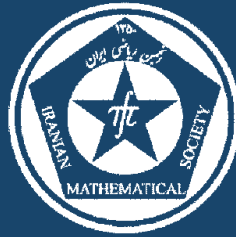


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A CERTAIN CONVOLUTION APPROACH FOR SUBCLASSES OF UNIVALENT HARMONIC FUNCTIONS

R. M. EL-ASHWAH* AND M. K. AOUF

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ABSTRACT. In the present paper we study convolution properties for subclasses of univalent harmonic functions in the open unit disc and obtain some basic properties such as coefficient characterization and extreme points.

Keywords: Analytic, harmonic, convolution.

MSC(2010): Primary: 30C45.

1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a simply connected complex domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D . It was shown by Clunie and Sheil-Small [5] that such harmonic function can be represented by $f = h + \bar{g}$, where h and g are analytic in D . Also, a necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $\left| h'(z) \right| > \left| g'(z) \right|$, (see also, [2, 9, 10] and [20]).

Denote by H the family of functions $f = h + \bar{g}$, which are harmonic univalent and sense-preserving in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ with normalization $f(0) = h(0) = f'_z(0) - 1 = 0$ and h, g are of the form:

$$(1.1) \quad h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \quad (0 \leq |b_1| < 1; z \in U).$$

Also denote by TH the subfamily of H consisting of harmonic functions $f = h + g$ of the form:

$$(1.2) \quad h(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \quad (a_n, b_n \geq 0; 0 \leq |b_1| < 1; z \in U).$$

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Let

$$(1.3) \quad \phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n \quad (\phi_n \geq 0; n \in \mathbb{N} = \{1, 2, \dots\}),$$

be a given analytic function in U , then

$$(1.4) \quad \begin{aligned} (\phi(z) + \overline{\phi(z)}) * (h(z) + \overline{g(z)}) &= z + \sum_{n=2}^{\infty} \phi_n a_n z^n + \sum_{n=1}^{\infty} \phi_n b_n \bar{z}^n \\ (\phi_1 = 1; 0 \leq |b_1| < 1; z \in U), \end{aligned}$$

where $*$ is the convolution operation.

Later on, various subclasses of H have been introduced and studied by several authors (see e. g. [2, 11–14]). We shall show in this note that these subclasses are special cases of the general class $H^0(\phi, \lambda, \sigma, \alpha)$ giving in the following definition.

Definition 1.1. Let $\phi(z)$ be given by (1.3). A harmonic function $f = h + \bar{g} \in H$ where h and g are given by (1.1) belongs to the class $H^0(\phi, \lambda, \sigma, \alpha)$ if

$$(1.5) \quad \operatorname{Re} \left\{ \frac{(z(h * \phi)'(z) - \sigma z \overline{(g * \phi)'(z)})}{(1 - \lambda)((h * \phi)(z) + \sigma \overline{(g * \phi)(z)}) + \lambda(z(h * \phi)'(z) - \sigma z \overline{(g * \phi)'(z)})} \right\} > \alpha$$

$$(0 \leq \alpha < 1; 0 \leq \lambda \leq 1; |\sigma| \leq 1; z \in U),$$

We note that

(i) $H^0(H_q^s[\alpha_1], \lambda, 1, \alpha) = R_H^{s,q}([\alpha_1], \lambda, \alpha)$ (where $H_q^s[\alpha_1]$ is the Dziok-Srivastava operator [6]) (see Murugusundaramoorthy et al. [12]);

(ii) $H^0(H_q^s[\alpha_1], 0, 1, \alpha) = R_H^{s,q}([\alpha_1], \alpha)$ (where $H_q^s[\alpha_1]$ is the Dziok-Srivastava operator [6]) (see Al-kharsani and Al-Khal [2]);

(iii) $H^0\left(\frac{z}{(1-z)}, \lambda, 1, \alpha\right) = S_H^*(\lambda, \alpha)$ (see Ozturk et al. [14]);

(iv) $H^0\left(\frac{z}{(1-z)}, 0, 1, \alpha\right) = S_H^*(\alpha)$ and $H^0\left(\frac{z}{(1-z)^2}, 0, -1, \alpha\right) = K_H(\alpha)$ (see Jahangiri [10]);

(v) $H^0\left(\frac{z}{(1-z)}, 0, 1, 0\right) = S_H^*$ and $H^0\left(\frac{z}{(1-z)^2}, 0, -1, 0\right) = K_H$ (see Silverman [19]).

Now we denote by $HP^0(\phi, \lambda, \sigma, k, \alpha)$ a general class of various subclasses of harmonic parabolic starlike functions introduced and studied by several authors (see e. g. [8, 16, 17] and [21]).

Definition 1.2. Let $\phi(z)$ be given by (1.3). A harmonic function $f = h + \bar{g} \in H$ belongs to the class $HP^0(\phi, \lambda, \sigma, k, \alpha)$ if

$$\operatorname{Re} \left\{ \frac{(1 + ke^{i\gamma})(z(h * \phi)'(z) - \sigma z \overline{(g * \phi)'(z)})}{(1 - \lambda)((h * \phi)(z) + \sigma \overline{(g * \phi)(z)}) + \lambda(z(h * \phi)'(z) - \sigma z \overline{(g * \phi)'(z)})} - ke^{i\gamma} \right\} > \alpha$$

$$(1.6) \quad (0 \leq \alpha < 1; 0 \leq \lambda \leq 1; |\sigma| \leq 1; k \geq 0; \gamma \text{ real}; z \in U).$$

We note that

(i) $HP^0 \left(z + \sum_{n=2}^{\infty} n^m z^n, \lambda, (-1)^m, 1, \alpha \right) = S_H^* (m+1, m, \lambda, \alpha)$ (see Sudharasan et al. [21]);

(ii) $HP^0 \left(\frac{z}{1-z}, \lambda, 1, 0, \alpha \right) = S_H^* (\lambda, \alpha)$ (see Ozturk et al. [14]);

(iii) $HP^0 \left(\frac{z}{1-z}, 0, 1, 1, \alpha \right) = G_H(\alpha)$ (see Rosy et al. [16]).

In this paper, we obtain necessary and sufficient convolution conditions for the classes $H^0(\phi, \lambda, \sigma, \alpha)$ and $HP^0(\phi, \lambda, \sigma, k, \alpha)$, sufficient coefficient bounds for functions in these two classes, these sufficient coefficient conditions are shown to be also necessary for functions with negative coefficients. Finally we determined growth estimates and extreme points for the class $TH^0(\phi, \lambda, \sigma, \alpha) = H^0(\phi, \lambda, \sigma, \alpha) \cap TH$. The class $TH^0(\phi, \lambda, \sigma, \alpha)$ includes as special cases several subclasses studied in [9] and [19].

2. Main results

We now derive the convolution characterization for functions in the class $H^0(\phi, \lambda, \sigma, \alpha)$.

Theorem 2.1. *Let $f = h + \bar{g} \in H$ and $\phi(z)$ be given by (1.3), then $f \in H^0(\phi, \lambda, \sigma, \alpha)$ if and only if*

$$(h * \phi)^* \left[\frac{z + \frac{(1-\lambda)(x+2\alpha-1)}{2(1-\alpha)} z^2}{(1-z)^2} \right] - \sigma \overline{(g * \phi)^*} \left[\frac{\frac{x+\alpha-\lambda(x+2\alpha-1)}{1-\alpha} \bar{z} - \frac{(1-\lambda)(x+2\alpha-1)}{2(1-\alpha)} \bar{z}^2}{(1-\bar{z})^2} \right] \neq 0$$

$$(2.1) \quad (|x| = 1; |z| \neq 0).$$

Proof. A necessary and sufficient condition for $f = h + \bar{g}$ to be in the class $H^0(\phi, \lambda, \sigma, \alpha)$, with h, g of the form (1.1), is given by (1.5). Since

$$\frac{\left(z(h * \phi)'(z) - \sigma z \overline{(g * \phi)'(z)} \right)}{(1-\lambda) \left((h * \phi)(z) + \sigma \overline{(g * \phi)(z)} \right) + \lambda \left(z(h * \phi)'(z) - \sigma z \overline{(g * \phi)'(z)} \right)} = 1$$

at $z = 0$, the condition (1.5) is equivalent to

$$\frac{1}{1-\alpha} \left\{ \frac{\left(z(h * \phi)'(z) - \sigma z \overline{(g * \phi)'(z)} \right)}{(1-\lambda) \left((h * \phi)(z) + \sigma \overline{(g * \phi)(z)} \right) + \lambda \left(z(h * \phi)'(z) - \sigma z \overline{(g * \phi)'(z)} \right)} - \alpha \right\} \neq \frac{x-1}{x+1}$$

$$(2.2) \quad (|x| = 1; x \neq -1; 0 < |z| < 1).$$

By a simple algebraic manipulation, (2.2) yields

$$\begin{aligned}
 &0 \neq (x + 1) \left[z (h * \phi)'(z) - \sigma z \overline{(g * \phi)'(z)} \right] \\
 &\quad - \alpha(x + 1) \left[(1 - \lambda) \left((h * \phi)(z) + \sigma \overline{(g * \phi)(z)} \right) \right. \\
 &\quad \left. + \lambda \left(z (h * \phi)'(z) - \sigma z \overline{(g * \phi)'(z)} \right) \right] \\
 &\quad - (x - 1)(1 - \alpha) \left[(1 - \lambda) \left((h * \phi)(z) + \sigma \overline{(g * \phi)(z)} \right) \right. \\
 &\quad \left. + \lambda \left(z (h * \phi)'(z) - \sigma z \overline{(g * \phi)'(z)} \right) \right] \\
 &= (h * \phi) * \left[\frac{2(1 - \alpha)z + (1 - \lambda)(x + 2\alpha - 1)z^2}{(1 - z)^2} \right] \\
 &\quad - \overline{\sigma(g * \phi)} * \left[\frac{2[x + \alpha - \lambda(x + 2\alpha - 1)]\bar{z} - (1 - \lambda)(x + 2\alpha - 1)\bar{z}^2}{(1 - \bar{z})^2} \right].
 \end{aligned}$$

The latter condition together with (1.5) establishes the result (2.1) for all $|x| = 1$. □

Remark 2.2. Taking $\lambda = 0$ and $b_1 = 0$ in Theorem 2.1 we obtain the result obtained by Ali et al. [3, Theorem 2];

(i) Taking $\lambda = 0, \sigma = 1, \alpha = 0$ and $\phi = \frac{z}{1 - z}$, in Theorem 2.1 we obtain the result obtained by Ahuja et al. [1, Theorem 2.6];

(ii) Taking $\lambda = 0, \sigma = 1, \alpha = 0$ and $\phi = \frac{z}{(1 - z)^2}$, in Theorem 2.1 we obtain the result obtained by Ahuja et al. [1, Theorem 2.8].

Necessary coefficient conditions for harmonic starlike functions were obtained in [5] (see also [18]).

Using the convolution characterization, we now derive a sufficient condition for harmonic functions to belong to the class $H^0(\phi, \lambda, \sigma, \alpha)$.

Theorem 2.3. Let $f = h + \bar{g} \in H$ and $\phi(z)$ be given by (1.3), then $f \in H^0(\phi, \lambda, \sigma, \alpha)$ if

$$\sum_{n=2}^{\infty} \frac{n(1 - \alpha\lambda) - \alpha(1 - \lambda)}{(1 - \alpha)} |a_n| |\phi_n| + |\sigma| \sum_{n=1}^{\infty} \frac{n(1 - \alpha\lambda) + \alpha(1 - \lambda)}{(1 - \alpha)} |b_n| |\phi_n| \leq 1.$$

Proof. For h and g given by (1.1), (2.1) gives

$$\begin{aligned} & \left| (h * \phi) * \left[\frac{z + \frac{(1-\lambda)(x+2\alpha-1)}{2(1-\alpha)} z^2}{(1-z)^2} \right] - \overline{\sigma(g * \phi)} * \left[\frac{\frac{x+\alpha-\lambda(x+2\alpha-1)}{1-\alpha} \bar{z} - \frac{(1-\lambda)(x+2\alpha-1)}{2(1-\alpha)} \bar{z}^2}{(1-\bar{z})^2} \right] \right| \\ &= \left| z + \sum_{n=2}^{\infty} \left\{ n + (n-1) \frac{(1-\lambda)(x+2\alpha-1)}{2(1-\alpha)} \right\} a_n \phi_n z^n \right. \\ & \quad \left. - \sigma \sum_{n=1}^{\infty} \left\{ n \frac{x+\alpha-\lambda(x+2\alpha-1)}{1-\alpha} - (n-1) \frac{(1-\lambda)(x+2\alpha-1)}{2(1-\alpha)} \right\} b_n \phi_n z^n \right| \\ & \geq |z| \left[1 - \sum_{n=2}^{\infty} \frac{n(1-\alpha\lambda) - \alpha(1-\lambda)}{(1-\alpha)} |a_n| |\phi_n| - |\sigma| \sum_{n=1}^{\infty} \frac{n(1-\alpha\lambda) + \alpha(1-\lambda)}{(1-\alpha)} |b_n| |\phi_n| \right]. \end{aligned}$$

The last expression is non-negative by hypothesis, and hence by Theorem 2.1, it follows that $f \in H_p^0(\phi, \lambda, \sigma, \alpha)$. \square

Remark 2.4. *i) Taking $\lambda = 0$ and $b_1 = 0$ in Theorem 2.3 we obtain the result obtained by Ali et al. [3, Theorem 3];*

(ii) Taking $\lambda = 0, \sigma = 1, \alpha = 0$ and $\phi = \frac{z}{1-z}$, in Theorem 2.3 we obtain the result obtained by Ahuja et al. [1, Corollary 2.7];

(iii) Taking $\lambda = 0, \sigma = 1, \alpha = 0$ and $\phi = \frac{z}{(1-z)^2}$, in Theorem 2.3 we obtain the result obtained by Ahuja et al. [1, Corollary 2.9].

Theorem 2.5. *Let $f = h + \bar{g} \in H$ and $\phi(z)$ be given by (1.3), then $f \in HP^0(\phi, \lambda, \sigma, k, \alpha)$ if and only if*

$$\begin{aligned} & (h * \phi) * \left[\frac{z + \frac{(1-\lambda)[(x+1)ke^{i\gamma} + x + 2\alpha - 1]}{2(1-\alpha)} z^2}{(1-z)^2} \right] \\ & - \overline{\sigma(g * \phi)} * \left[\frac{\frac{(x+1)ke^{i\gamma} + x + \alpha - \lambda((x+1)ke^{i\gamma} + x + 2\alpha - 1)}{1-\alpha} \bar{z} - \frac{(1-\lambda)[(x+1)ke^{i\gamma} + x + 2\alpha - 1]}{2(1-\alpha)} \bar{z}^2}{(1-\bar{z})^2} \right] \\ & \neq 0 \quad (|x| = 1; |z| \neq 0). \end{aligned}$$

Proof. A necessary and sufficient condition for $f = h + \bar{g}$ to be in the class $HP^0(\phi, \lambda, \sigma, k, \alpha)$, with h, g of the form (1.1), is given by (1.6). Since

$$\frac{(1 + ke^{i\gamma}) \left(z(h * \phi)'(z) - \overline{\sigma z(g * \phi)'(z)} \right)}{(1-\lambda) \left((h * \phi)(z) + \overline{\sigma(g * \phi)(z)} \right) + \lambda \left(z(h * \phi)'(z) - \overline{\sigma z(g * \phi)'(z)} \right)} - ke^{i\gamma} = 1$$

at $z = 0$, the condition (1.6) is equivalent to

$$\frac{1}{1-\alpha} \left\{ \frac{(1+ke^{i\gamma})(z(h * \phi)'(z) - \overline{\sigma z(g * \phi)'(z)})}{(1-\lambda)((h * \phi)(z) + \overline{\sigma(g * \phi)(z)}) + \lambda(z(h * \phi)'(z) - \overline{\sigma z(g * \phi)'(z)})} - ke^{i\gamma} - \alpha \right\} \neq \frac{x-1}{x+1}$$

$$(2.3) \quad (|x| = 1; x \neq -1; z \neq 0).$$

By a simple algebraic manipulation similar to that used in Theorem 2.1, we obtain the desired result. □

Proceeding similarly as in Theorem 2.3, the following sufficient condition for the class $HP^0(\phi, \lambda, \sigma, k, \alpha)$ is easily derived.

Theorem 2.6. *Let $f = h + \bar{g} \in H$ and $\phi(z)$ be given by (1.3), then $f \in HP^0(\phi, \lambda, \sigma, k, \alpha)$ if*

$$\sum_{n=2}^{\infty} \frac{n[(k+1) - \lambda(k+\alpha)] - (1-\lambda)(k+\alpha)}{(1-\alpha)} |a_n| |\phi_n| + |\sigma| \sum_{n=1}^{\infty} \frac{n[(k+1) - \lambda(k+\alpha)] + (1-\lambda)(k+\alpha)}{(1-\alpha)} |b_n| |\phi_n| \leq 1.$$

Theorem 2.7. *Let $\phi(z)$ be given by (1.3) and $f = h + \bar{g}$ where h, g are given by (1.2). Then $f \in TH^0(\phi, \lambda, \sigma, \alpha)$ if and only if*

$$(2.4) \quad \sum_{n=2}^{\infty} \frac{n(1-\alpha\lambda) - \alpha(1-\lambda)}{(1-\alpha)} |a_n| |\phi_n| + |\sigma| \sum_{n=1}^{\infty} \frac{n(1-\alpha\lambda) + \alpha(1-\lambda)}{(1-\alpha)} |b_n| |\phi_n| \leq 1.$$

Proof. If f belongs to the class $TH^0(\phi, \lambda, \sigma, \alpha)$, then (1.5) is equivalent to

$$\operatorname{Re} \left\{ \frac{(1-\alpha)z - \sum_{n=2}^{\infty} [n(1-\alpha\lambda) - \alpha(1-\lambda)] a_n \phi_n z^n - \sigma \sum_{n=1}^{\infty} [n(1-\alpha\lambda) + \alpha(1-\lambda)] b_n \phi_n \bar{z}^n}{z - \sum_{n=2}^{\infty} [(1-\lambda) + n\lambda] a_n \phi_n z^n + \sigma \sum_{n=1}^{\infty} [(1-\lambda) - n\lambda] b_n \phi_n \bar{z}^n} \right\} > 0,$$

for $z \in U$. Letting $z \rightarrow 1^-$ through real values yields condition (2.4). □

Conversely, for h, g given by (1.2), we have

$$\begin{aligned} & (h * \phi) * \left[\frac{z + \frac{(1-\lambda)(x+2\alpha-1)}{2(1-\alpha)} z^2}{(1-z)^2} \right] - \sigma \overline{(g * \phi)} * \left[\frac{\frac{x+\alpha-\lambda(x+2\alpha-1)}{1-\alpha} \bar{z} - \frac{(1-\lambda)(x+2\alpha-1)}{2(1-\alpha)} \bar{z}^2}{(1-\bar{z})^2} \right] \\ & \geq |z| \left[1 - \sum_{n=2}^{\infty} \frac{n(1-\alpha\lambda) - \alpha(1-\lambda)}{(1-\alpha)} |a_n| |\phi_n| - |\sigma| \sum_{n=1}^{\infty} \frac{n(1-\alpha\lambda) + \alpha(1-\lambda)}{(1-\alpha)} |b_n| |\phi_n| \right] \end{aligned}$$

which is non-negative by hypothesis, this completes the proof of Theorem 2.7.

Remark 2.8. (i) Taking $\sigma = 1$ and $\phi(z) = z + \sum_{n=2}^{\infty} \Gamma_{n-1}(\alpha_1) z^n$, where

$$(2.5) \quad \Gamma_{n-1}(\alpha_1) = \frac{(\alpha_1)_{n-1} \dots (\alpha_q)_{n-1}}{(\beta_1)_{n-1} \dots (\beta_s)_{n-1} (1)_{n-1}} \quad (q \leq s+1, \alpha_i \in \mathbb{C} (i = 1, 2, \dots, q) \text{ and } \beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^- (j = 1, 2, \dots),$$

in Theorem 2.7 we obtain the result obtained by Murugusundaramoorthy et al. [13].

Corollary 2.9. Let $\phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n$ with $\phi_n \geq \phi_2 (n \geq 2)$, and $|\sigma| \geq \frac{2-\alpha(1+\lambda)}{2+\alpha(1-3\lambda)}$. If $f \in TH^0(\phi, \lambda, \sigma, \alpha)$ satisfies $\frac{2-\alpha(1+\lambda)}{2+\alpha(1-3\lambda)} \left(\frac{1+\alpha(1-\lambda)}{1-\alpha} \right) |b_1| < 1$, then for $|z| = r < 1$,

$$|f(z)| \leq (1 + |b_1|)r + \frac{1-\alpha}{(2-\alpha(1+\lambda))\phi_2} \left(1 - \frac{1+\alpha(1-\lambda)}{1-\alpha} \cdot \frac{2-\alpha(1+\lambda)}{2+\alpha(1-3\lambda)} |b_1| \right) r^2$$

and

$$|f(z)| \geq (1 - |b_1|)r - \frac{1-\alpha}{(2-\alpha(1+\lambda))\phi_2} \left(1 - \frac{1+\alpha(1-\lambda)}{1-\alpha} \cdot \frac{2-\alpha(1+\lambda)}{2+\alpha(1-3\lambda)} |b_1| \right) r^2.$$

The function

$$f(z) = z + b_1 \bar{z} + \frac{1-\alpha}{(2-\alpha(1+\lambda))\phi_2} \left(1 - \frac{1+\alpha(1-\lambda)}{1-\alpha} \cdot \frac{2-\alpha(1+\lambda)}{2+\alpha(1-3\lambda)} b_1 \right) \bar{z}^2$$

and its rotations show that the bounds are sharp.

The following covering result follows from the left side inequality in Corollary 2.9.

Corollary 2.10. Let $f \in TH^0(\phi, \lambda, \sigma, \alpha)$, the

$$\left\{ w : |w| < 1 - \frac{1-\alpha}{(2-\alpha(1+\lambda))\phi_2} - \left(1 - \frac{1+\alpha(1-\lambda)}{(2+\alpha(1-3\lambda))\phi_2} \right) |b_1| \right\} \subset f(U).$$

Theorem 2.11. Let $h_1(z) = z$, $h_n(z) = z - \frac{(1-\alpha)}{(n(1-\alpha\lambda)-\alpha(1-\lambda))\phi_n} z^n$, and $g_n(z) = z + \frac{(1-\alpha)}{\sigma(n(1-\alpha\lambda)+\alpha(1-\lambda))\phi_n} \bar{z}^n$ ($n \geq 1$). A function $f \in TH^0(\phi, \lambda, \sigma, \alpha)$ if and only if f can be expressed in the form $f(z) = \sum_{n=1}^{\infty} (\lambda_n h_n + \gamma_n g_n)$, where $\lambda_n \geq 0, \gamma_n \geq$

$$0, \lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n - \sum_{n=1}^{\infty} \gamma_n.$$

In particular, the extreme points of the class $TH^0(\phi, \lambda, \sigma, \alpha)$ are $\{h_n\}$ and $\{g_n\}$, respectively.

Proof. Let

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} (\lambda_n h_n + \gamma_n g_n) \\ &= z - \sum_{n=2}^{\infty} \lambda_n \frac{(1-\alpha)}{(n(1-\alpha\lambda)-\alpha(1-\lambda))\phi_n} z^n + \sum_{n=1}^{\infty} \gamma_n \frac{(1-\alpha)}{(n(1-\alpha\lambda)+\alpha(1-\lambda))\phi_n} \bar{z}^n. \end{aligned}$$

Since

$$\begin{aligned} & \sum_{n=2}^{\infty} \frac{n(1-\alpha\lambda) - \alpha(1-\lambda)}{(1-\alpha)} \lambda_n \frac{(1-\alpha)}{(n(1-\alpha\lambda) - \alpha(1-\lambda)) \phi_n} \phi_n \\ & + \sigma \sum_{n=1}^{\infty} \frac{n(1-\alpha\lambda) + \alpha(1-\lambda)}{(1-\alpha)} \gamma_n \frac{(1-\alpha)}{\sigma(n(1-\alpha\lambda) + \alpha(1-\lambda)) \phi_n} \phi_n \\ & = \sum_{n=2}^{\infty} \lambda_n + \sum_{n=1}^{\infty} \gamma_n = 1 - \lambda_1 \leq 1, \end{aligned}$$

it follows from Theorem 2.7 that $f \in TH^0(\phi, \lambda, \sigma, \alpha)$.

Conversely, if $f \in TH^0(\phi, \lambda, \sigma, \alpha)$, then $a_n \leq \frac{(1-\alpha)}{(n(1-\alpha\lambda) - \alpha(1-\lambda)) \phi_n}$ and $b_n \leq \frac{(1-\alpha)}{(n(1-\alpha\lambda) + \alpha(1-\lambda)) \phi_n}$.

Set $\lambda_n = \frac{n(1-\alpha\lambda) - \alpha(1-\lambda)}{(1-\alpha)} a_n \phi_n$, $\gamma_n = \frac{(n(1-\alpha\lambda) + \alpha(1-\lambda)) \sigma}{(1-\alpha)} b_n \phi_n$, $\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n -$

$\sum_{n=1}^{\infty} \gamma_n$, then

$$\sum_{n=1}^{\infty} (\lambda_n h_n + \gamma_n g_n) = z - \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \bar{z}^n = f(z).$$

This proves Theorem 2.11. □

3. Applications

We can derive new results by taking $\phi(z)$ as follows:

1- $\phi(z) = z + \sum_{n=2}^{\infty} \left[\frac{1+\ell+\gamma(n-1)}{1+\ell} \right]^m z^n$ (see [4, 7] and [15] with $p = 1$), where $\ell > -1, \gamma \geq 0$ and $m \in \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$.

2- $\phi(z) = z + \sum_{n=2}^{\infty} \Gamma_{n-1}(\alpha_1) z^n$ (see [6]), where $\Gamma_{n-1}(\alpha_1)$ is given by (2.5).

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