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NOTES ON AMALGAMATED DUPLICATION OF A RING ALONG AN IDEAL

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Abstract. In this paper, we study some ring theoretic properties of the amalgamated duplication ring $R \bowtie I$ of a commutative Noetherian ring R along an ideal I of R which was introduced by D'Anna and Fontana. Indeed, it is determined that when $R \bowtie I$ satisfies Serre's conditions (R_n) and (S_n), and when is a normal ring, a generalized Cohen-Macaulay ring and finally a filter ring.
Keywords: Amalgamated ring, Cohen-Macaulay ring, Serre condition, normal ring, filter ring.
MSC(2010): Primary: 13A15; Secondary: 13C14, 13D45, 13E05.

1. **Introduction**

In [6], D'Anna and Fontana considered a construction obtained involving a ring R and an ideal $I \subset R$ that is denoted by $R \bowtie I$, called amalgamated duplication of R along I.

D'Anna in [5] studied that when $R \bowtie I$ is Cohen-Macaulay or a Gorenstein ring. In [1], the authors determined when $R \bowtie I$ is a quasi-Gorenstein ring. In Section 2 of this paper, we provide an answer to the question [1, Remark 3.7], asking that when the ring $R \bowtie I$ satisfies Serre's conditions (R_n) (see Theorem 2.3). Also we determine when $R \bowtie I$ satisfies Serre's condition (S_n). In particular, we show that if R is an integral domain then $R \bowtie I$ satisfies (S_n) if and only if R and I satisfy (S_n) (see Theorem 2.5). Then, in Corollary 2.9, we determine when $R \bowtie I$ is a normal ring. In Section 3, we generalize D'Anna's result to generalized Cohen-Macaulay rings and filter rings (we recall the definitions of generalized Cohen-Macaulay rings and filter rings in suitable places).

Next, we deal with some applications of a general construction, introduced in [6], called amalgamated duplication of a ring along an ideal.

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Let R be a commutative ring with unit element 1 and let I be a proper ideal of R. Set

$$R \bowtie I = \{(r,s) | r,s \in R, s-r \in I\}.$$

It is easy to check that $R \bowtie I$ is a subring of $R \times R$ with unit element $(1,1)$ (with the usual componentwise operations) and that $R \bowtie I = \{(r, r+i) | r \in R, i \in I\}$. In the following proposition, we collect some of the main properties of the ring $R \bowtie I$ from [5, 6].

Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a finitely generated R-module. We will denote the common length of the maximal M-regular sequences in $\mathfrak{m}$ by $\operatorname{depth} M$. Also, we will denote the Krull dimension of M by $\dim M$.

Proposition 1.1. *Let R be a ring and let I be an ideal of R. Then the following statements hold.*

(1) *([5, Page 509]) By introducing a multiplicative structure in the R-module direct sum $R \oplus I$ by setting*

$$(r,i)(s,j) = (rs, rj + si + ij),$$

one can show that the map $f : R \oplus I \to R \bowtie I$ defined by $f((r,i)) = (r, r+i)$ is a ring isomorphism and R-module isomorphism too. Moreover, there is a split exact sequence of R-modules

$$0 \to R \xrightarrow{\varphi} R \bowtie I \xrightarrow{\psi} I \to 0$$

where $\varphi(r) = (r,r)$ for all $r \in R$, and $\psi((r,s)) = s - r$, for all $(r,s) \in R \bowtie I$.

(2) *([5, Propositions 5 and 7]) Let $\mathfrak{p}$ be a prime ideal of R and set:*

$$\mathfrak{p}_0 = \{(p, p+i) | p \in \mathfrak{p}, i \in I \cap \mathfrak{p}\},$$

$$\mathfrak{p}_1 = \{(p, p+i) | p \in \mathfrak{p}, i \in I\} \text{ and}$$

$$\mathfrak{p}_2 = \{(p+i, p) | p \in \mathfrak{p}, i \in I\}.$$

(a) *If $I \subseteq \mathfrak{p}$, then $\mathfrak{p}_0 = \mathfrak{p}_1 = \mathfrak{p}_2$ is a prime ideal of $R \bowtie I$ and it is the unique prime ideal of $R \bowtie I$ lying over $\mathfrak{p}$ and $(R \bowtie I)_{\mathfrak{p}_0} \cong R_{\mathfrak{p}} \bowtie I_{\mathfrak{p}}$.*

(b) *If $I \nsubseteq \mathfrak{p}$, then $\mathfrak{p}_1 \neq \mathfrak{p}_2$, $\mathfrak{p}_0 = \mathfrak{p}_1 \cap \mathfrak{p}_2$ and $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are the only prime ideals of $R \bowtie I$ lying over $\mathfrak{p}$.*

Moreover we have, in case (a), $R/\mathfrak{p} \cong (R \bowtie I)/\mathfrak{p}_0$ and, in case (b), $R/\mathfrak{p} \cong (R \bowtie I)/\mathfrak{p}_i$ (for $i = 1, 2$) and $(R \bowtie I)_{\mathfrak{p}_1} \cong R_{\mathfrak{p}} \cong (R \bowtie I)_{\mathfrak{p}_2}$.

(3) *([5, Corollary 6] and [6, Corollary 2.11]) R and $R \bowtie I$ have the same Krull dimension and if R is a local ring with maximal ideal $\mathfrak{m}$, then $R \bowtie I$ is local with maximal ideal $\mathfrak{m}_0 = \{(r, r+i) | r \in \mathfrak{m}, i \in I\}$. Also, if R is a Noetherian ring, then $R \bowtie I$ is a finitely generated R-module.*

Hence the extension $R \to R \bowtie I$ is integral. Moreover, if R is local, then

$$\operatorname{depth}(R \bowtie I) = \min\{\operatorname{depth} R, \operatorname{depth}_R I\} \leq \operatorname{depth}_R I.$$

Throughout R stands for a commutative Noetherian ring with identity and I denotes a proper ideal of R. Also, in the course of the paper, we will denote the height of I by $\operatorname{ht} I$.

2. Serre's conditions (R_n) and (S_n)

In this section, we describe when the amalgamated duplication of a ring along an ideal satisfies the Serre's conditions (R_n) and (S_n) for a non-negative integer n. Also in [1, Remark 3.7], the authors asked that when $R \bowtie I$ satisfies Serre's condition (R_n). Recall that a Noetherian ring R satisfies Serre's condition (R_n) if, for all $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\operatorname{ht} \mathfrak{p} \leq n$, the local ring $R_\mathfrak{p}$ is regular. By a regular local ring we mean a local Noetherian ring $(R, \mathfrak{m})$ such that the maximal ideal $\mathfrak{m}$ can be generated by $\dim R$ elements.

Lemma 2.1. *If $R \bowtie I$ is an integral domain, then $I = 0$.*

Proof. By [6, Corollary 2.5], $O_1 := \{(0, i) | i \in I\}$ and $O_2 := \{(i, 0) | i \in I\}$ are the only minimal prime ideals of $R \bowtie I$. Then, by assumption, $O_1 = O_2 = 0$. Therefore $I = 0$. □

The following corollary shows that $R \bowtie I$ can never be a regular ring except in the trivial case that $I = 0$.

Corollary 2.2. *Let R be a local ring. If $R \bowtie I$ is a regular local ring, then $I = 0$.*

Proof. It is well known that a regular local ring is an integral domain. So the assertion is clear from Lemma 2.1. □

The above corollary enables us to determine when $R \bowtie I$ satisfies Serre's condition (R_n).

Theorem 2.3. *The ring $R \bowtie I$ satisfies Serre's condition (R_n) if and only if R satisfies (R_n) and $I_\mathfrak{p} = 0$ for each $\mathfrak{p} \supseteq I$ such that $\operatorname{ht} \mathfrak{p} \leq n$.*

Proof. Assume that $R \bowtie I$ satisfies (R_n). Then, by [1, Proposition 3.7], R also satisfies (R_n). Now let $\mathfrak{p} \supseteq I$ be such that $\operatorname{ht} \mathfrak{p} \leq n$. Then, by Proposition 1.1, $\mathfrak{q} := \mathfrak{p}_0 = \mathfrak{p}_1 = \mathfrak{p}_2$ and $\operatorname{ht} \mathfrak{q} \leq n$. So that $(R \bowtie I)_\mathfrak{q} \cong R_\mathfrak{p} \bowtie I_\mathfrak{p}$ is a regular local ring. Therefore Corollary 2.2 shows that $I_\mathfrak{p} = 0$. Conversely assume that R satisfies (R_n) and $I_\mathfrak{p} = 0$ for each $\mathfrak{p} \supseteq I$ such that $\operatorname{ht} \mathfrak{p} \leq n$. In order to prove the assertion, let $\mathfrak{q} \in \operatorname{Spec}(R \bowtie I)$ be such that $\operatorname{ht} \mathfrak{q} \leq n$ and set $\mathfrak{p} := \mathfrak{q} \cap R$. Hence $\operatorname{ht} \mathfrak{p} \leq n$. If $I \subseteq \mathfrak{p}$, then $\mathfrak{q} = \mathfrak{p}_0 = \mathfrak{p}_1 = \mathfrak{p}_2$. Thus, by the hypothesis, $I_\mathfrak{p} = 0$. So that $(R \bowtie I)_\mathfrak{q} \cong R_\mathfrak{p}$ is regular by assumption. Finally if $I \nsubseteq \mathfrak{p}$, then $\mathfrak{q} = \mathfrak{p}_1$ or $\mathfrak{p}_2$ and we have $(R \bowtie I)_\mathfrak{q} \cong R_\mathfrak{p}$ which is regular by assumption. □

Corollary 2.4. *Suppose that* $\operatorname{ht} I > n$. *Then* $R \bowtie I$ *satisfies Serre's condition* (R_n) *if and only if* R *satisfies* (R_n).

A finitely generated module M over a Noetherian ring R satisfies Serre's condition (S_n) if $\operatorname{depth} M_{\mathfrak{p}} \geq \min\{n, \dim M_{\mathfrak{p}}\}$, for all $\mathfrak{p} \in \operatorname{Spec}(R)$. In the following theorem, we determine when $R \bowtie I$ satisfies Serre's condition (S_n), which is also a generalization of [1, Theorem 3.1].

Theorem 2.5. *If* $R \bowtie I$ *satisfies Serre's condition* (S_n), *then so does* R *and* I. *The converse holds if* $I_{\mathfrak{p}}$ *is of maximal Krull dimension for all* $\mathfrak{p} \in \operatorname{Spec}(R)$. *In particular, if* $\operatorname{Ann}(I) = 0$ *(e.g.* R *is an integral domain), then* $R \bowtie I$ *satisfies* (S_n) *if and only if* R *and* I *satisfy* (S_n).

Proof. Suppose that $R \bowtie I$ satisfies (S_n). Then, by [1, Theorem 3.1(2)], R satisfies (S_n). Now let $\mathfrak{p} \in \operatorname{Spec}(R)$. If $I \nsubseteq \mathfrak{p}$, then $I_{\mathfrak{p}} = R_{\mathfrak{p}}$. Hence, we have

$$\operatorname{depth} I_{\mathfrak{p}} = \operatorname{depth} R_{\mathfrak{p}} \geq \min\{n, \dim R_{\mathfrak{p}}\} = \min\{n, \dim I_{\mathfrak{p}}\}.$$

If $I \subseteq \mathfrak{p}$, then $(R \bowtie I)_{\mathfrak{p}_0} \cong R_{\mathfrak{p}} \bowtie I_{\mathfrak{p}}$. Thus, by assumption, we have

$$\begin{aligned} \operatorname{depth} I_{\mathfrak{p}} &\geq \operatorname{depth}(R \bowtie I)_{\mathfrak{p}_0} \\ &\geq \min\{n, \dim(R \bowtie I)_{\mathfrak{p}_0}\} \\ &= \min\{n, \dim I_{\mathfrak{p}}\}. \end{aligned}$$

Therefore I satisfies (S_n).

Conversely suppose that R and I satisfy (S_n) and $I_{\mathfrak{p}}$ is of maximal Krull dimension for all $\mathfrak{p} \in \operatorname{Spec}(R)$. Let $\mathfrak{q} \in \operatorname{Spec}(R \bowtie I)$ and put $\mathfrak{p} := \mathfrak{q} \cap R$. If $I \nsubseteq \mathfrak{p}$, then $\mathfrak{q} = \mathfrak{p}_1$ or $\mathfrak{p}_2$ and $(R \bowtie I)_{\mathfrak{q}} \cong R_{\mathfrak{p}}$. Thus

$$\operatorname{depth}(R \bowtie I)_{\mathfrak{q}} = \operatorname{depth} R_{\mathfrak{p}} \geq \min\{n, \dim R_{\mathfrak{p}}\} = \min\{n, \dim(R \bowtie I)_{\mathfrak{q}}\}.$$

If $I \subseteq \mathfrak{p}$, then $(R \bowtie I)_{\mathfrak{q}} \cong R_{\mathfrak{p}} \bowtie I_{\mathfrak{p}}$. Thus

$$\begin{aligned} \operatorname{depth}(R \bowtie I)_{\mathfrak{q}} &= \operatorname{depth}(R_{\mathfrak{p}} \bowtie I_{\mathfrak{p}}) \\ &= \min\{\operatorname{depth} R_{\mathfrak{p}}, \operatorname{depth} I_{\mathfrak{p}}\} \\ &\geq \min\{n, \dim R_{\mathfrak{p}}, \dim I_{\mathfrak{p}}\} \\ &= \min\{n, \dim R_{\mathfrak{p}}\} \\ &= \min\{n, \dim(R \bowtie I)_{\mathfrak{q}}\}. \end{aligned}$$

Therefore $R \bowtie I$ satisfies (S_n).

For the in particular case, note that if $\operatorname{Ann}(I) = 0$, then $\dim I = \dim(R/\operatorname{Ann}(I)) = \dim R$. That is I is of maximal Krull dimension. □

Corollary 2.6. *Let* $x \in R$ *be a regular element. Then* $R \bowtie Rx$ *satisfies* (S_n) *if and only if* R *satisfies* (S_n).

Note that if the ring R is Cohen-Macaulay, then R satisfies Serre's condition (S_n) for any integer n. Also, when $\dim R = d$ and R satisfies Serre's condition (S_d), then R is Cohen-Macaulay. Thus we obtain D'Anna's result [5, Page 512].

Corollary 2.7. *Let $(R, \mathfrak{m})$ be a local ring. Then $R \bowtie I$ is Cohen-Macaulay if and only if R is Cohen-Macaulay and I is maximal Cohen-Macaulay.*

Recall that a ring R is called *normal* if all its localizations are integrally closed domains (see [3]). In [7] (see also [3, Theorem 2.2.22]) Serre characterized normal Noetherian rings: a Noetherian ring is normal if and only if it satisfies conditions (R_1) and (S_2). The next theorem describes the behavior of normality under amalgamated duplication.

Lemma 2.8. *Let I be an ideal of R such that $\operatorname{ht} I \leq 1$ and $R \bowtie I$ is normal. Then $I = 0$.*

Proof. Using Serre's characterization of normality for $R \bowtie I$ in conjunction with Theorems 2.3 and 2.5, one can deduce that R is normal and $I_\mathfrak{p} = 0$ for each $\mathfrak{p} \supseteq I$ such that $\operatorname{ht} \mathfrak{p} \leq 1$. Then $R \cong R_1 \times \cdots \times R_n$ for integrally closed integral domains R_i by [3, Page 71]; so that one can write $I \cong I_1 \times \cdots \times I_n$ for some ideals I_i of R_i such that $\operatorname{ht} I_i \leq 1$ for all $i = 1, \ldots, n$. Now for each $1 \leq i \leq n$, choose $\mathfrak{p}_i \supseteq I_i$ such that $\operatorname{ht} \mathfrak{p}_i \leq 1$ and set $\mathfrak{p} \cong R_1 \times \cdots \times \mathfrak{p}_i \times \cdots \times R_n$. Then $\mathfrak{p} \supseteq I$ and $\operatorname{ht} \mathfrak{p} \leq 1$. Thus $0 = I_\mathfrak{p} \cong (I_i)_{\mathfrak{p}_i}$ and since R_i is an integral domain, we see that $I_i = 0$ for all $i = 1, \ldots, n$. Therefore $I = 0$. □

Theorem 2.9. *Let I be a nontrivial ideal of R. If $R \bowtie I$ is normal, then R is normal, $\operatorname{ht} I \geq 2$ and I satisfies (S_2). The converse holds if $I_\mathfrak{p}$ is of maximal Krull dimension for all $\mathfrak{p} \in \operatorname{Spec}(R)$. In particular, if $\operatorname{Ann}(I) = 0$ (e.g. R is an integral domain), then $R \bowtie I$ is normal if and only if R is normal, $\operatorname{ht} I \geq 2$ and I satisfies (S_2).*

Proof. Suppose that $R \bowtie I$ is normal. Then, one can use Serre's characterization of normality together with Theorems 2.3 and 2.5 to deduce that R is normal and that I satisfies (S_2). Now if $\operatorname{ht} I \leq 1$, then, by Lemma 2.8, we have $I = 0$ which is a contradiction. Conversely, again, Theorems 2.3 and 2.5 imply that $R \bowtie I$ satisfies (R_1) and (S_2). Thus $R \bowtie I$ is a normal ring. □

3. Cohen-Macaulay rings

In [5], D'Anna showed that $R \bowtie I$ is Cohen-Macaulay if and only if R is Cohen-Macaulay and I is maximal Cohen-Macaulay. In this section, we are interested in establishing a similar result for generalized Cohen-Macaulay rings and filter rings.

Let us recall that a finitely generated module M over a Noetherian local ring $(R, \mathfrak{m})$ is said to be a generalized Cohen-Macaulay R-module if $H^i_\mathfrak{m}(M)$, the i-th local cohomology module of M with respect to $\mathfrak{m}$, is of finite length for all $i < \dim M$. A local ring is called generalized Cohen-Macaulay if it is a generalized Cohen-Macaulay module over itself. It is known that over a local ring $(R, \mathfrak{m})$, a finitely generated R-module M is Cohen-Macaulay if and only if $H^i_{\mathfrak{m}}(M) = 0$ for all $i < \dim M$ (see [2, Corollary 6.2.8]). Therefore every Cohen-Macaulay module is a generalized Cohen-Macaulay module.

Remark 3.1. *Let $(R, \mathfrak{m})$ be a local ring and I be a nontrivial ideal of R. Consider the local ring homomorphism $\varphi : R \to R \bowtie I$, where $\varphi(r) = (r, r)$. By [6, Theorem 3.5(a)(v)], we have $\mathfrak{m}_0 = \sqrt{\mathfrak{m}(R \bowtie I)}$. Thus, by the Independence Theorem of local cohomology [2, Theorem 4.2.1], we have*

$$\begin{aligned} H^i_{\mathfrak{m}_0}(R \bowtie I) \cong & H^i_{\mathfrak{m}}(R \bowtie I) \\ \cong & H^i_{\mathfrak{m}}(R \oplus I) \\ \cong & H^i_{\mathfrak{m}}(R) \oplus H^i_{\mathfrak{m}}(I) \end{aligned}$$

as R-modules; so that $R \bowtie I$ is Cohen-Macaulay if and only if $H^i_{\mathfrak{m}_0}(R \bowtie I) = 0$ for all $i < \dim R \bowtie I$ if and only if $H^i_{\mathfrak{m}}(R) = 0 = H^i_{\mathfrak{m}}(I)$ for all $i < \dim R$ if and only if R is Cohen-Macaulay and I is maximal Cohen-Macaulay. Thus, we obtain a second alternate proof of D'Anna's result.

Next we generalize D'Anna's result to generalized Cohen-Macaulay rings. To this end, we need an auxiliary lemma.

In the course of next lemma and its proof, for a finite length R-module M, we use $\ell_R(M)$ to denote the length of M over R.

Lemma 3.2. *Let $(R, \mathfrak{m})$ be a local ring and M be an $R \bowtie I$-module. If $\ell_{R \bowtie I}(M) < \infty$, then, $\ell_R(M) < \infty$ and $\ell_{R \bowtie I}(M) = \ell_R(M)$. Here M is considered as an R-module via $\varphi : R \to R \bowtie I$.*

Proof. Let $n := \ell_{R \bowtie I}(M)$. Then, there is a composition series $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ of $R \bowtie I$-modules such that $M_i/M_{i-1} \cong R \bowtie I/\mathfrak{m}_0$ for all $i = 1, \ldots, n$. On the other hand, $R \bowtie I/\mathfrak{m}_0 \cong R/\mathfrak{m}$ as R-modules. Thus, we have a composition series of R-modules $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ such that $M_i/M_{i-1} \cong R/\mathfrak{m}$ for all $i = 1, \ldots, n$. Therefore $\ell_R(M) < \infty$ and $\ell_R(M) = n$. □

Theorem 3.3. *Let $(R, \mathfrak{m})$ be a local ring and I be a nontrivial ideal of R. Then $R \bowtie I$ is generalized Cohen-Macaulay if and only if R and I are generalized Cohen-Macaulay and $\dim I \in \{0, \dim R\}$.*

Proof. Suppose that $R \bowtie I$ is generalized Cohen-Macaulay. Then the local cohomology module $H^i_{\mathfrak{m}_0}(R \bowtie I)$ is of finite length over $R \bowtie I$ for all $i < \dim(R \bowtie I)$. Thus, by the previous lemma, $H^i_{\mathfrak{m}_0}(R \bowtie I)$ has finite length over R for $i < \dim R$. Notice that we have the R-isomorphism $H^i_{\mathfrak{m}_0}(R \bowtie I) \cong H^i_{\mathfrak{m}}(R) \oplus H^i_{\mathfrak{m}}(I)$. Hence $H^i_{\mathfrak{m}}(R)$ and $H^i_{\mathfrak{m}}(I)$ have finite length over R

for all $i < \dim R$. Therefore R and I are generalized Cohen-Macaulay and $\dim I = \dim R$ or 0 by [2, Corollary 7.3.3]. Conversely suppose that R and I are generalized Cohen-Macaulay and $\dim I = \dim R$ or 0. Thus, there exists a positive integer t such that $\mathfrak{m}^t H^i_{\mathfrak{m}}(R) = 0 = \mathfrak{m}^t H^i_{\mathfrak{m}}(I)$ for all $i < \dim R$. Hence $\mathfrak{m}^t H^i_{\mathfrak{m}_0}(R \bowtie I) = 0$ for all $i < \dim(R \bowtie I)$. On the other hand, by [6, Theorem 3.5(a)(v)], we know $\mathfrak{m}_0 = \sqrt{\mathfrak{m}(R \bowtie I)}$; so that there exists a positive integer s such that $\mathfrak{m}_0^s \subseteq \mathfrak{m}(R \bowtie I)$. Consequently $\mathfrak{m}_0^{st} H^i_{\mathfrak{m}_0}(R \bowtie I) = 0$ for all $i < \dim(R \bowtie I)$. Therefore $R \bowtie I$ is generalized Cohen-Macaulay. □

In [4], Cuong, Schenzel, and Trung introduced the notion of filter regular sequence as an extension of the more known concept of regular sequences. By using this notion they defined the filter modules.

Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated R-module. Recall from [4] that a sequence $x_1, \dots, x_n$ of elements in $\mathfrak{m}$ is an M-filter regular sequence if $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}(M/(x_1, \dots, x_{i-1})M) \backslash \{\mathfrak{m}\}$ and for all $i = 1, \dots, n$. M is called a *filter module* if every system of parameters of M is an M-filter regular sequence. A ring is called filter ring if it is a filter module over itself.

In general, every generalized Cohen-Macaulay module is a filter module.

Proposition 3.4. *([4, Satz 2.5]) Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated R-module such that $\dim M > 0$. Then the following are equivalent:*

(1) *M is a filter module.*
(2) $\operatorname{depth} M_{\mathfrak{p}} = \dim M - \dim R/\mathfrak{p}$, *for all* $\mathfrak{p} \in \operatorname{Supp}(M) \backslash \{\mathfrak{m}\}$.
(3) *$M_{\mathfrak{p}}$ is Cohen-Macaulay, for all $\mathfrak{p} \in \operatorname{Supp}(M) \backslash \{\mathfrak{m}\}$, and $\operatorname{Supp}(M)$ is catenary and equidimensional.*
(4) *$M_{\mathfrak{p}}$ is Cohen-Macaulay, of dimension $\dim M_{\mathfrak{p}} = \dim M - \dim R/\mathfrak{p}$, for all $\mathfrak{p} \in \operatorname{Supp}(M) \backslash \{\mathfrak{m}\}$.*

Finally, using above proposition, we obtain the following generalization of D'Anna's result to filter rings.

Theorem 3.5. *Let $(R, \mathfrak{m})$ be a local ring and I be a nontrivial ideal of R. Then $R \bowtie I$ is a filter ring if and only if R is a filter ring and $I_{\mathfrak{p}}$ is maximal Cohen-Macaulay for all $\mathfrak{p} \in \operatorname{Spec}(R) \backslash \{\mathfrak{m}\}$.*

Proof. First suppose that $R \bowtie I$ is a filter ring. Let $\mathfrak{p} \in \operatorname{Spec}(R) \backslash \{\mathfrak{m}\}$. Consider the following two cases.

Case 1. If $I \subseteq \mathfrak{p}$, then $\mathfrak{p}_0$ is the only prime ideal of $R \bowtie I$ lying over $\mathfrak{p}$, and we have the isomorphisms $(R \bowtie I)_{\mathfrak{p}_0} \cong R_{\mathfrak{p}} \bowtie I_{\mathfrak{p}}$ and $(R \bowtie I)/\mathfrak{p}_0 \cong R/\mathfrak{p}$. Note that in this case $\mathfrak{p}_0 \neq \mathfrak{m}_0$. Since $R \bowtie I$ is a filter ring, then, by Proposition 3.4, $(R \bowtie I)_{\mathfrak{p}_0}$ is Cohen-Macaulay and

$$\dim(R \bowtie I)_{\mathfrak{p}_0} = \dim(R \bowtie I) - \dim(R \bowtie I)/\mathfrak{p}_0.$$

Therefore, the Cohen-Macaulayness of $R_{\mathfrak{p}} \bowtie I_{\mathfrak{p}}$ in conjunction with Proposition 2.7 implies that $R_{\mathfrak{p}}$ is Cohen-Macaulay and $I_{\mathfrak{p}}$ is maximal Cohen-Macaulay. Also one has

$$\begin{aligned}\dim R_{\mathfrak{p}} &= \dim(R_{\mathfrak{p}} \bowtie I_{\mathfrak{p}}) \\ &= \dim(R \bowtie I)_{\mathfrak{p}_0} \\ &= \dim(R \bowtie I) - \dim(R \bowtie I)/\mathfrak{p}_0 \\ &= \dim R - \dim R/\mathfrak{p}.\end{aligned}$$

Case 2. If $I \nsubseteq \mathfrak{p}$, then $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are the only prime ideals of $R \bowtie I$ lying over $\mathfrak{p}$, and we have the isomorphisms $(R \bowtie I)_{\mathfrak{p}_i} \cong R_{\mathfrak{p}}$ and $(R \bowtie I)/\mathfrak{p}_i \cong R/\mathfrak{p}$ for $i = 1, 2$. Again note that $\mathfrak{p}_i \neq \mathfrak{m}_0$ for $i = 1, 2$. Thus, $(R \bowtie I)_{\mathfrak{p}_i} \cong R_{\mathfrak{p}}$ is Cohen-Macaulay and

$$\dim(R \bowtie I)_{\mathfrak{p}_i} = \dim(R \bowtie I) - \dim(R \bowtie I)/\mathfrak{p}_i.$$

Note also that $I_{\mathfrak{p}} = R_{\mathfrak{p}}$ is maximal Cohen-Macaulay. Hence, we obtain that

$$\dim R_{\mathfrak{p}} = \dim R - \dim R/\mathfrak{p}.$$

Therefore R is a filter ring by Proposition 3.4.

Conversely assume that R is a filter ring and $I_{\mathfrak{p}}$ is maximal Cohen-Macaulay for all $\mathfrak{p} \in \mathrm{Spec}(R)\backslash\{\mathfrak{m}\}$. Let $\mathfrak{q} \in \mathrm{Spec}(R \bowtie I)\backslash\{\mathfrak{m}_0\}$ and set $\mathfrak{p} = \mathfrak{q} \cap R$. Consider the following two cases.

Case 1. If $I \subseteq \mathfrak{p}$, then we have $(R \bowtie I)_{\mathfrak{q}} \cong R_{\mathfrak{p}} \bowtie I_{\mathfrak{p}}$ and $(R \bowtie I)/\mathfrak{q} \cong R/\mathfrak{p}$. Since $\mathfrak{p} \neq \mathfrak{m}$, by assumption, we have that $R_{\mathfrak{p}}$ is Cohen-Macaulay and $I_{\mathfrak{p}}$ is maximal Cohen-Macaulay. Consequently $(R \bowtie I)_{\mathfrak{q}}$ is Cohen-Macaulay and we have the following equalities

$$\begin{aligned}\dim(R \bowtie I)_{\mathfrak{q}} &= \dim(R_{\mathfrak{p}} \bowtie I_{\mathfrak{p}}) \\ &= \dim R_{\mathfrak{p}} \\ &= \dim R - \dim R/\mathfrak{p} \\ &= \dim(R \bowtie I) - \dim(R \bowtie I)/\mathfrak{q}.\end{aligned}$$

Case 2. If $I \nsubseteq \mathfrak{p}$, then $\mathfrak{q} = \mathfrak{p}_1$ or $\mathfrak{p}_2$ and we have the isomorphisms $(R \bowtie I)_{\mathfrak{q}} \cong R_{\mathfrak{p}}$ and $(R \bowtie I)/\mathfrak{q} \cong R/\mathfrak{p}$. Since $\mathfrak{p} \neq \mathfrak{m}$, by assumption, we have that $R_{\mathfrak{p}}$ is Cohen-Macaulay. Consequently $(R \bowtie I)_{\mathfrak{q}}$ is Cohen-Macaulay and we have the following equality

$$\dim(R \bowtie I)_{\mathfrak{q}} = \dim(R \bowtie I) - \dim(R \bowtie I)/\mathfrak{q}.$$

Therefore $R \bowtie I$ is a filter ring. □

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