big\}\mathrm{d}s \\ &\leq \int_0^t e^s\Big\{V(X(s)) + \frac{p}{V(X(s))}\sum_{i=1}^n X_i^p(s)|F(X_s)|\Big\}\mathrm{d}s \\ &= \int_0^t e^s\Big\{V(X(s)) + p|F(X_s)|\Big\}\mathrm{d}s \\ &\leq \int_0^t e^s\Big\{n^{1-p/2}|X(s)|^p + p\delta\Big(1 + |X(s)|^{\alpha_1} + \int_{-\tau}^0 |X(s+\theta)|^{\alpha_2}\rho_1(\mathrm{d}\theta)\Big)\Big\}\mathrm{d}s. \end{aligned}$$

Next, applying the inequality $\ln x \leq x - 1$ and using (**H4**), we deduce that

(3.8)

$$\begin{aligned} J_2(t) &= \int_0^t \int_{\mathbb{Y}} e^s\Big\{\log Q(X_s,u) - Q(X_s,u) + 1 \\ &\quad + Q(X_s,u) - \frac{p}{V(X(s))}\sum_{i=1}^n X_i^p(s)H_i(X_s,u) - 1\Big\}\lambda(\mathrm{d}u)\mathrm{d}s \\ &\leq \int_0^t \int_{\mathbb{Y}} e^s\Big\{Q(X_s,u) - \frac{p}{V(X(s))}\sum_{i=1}^n X_i^p(s)H_i(X_s,u) - 1\Big\}\lambda(\mathrm{d}u)\mathrm{d}s \\ &= \int_0^t \frac{e^s}{V(X(s))}\Big\{\sum_{i=1}^n \int_{\mathbb{Y}}((1+H_i(X_s,u))^p - 1 - pH_i(X_s,u))\lambda(\mathrm{d}u)\Big\}X_i^p(s)\mathrm{d}s \\ &\leq \int_0^t \frac{e^s}{V(X(s))}\Big\{\sum_{i=1}^n \Big(\delta_2 - \delta_3|X(s)|^{\beta_1} + \delta_4\int_{-\tau}^0 |X(s+\theta)|^{\beta_2}\rho_2(\mathrm{d}\theta)\Big)\Big\}X_i^p(s)\mathrm{d}s \\ &\leq \int_0^t e^s\Big\{\delta_2 - \delta_3|X(s)|^{\beta_1} + \delta_4\int_{-\tau}^0 |X(s+\theta)|^{\beta_2}\rho_2(\mathrm{d}\theta)\Big\}\mathrm{d}s. \end{aligned}$$

Furthermore, in the light of a Taylor's series expansion

$$\begin{aligned} J_3(t) &= \frac{\varepsilon e^{2s-k}}{2}\int_0^t \int_{\mathbb{Y}} (\ln Q(X_s,u))^2 Q^\eta(X_s,u)\lambda(\mathrm{d}u)\mathrm{d}s \\ &= \frac{\varepsilon e^{2s-k}}{2}\int_0^t \int_{\mathbb{Y}} \{(\ln Q(X_s,u))^2 Q^\eta(X_s,u)\}1_{\{0<Q\leq 1\}}\lambda(\mathrm{d}u)\mathrm{d}s \\ &\quad + \frac{\varepsilon e^{2s-k}}{2}\int_0^t \int_{\mathbb{Y}} (\{\ln Q(X_s,u))^2 Q^\eta(X_s,u)\}1_{\{Q\geq 1\}}\lambda(\mathrm{d}u)\mathrm{d}s, \end{aligned} \tag{3.9}$$

where η lies between 0 and ε. On the other hand, for $\eta \in (0, \frac{1}{2})$ note that $Q^\eta \leq 1$ whenever $0 < Q < 1$, and that $Q^\eta \leq Q^{\frac{1}{2}}$ with $Q \geq 1$. Hence, by the inequality

$$\ln x \leq 4(x^{\frac{1}{4}} - 1), \quad x \geq 1, \tag{3.10}$$

we get from (3.9) that

$$
J_3(t) \le \frac{\varepsilon e^{2s-k}}{2} \int_0^t \int_{\mathbb{Y}} \{(\ln Q(X_s,u))^2 + 16Q(X_s,u)\}\lambda(\mathrm{d}u)\mathrm{d}s
\le 8\delta_6\varepsilon e^{2s-k} \int_0^t \Big\{1 + |X(s)|^\kappa + \int_{-\tau}^0 |X(s+\theta)|^\kappa \mathrm{d}\theta\Big\}\mathrm{d}s. \tag{3.11}
$$

Substituting (3.7), (3.8) and (3.11) into (3.5) gives that for any $\omega \in \Omega_0$, $0 \le t \le k$ and $k \ge k_0(\omega)$

$$
\begin{aligned}
e^t \ln V(X(t)) \le{}& \ln V(\xi(0)) + 2\varepsilon^{-1} e^k \ln k \\
&+ \int_0^t e^s \Big\{ n^{1-p/2}|X(s)|^p + p\delta\Big(1 + |X(s)|^{\alpha_1} \\
&+ \int_{-\tau}^0 |X(s+\theta)|^{\alpha_2}\rho_1(\mathrm{d}\theta)\Big) \\
&+ \delta_2 - \delta_3|X(s)|^{\beta_1} + \delta_4 \int_{-\tau}^0 |X(s+\theta)|^{\beta_2}\rho_2(\mathrm{d}\theta) \\
&+ 8\delta_6\varepsilon\Big(1 + |X(s)|^\kappa + \int_{-\tau}^0 |X(s+\theta)|^\kappa \mathrm{d}\theta\Big)\Big\}\mathrm{d}s \\
\le{}& \ln V(\xi(0)) + 2\varepsilon^{-1}e^k \ln k + Ce^t
\end{aligned}
$$

for some $C > 0$, where in the last step we have used the fact that the polynomial in the integrator is bounded since the leading term of the polynomial is negative. Thus, for any $\omega \in \Omega_0$, $k-1 \le t \le k$ and $k \ge k_0(\omega)$,

$$
\frac{1}{\ln t} \ln V(X(t)) \le \frac{1}{\ln t}\{C + e^{-t}\ln V(\xi(0)) + 2\varepsilon^{-1}e^{k-t}\ln k\}.
$$

Taking $k \uparrow \infty$ and using (2.2) yields that

$$
\limsup_{t\to\infty} \frac{\ln(|X(t)|)}{\ln t} \le \frac{2e}{p\varepsilon}, \quad \text{a.s.} \tag{3.12}
$$

Then the desired assertion (3.2) follows by letting $\varepsilon \uparrow \frac{1}{2}$ and noting that $\lim_{t\to\infty} \frac{\ln t}{t} = 0$. □

Remark 3.1. From (3.12), we conclude that the population size is at most of polynomial growth almost surely.

Remark 3.2. For Example 2.2, by (2.3), (3.10) and the nonnegative property of $H(\varphi, u)$, we get that

$$\begin{aligned}
\int_0^\infty \{(\ln Q(\varphi,u))^2 + Q(\varphi,u)\}\lambda(\mathrm{d}u) \\
&\le \int_0^\infty \{16Q^{\frac{1}{2}}(\varphi,u) + Q(\varphi,u)\}\lambda(\mathrm{d}u) \\
&\le C\int_0^\infty \{1 + Q(\varphi,u)\}\lambda(\mathrm{d}u) \\
&\le C\int_0^\infty \{1 + (1+H(\varphi,u))^p\}\lambda(\mathrm{d}u) \\
&= C\int_0^\infty \left\{1 + \left[1 + \Big(|\varphi(0)|^\beta + \int_{-\tau}^0 |\varphi(\theta)|^\gamma \mathrm{d}\theta\Big)u\right]^p\right\}\lambda(\mathrm{d}u),
\end{aligned}$$

where $C > 0$ is some constant. By the elemental inequality $(a+b)^\kappa \le a^\kappa + b^\kappa, a, b \ge 0, \kappa \in (0,1)$, we have

$$\begin{aligned}
&\int_0^\infty \{(\ln Q(\varphi,u))^2 + Q(\varphi,u)\}\lambda(\mathrm{d}u) \\
&\le C\int_0^\infty \Big\{1 + \Big[|\varphi(0)|^{p\beta} + \Big(\int_{-\tau}^0 |\varphi(\theta)|^\gamma \mathrm{d}\theta\Big)^p\Big]u^p\Big\}\lambda(\mathrm{d}u).
\end{aligned}$$

By virtue of $\beta > \gamma > 0$, we choose $\delta \in (1, \beta/\gamma]$. Applying the Young inequality (2.3) and the Hölder inequality leads to

$$\begin{aligned}
\Big(\int_{-\tau}^0 \varphi^\gamma(\theta)d\theta\Big)^p &= \Big(\Big(\int_{-\tau}^0 \varphi^\gamma(\theta)d\theta\Big)^\delta\Big)^{p/\delta} \le \varepsilon\tau^{1/\delta-1}\Big(\int_{-\tau}^0 \varphi^\gamma(\theta)d\theta\Big)^\delta + C \\
&\le \varepsilon\int_{-\tau}^0 \varphi^{\delta\gamma}(\theta)d\theta + C, \quad \varepsilon \in (0,1).
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_0^\infty \{(\ln Q(\varphi,u))^2 + Q(\varphi,u)\}\lambda(\mathrm{d}u) \\
&\le C\int_0^\infty (1+u^p)\lambda(\mathrm{d}u) + C\int_0^\infty u^p\lambda(\mathrm{d}u)|\varphi(0)|^{p\beta} + \varepsilon C\int_0^\infty u^p\lambda(\mathrm{d}u)\int_{-\tau}^0 \varphi^{\delta\gamma}(\theta)\mathrm{d}\theta,
\end{aligned}$$

and therefore (3.1) follows.

Remark 3.3. Note from ($\mathbf{H}4'$), the Hölder inequality and (3.6) that

$$\begin{aligned}
&\int_0^t \frac{1}{V(X(s))}\Big\{\sum_{i=1}^n\int_{\mathbb{Y}}((1+H_i(X_s,u))^p-1-pH_i(X_s,u))\lambda(\mathrm{d}u)\Big\}X_i^p(s)\mathrm{d}s\\
&\le \int_0^t \frac{1}{V(X(s))}\Big\{\delta_2'V(X(s))-\delta_3'|X(s)|^{p+\beta}\\
&\quad+\delta_4'\int_{-\tau}^0 V(X(s))\sum_{i=1}^n X_i^\beta(s+\theta)\rho_2'(\mathrm{d}\theta)\Big)\Big\}\mathrm{d}s\\
&\le \int_0^t\Big\{\delta_2'-\frac{\delta_3'}{V(X(s))}|X(s)|^{p+\beta}+\delta_4'\int_{-\tau}^0\sum_{i=1}^n X_i^\beta(s+\theta)\rho_2'(\mathrm{d}\theta)\Big)\Big\}\mathrm{d}s\\
&\le \int_0^t\Big\{\delta_2'+\delta_4'n^{1-p/2}\int_{-\tau}^0|X(\theta)|^\beta\mathrm{d}\theta-\frac{\delta_3'-\delta_4'n^{2-p}}{n^{1-p/2}}|X(s)|^\beta\Big\}\mathrm{d}s.
\end{aligned}$$

Then, if $\delta_3' > \delta_4' n^{2-p}$, following the argument of Theorem 3.1, we conclude that (3.2) still holds under (**H1**)-(**H3**), ($\mathbf{H}4'$), (**H5**) and (3.1).

Remark 3.4. In this paper we discuss the existence and uniqueness of global positive solution for a class of functional stochastic population dynamical systems with jumps and carry out the corresponding pathwise analysis. However, for model (1.2), even for the case independent of after-effect, the extinction problem remains as an interesting open problem currently under investigation.

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