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OPTIMALITY CONDITIONS FOR PARETO EFFICIENCY AND PROPER IDEAL POINT IN SET-VALUED NONSMOOTH VECTOR OPTIMIZATION USING CONTINGENT CONE

Y. F. CHAI* AND S. Y. LIU

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ABSTRACT. In this paper, we first present a new important property for Bouligand tangent cone (contingent cone) of a star-shaped set. We then establish optimality conditions for Pareto minima and proper ideal efficiencies in nonsmooth vector optimization problems by means of Bouligand tangent cone of image set, where the objective is generalized cone convex set-valued map, in general real normed spaces.

Keywords: Star-shaped set, Bouligand tangent cone, generalized cone convex maps, optimality conditions.

MSC(2010): Primary: 90C30; Secondary: 90C46.

1. Introduction

In the last several decades, nonsmooth set-valued vector optimization problem has attracted increasing attentions. Various notions of efficiency for vector optimization problems with set-valued maps have been introduced, such as Pareto minima [5], weak efficient points, ε -weak efficient points [6, 8, 15, 16], strong minima [2, 10], proper efficiencies [12] and isolated minima [4]. To consider optimality conditions for these efficiencies, many generalized derivatives of objective maps have been introduced with fruitful applications. Such as the generalized subderivative [2], the contingent derivative and the Dini derivative [5, 6], the Clarke derivative and adjacent derivative [16], the contingent epiderivative [8, 10] and the generalized contingent epiderivative [3].

In most of references, both necessary and sufficient optimality conditions obtained in terms of generalized derivatives are only for weak Pareto efficiencies of vector optimization problems (see, for examples, [4–6, 10, 12, 15] and

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references), and the well-known optimality conditions for Pareto efficiency by means of Bouligand tangent cone of image set are only sufficient (see, [9]). Under what assumptions, do Bouligand tangent cone of image set present also necessary conditions for Pareto efficiency in vector optimization problems? On the other hand, we observe also that the notion of proper ideal points for vector optimization problem was introduced in [1], but rather few results on it along with optimality conditions have been developed. So the main results in this paper are based on Bouligand tangent cone of image set to establish optimality conditions for Pareto minima and proper ideal points in vector optimization problems with generalized cone convex set-valued map. Before this, some useful properties and new results (see, Proposition 2.3 and Corollary 2.7) for contingent cone of a star-shaped set are also highlighted, which are stronger than the corresponding results obtained earlier by Jahn ([11], Chapter 4) and can help us obtain optimality conditions for vector optimization problems.

The organization of the paper is as follows. The preliminaries and the notations, especially an important property for Bouligand tangent cone are presented in section 2. Necessary and sufficient conditions for the existence of Pareto minimum and proper ideal efficiency by means of Bouligand tangent cone of image set in nonsmooth set-valued vector optimization problems, the main results, are established in section 3. Section 4 contains some concluding remarks.

2. Preliminaries and notations

Throughout the paper, if not otherwise stated, let X, Y be real normed spaces, $D \subset Y$ be a closed, convex and pointed cone. $F : X \rightarrow 2^Y$ be a set-valued map. The graph, the epigraph and the domain of F are defined respectively by

$$\begin{aligned} grF &= \{(x, y) \in X \times Y : y \in F(x)\}, \\ epiF &= \{(x, y) \in X \times Y : y \in F(x) + D\}, \end{aligned}$$

and

$$domF = \{x \in X : F(x) \neq \emptyset\}.$$

For $A \subset X$, the sets clA , $intA$, $coneA$ and $extA$ denote respectively by closure, the interior, the cone hull and the extreme point of A .

Furthermore,

$$coneA = \{\lambda a : \lambda \geq 0, a \in A\},$$

and a point $x_0 \in A$ is called an extreme point of A , if

$$x_0 = \lambda x_1 + (1 - \lambda)x_2,$$

for some $x_1, x_2 \in A$ and some $\lambda \in (0, 1)$ implies that $x_1 = x_2 = x_0$.

It is well known that if A is a convex set, so are clA and $coneA$.

A is called star-shaped at x_0 ($x_0 \in A$), if for all $x \in A$ and $\lambda \in [0, 1]$, one has $(1 - \lambda)x_0 + \lambda x \in A$.

The Bouligand tangent cone (contingent cone) of A at x_0 is defined as,

$$T(A, x_0) = \{u \in X : \exists t_n \rightarrow 0^+, \exists u_n \rightarrow u, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, x_0 + t_n u_n \in A\}.$$

Proposition 2.1 (see [11], Chapter 4). *Let $x_0 \in A$, if A is star-shaped at x_0 , then $A - x_0 \subseteq T(A, x_0)$.*

In the following sections, we always denote the origin as θ .

We say that $A \subset X$ satisfies the property F , if for any $x \in A$ and $\lambda \in [0, 1]$, one has $\lambda x \in A$. Let $\theta \in A$, we say that A satisfies the property Λ near θ , if there exists a neighbourhood $B(\theta, \varepsilon)$ of θ such that for any $x \in B(\theta, \varepsilon) \cap (clA \setminus intA)$ and $\lambda \in [0, 1]$, one has $\lambda x \in clA \setminus intA$, where $B(\theta, \varepsilon)$ denotes the ball centered at θ with radius ε .

Corollary 2.2. *Let $A \subset X$ be star-shaped at x_0 , then $A - x_0$ satisfies the property F .*

Proof. Take any $x \in A$ and $t \in [0, 1]$, we have $t(x - x_0) + x_0 = tx + (1 - t)x_0$. Therefore $t(x - x_0) + x_0 \in A$ follows immediately from the assumption that A star-shaped at x_0 , which implies $t(x - x_0) \in A - x_0$. The proof is complete. \square

In the following sections, the n is always in a index set Ξ .

Proposition 2.3. *Let $A \subseteq X$ and $intA \neq \emptyset$. If A satisfies the property F and Λ near θ or the origin θ , then $cone(clA) = cl(coneA)$.*

Proof. Obviously, $\theta \in cone(clA)$ and $\theta \in cl(coneA)$, so we consider only $u \neq \theta$ in the sequel. We first prove the inclusion $cone(clA) \subseteq cl(coneA)$. Let $u \in cone(clA)$, then there exists $a \in clA$, $t > 0$ such that $u = ta$. Furthermore, following $a \in clA$, so there exists $\{a_n\} \subset A$ such that $a_n \rightarrow a$. For each n , set $u_n = ta_n$, then $u_n \in coneA$ and $u_n = ta_n \rightarrow ta = u$, i.e., $u_n \rightarrow u$, which implies $u \in cl(coneA)$.

For the contrary inclusion, we should only prove $cone(clA)$ is a closed cone. Let $\{u_n\} \subset cone(clA)$ and $u_n \rightarrow u$. Next, we will prove $u \in cone(clA)$. In fact, we conclude that there exist $x_n \in X$ and $x_n \rightarrow \theta$ such that $u + x_n = u_n \in cone(clA)$. So, for each n , there exist $t_n > 0$, $a_n \in clA$ such that

$$(2.1) \quad u + x_n = u_n = t_n a_n.$$

If $a_n \rightarrow \theta$, then $t_n \rightarrow +\infty$ as $n \rightarrow +\infty$. For each n , we have

$$\frac{u}{t_n} + \frac{x_n}{t_n} = a_n \in clA.$$

For given $\varepsilon > 0$, there exists n_0 such that $a_n \in B(\theta, \varepsilon)$, $\forall n \geq n_0$. We consider $\{a_n\}$ into two cases to discuss.

(i) If $\{a_n\} \subset clA \setminus intA$ as $n \geq n_0$, by assumption, we can conclude that there exists a subsequence $\{a_{n_m}\} \subseteq \{a_n\}$ such that

$$(2.2) \quad a_{n_m} = \lambda_{n_m} a_{\bar{n}_0},$$

with $\lambda_{n_m} \in (0, 1]$ and $\bar{n}_0 \geq n_0$. In fact, since $t_n a_n \rightarrow u$, so for any $\varepsilon_1 > 0$, there exists $n_1 \geq n_0$ such that $\|t_n a_n - u\| \leq \varepsilon_1$ as $n \geq n_1$. Otherwise, if for any given $\bar{n}_0 \geq n_1$ and $\lambda \in (0, 1]$,

$$(2.3) \quad \lambda a_{\bar{n}_0} \notin \{a_n\},$$

then we can take some t_n from $\{t_n\}$ such that $\frac{t_{\bar{n}_0}}{t_n} < 1$ as $n \geq \bar{n}_0$. From (2.3), it follows that $\frac{t_{\bar{n}_0}}{t_n} a_{\bar{n}_0} \notin \{a_n\}$ as $n \geq \bar{n}_0$. Thus $\|t_n(\frac{t_{\bar{n}_0}}{t_n} a_{\bar{n}_0}) - u\| = \|t_{\bar{n}_0} a_{\bar{n}_0} - u\| > \varepsilon_1$, a contradiction. So (2.2) holds. Again from $t_n a_n \rightarrow u$, so, $t_{n_m} a_{n_m} = t_{n_m} \lambda_{n_m} a_{\bar{n}_0} \rightarrow u$. Furthermore, $a_{\bar{n}_0}$ is fixed, so, $t_{n_m} \lambda_{n_m} \rightarrow t_0$. That is, $t_{n_m} \lambda_{n_m} = t_0 + \bar{t}_{n_m}$ with $\bar{t}_{n_m} \rightarrow 0$. Thus, $t_{n_m} a_{n_m} = (t_0 + \bar{t}_{n_m}) a_{\bar{n}_0} = t_0 a_{\bar{n}_0} + \bar{t}_{n_m} a_{\bar{n}_0} \rightarrow t_0 a_{\bar{n}_0}$. Considering the uniqueness of limits, we have $u = t_0 a_{\bar{n}_0}$ with $a_{\bar{n}_0} \in clA \setminus intA$, which implies $u \in cone(clA)$.

(ii) If $\{a_n\} \subset intA$, then for any n , $\exists \varepsilon_n$ such that $B(a_n, \varepsilon_n) \subset A$. From $t_n a_n = u + x_n$, one has $a_n - \frac{u}{t_n} = \frac{x_n}{t_n}$ and $a_n \in B(\theta, \varepsilon)$ as n large enough. If there exists n_0 such that $\|a_{n_0} - \frac{u}{t_{n_0}}\| = \frac{\|x_{n_0}\|}{t_{n_0}} \leq \varepsilon_{n_0}$, then $\frac{u}{t_{n_0}} \in A$, so, $u \in cone(clA)$. If, for any n , $\|a_n - \frac{u}{t_n}\| = \frac{\|x_n\|}{t_n} > \varepsilon_n$, then this implies there exists $\bar{a}_n \in clA \setminus intA$ such that $\|\bar{a}_n - a_n\| \leq \frac{\|x_n\|}{t_n}$. But, $\|t_n \bar{a}_n - u\| = \|t_n \bar{a}_n - t_n a_n + t_n a_n - u\| \leq t_n \frac{\|x_n\|}{t_n} + \|x_n\| = 2\|x_n\| \rightarrow 0$. Thus, $t_n \bar{a}_n \rightarrow u$. Considering $\bar{a}_n \in clA \setminus intA$ and (i), we conclude that $u \in cone(clA)$.

If $a_n \rightarrow \theta$, then $\{t_n\}$ is bounded. Set $t = \sup_n \{t_n\}$, then $t \geq t_n$ and $t \in R^+$. Dividing t by (2.1) and, for each n , set $b_n = \frac{t_n a_n}{t} = \frac{u}{t} + \frac{x_n}{t}$. By the assumption that A satisfies the property F , we get $b_n \in A$ and $b_n \rightarrow \frac{u}{t}$ as $n \rightarrow \infty$. Thus, $\frac{u}{t} \in clA$, i.e., $u \in cone(clA)$. From the above two cases, we conclude that $cone(clA)$ is a closed cone, so, $cl(coneA) \subseteq cone(clA)$. The proof is complete. \square

The following two examples will illustrate the importance of the property Λ for a set.

Example 2.4. Consider the set $A = \{(x, y) \in R^2 : 0 \leq y \leq \sqrt{x}, x \geq 0\}$, then one has

$$clconeA = \{(x, y) \in R^2 : x \geq 0, y \geq 0\},$$

but,

$$cone(clA) = \{(x, y) \in R^2 : x > 0, y \geq 0\} \cup \{(0, 0)\}.$$

Obviously, $clconeA \neq cone(clA)$.

We note that A satisfies the property F , but not satisfies Λ .

Example 2.5. Consider $A = \{(x, y) : 0 \leq y \leq 2x \leq \frac{1}{2}\} \cup \{(x, y) : 0 \leq y \leq \sqrt{x}, x \geq \frac{1}{4}\}$, it is easy to see that

$$cone(clA) = \{(x, y) \in R^2 : 0 \leq y \leq 2x, x \geq 0\} = clconeA.$$

Obviously, the set A satisfies both F and Λ .

Corollary 2.6. *Let A be a cone in X and satisfy the property Λ near the set θ , then $\text{cone}(clA) = clA$.*

Corollary 2.7. *Let $x_0 \in A$. If A is a closed subset of X and star-shaped at x_0 with $\text{int}A \neq \emptyset$ such that $A - x_0$ satisfies the property Λ near the θ , then $\text{cone}A = cl(\text{cone}A)$.*

Proposition 2.8 ([11], Chapter 4). *Let $x_0 \in A \subset X$, if A is star-shaped at x_0 , then $T(A, x_0) = cl(\text{cone}(A - x_0))$.*

The following corollary states a new property for Bouligand tangent cone (contingent cone) of a star-shaped set.

Corollary 2.9. *Let $x_0 \in A \subset X$. If A is a closed set and star-shaped at x_0 with $\text{int}A \neq \emptyset$ such that $A - x_0$ satisfies the property Λ near θ , then $T(A, x_0) = \text{cone}(A - x_0)$.*

Proof. By Proposition 2.6, one has $T(A, x_0) = cl(\text{cone}(A - x_0))$. Furthermore, applying Corollary 2.5, one has

$$cl(\text{cone}(A - x_0)) = \text{cone}(A - x_0),$$

which implies $T(A, x_0) = \text{cone}(A - x_0)$. □

Let A be a subset of X , $F : A \rightarrow 2^Y$ be a set-valued map, $D \subset Y$ be a closed convex and pointed cone. We recall that F is closed-valued on A , if $F(A)$ is a closed subset of Y , and we call that F is D -closed-valued on A , if $F(A) + D$ is a closed subset of Y . We say that F is D -convex-along-rays at $(x_0, y_0) \in grF$, if A is star-shaped at x_0 and $(1 - t)y_0 + tF(x) \subseteq F((1 - t)x_0 + tx) + D$ for all $x \in A$ and $0 < t < 1$, and F is convex-along-rays at $(x_0, y_0) \in grF$ if A is star-shaped at x_0 and $(1 - t)y_0 + tF(x) \subseteq F((1 - t)x_0 + tx)$ for all $x \in A$ and $0 < t < 1$. Obviously, if F is D -convex-along-rays at (x_0, y_0) , then $F(A) + D$ is star-shaped at y_0 . Similarly, if F convex-along-rays at (x_0, y_0) , then $F(A)$ is star-shaped at y_0 .

Now we turn attention to a D -convexlike set-valued maps F . Let A be a nonempty subset of X . F is called D -convexlike, if for all $x_1, x_2 \in A$ and $\lambda \in [0, 1]$,

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subseteq F(A) + D.$$

Remark 2.10. *F is D -convexlike on A if and only if $F(A) + D$ is a convex subset of Y .*

Let A be a convex set. We recall that a set-valued map $F : A \rightarrow 2^Y$ is called convex map on convex set A , if for all $x_1, x_2 \in A$ and $\lambda \in [0, 1]$,

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subseteq F((\lambda x_1) + (1 - \lambda x_2)).$$

3. Main results

We consider the following set-valued vector optimization problem:

$$(3.1) \quad \text{Min}\{F(x) : x \in A\},$$

where $F : X \rightarrow 2^Y$, $F(A) = \bigcup_{x \in A} F(x)$.

In this section we restrict ourselves to dealing with necessary and sufficient conditions for the existence of the Pareto minimal point and the proper ideal efficiency of vector optimization problems (3.1) by means of contingent cone of image set. If not otherwise stated, we always assume $D \subset Y$ is a closed convex and pointed cone.

Definition 3.1. Consider the above problem (3.1), let $x_0 \in A$, $y_0 \in F(x_0)$,

(i) A pair $(x_0, y_0) \in \text{gr}F$ is called Pareto minimal point of F on A , if $(F(A) - y_0) \cap (-D) = \{\theta\}$.

(ii) A pair $(x_0, y_0) \in \text{gr}F$ is called proper ideal point of F on A , if there exists a closed, convex, pointed cone P of Y such that $D \subset P$ and $F(A) - y_0 \subset P$.

The set of all minimal points and proper ideal points of (3.1) are denoted by $\text{Min}(F, A)$ and $\text{PI}(F, A)$, respectively.

Obviously, $\text{PI}(F, A) \subseteq \text{Min}(F, A)$.

Now, let us first consider the optimality conditions for Pareto minimal points of (3.1).

In the following, we suppose $\text{int}(F(A)) \neq \emptyset$.

Theorem 3.2. Let $(x_0, y_0) \in \text{gr}F$, $x_0 \in A$ and A be star-shaped at x_0 . If $F : A \rightarrow 2^Y$ is convex-along-rays at (x_0, y_0) and closed-valued on A such that $F(A) - y_0$ satisfies the property Λ near θ , then $(x_0, y_0) \in \text{Min}(F, A)$ if and only if $T(F(A), y_0) \cap (-D) = \{\theta\}$.

Proof. F is closed-valued on A , i.e., $F(A)$ is a closed subset of Y , so is $F(A) - y_0$. F is convex-along-rays at (x_0, y_0) , thus, $F(A)$ is star-shaped at y_0 . From Corollary 2.2, $F(A) - y_0$ satisfies the property F and also the property Λ near θ by assumption. Applying Proposition 2.6 and Proposition 2.3, we get $T(F(A), y_0) = \text{cl}(\text{cone}(F(A) - y_0)) = \text{cone}(F(A) - y_0)$. Therefore, $(F(A) - y_0) \cap (-D) = \{\theta\}$ if and if $T(F(A), y_0) \cap (-D) = \{\theta\}$. The proof is complete. \square

Lemma 3.3 ([9], Proposition 3.2). Let $x_0 \in A$, $y_0 \in F(x_0)$. If D is a pointed cone and $T(F(A) + D, y_0) \cap (-D) = \{\theta\}$, then $(x_0, y_0) \in \text{Min}(F, A)$.

Theorem 3.4. Let $(x_0, y_0) \in \text{gr}F$, $x_0 \in A$ and A be star-shaped at x_0 . If $F : A \rightarrow 2^Y$ is D -convex-along-rays at (x_0, y_0) , D -closed valued on A and

$(x_0, y_0) \in \text{Min}(F, A)$ such that $(F(A) + D - y_0)$ satisfies the property Λ near θ , then $T(F(A) + D, y_0) \cap (-D) = \{\theta\}$.

Proof. Since $(x_0, y_0) \in \text{Min}(F, A)$, therefore $(F(A) - y_0) \cap (-D) = \{\theta\}$. Since D is a pointed cone, so $(F(A) + D - y_0) \cap (-D) = \{\theta\}$. By virtue of Corollary 2.7, we conclude that $T(F(A) + D, y_0) = \text{cone}(F(A) + D - y_0)$, which yields $T(F(A) + D, y_0) \cap (-D) = \{\theta\}$. \square

From Lemma 3.3 and Theorem 3.4, we get the following corollary.

Corollary 3.5. *Let $x_0 \in A$, $y_0 \in F(x_0)$. If $F : A \rightarrow 2^Y$ is D -closed valued on A and $F(A) + D$ is also star-shaped at y_0 such that $(F(A) + D - y_0)$ satisfies the property Λ near θ , then $(x_0, y_0) \in \text{Min}(F, A)$ if and only if $T(F(A) + D, y_0) \cap (-D) = \{\theta\}$.*

If F is a convex map on A and D is a pointed convex cone (not necessarily closed) of Y , we will obtain another necessary conditions for Pareto minimal points of (3.1).

We recall that if A is a convex set and $\text{int}A \neq \emptyset$, then $\text{int}A$ is a convex set and $\text{int}A = \text{int}(clA)$, see [14].

Theorem 3.6. *Let $(x_0, y_0) \in \text{gr}F$ and $x_0 \in A$. If F is a convex map on convex set A and D is a convex pointed cone of Y such that $(x_0, y_0) \in \text{Min}(F, A)$, $y_0 \in \text{ext}(F(A))$ and $D \subset (\text{int}(T(F(A) + D, y_0) \cup \{\theta\}))$, then $T(F(A) + D, y_0) \cap (-D) = \{\theta\}$.*

Proof. From the assumption that F is a convex map on convex set A and D is a convex pointed cone, we conclude that $F(A) + D$ is a convex set. It is clear that

$$T(F(A) + D, y_0) = cl(\text{cone}(F(A) + D - y_0)).$$

Therefore, $\text{int}(T(F(A) + D, y_0) \cap (-D)) = \text{int}(cl(\text{cone}(F(A) + D - y_0) \cap (-D))) = \text{int}(\text{cone}(F(A) + D - y_0) \cap (-D))$. Suppose that $T(F(A) + D, y_0) \cap (-D) = \{\theta\}$ is not true, i.e., there is $u \in T(F(A) + D, y_0) \cap (-D)$ and $u \neq \theta$. Since $D \subset (\text{int}(T(F(A) + D, y_0) \cup \{\theta\}))$, we can assume that

$$u \in T(F(A) + D, y_0) \cap (-\text{int}(T(F(A) + D, y_0))).$$

Then for n sufficiently large, there exists $u_n \in \text{cone}(F(A) + D - y_0) \cap (-\text{cone}(F(A) + D - y_0))$ such that $u_n \rightarrow u$. Thereby, there exists $t_n, s_n > 0$, $y_n, w_n \in F(A)$ and $d_n, q_n \in D$ such that

$$u_n = t_n(y_n + d_n - y_0),$$

and

$$(3.2) \quad u_n = -s_n(w_n + q_n - y_0),$$

i.e.,

$$(3.3) \quad t_n(y_n + d_n - y_0) + s_n(w_n + q_n - y_0) = \theta.$$

Obviously, $y_n \neq y_0 \neq w_n$.

Otherwise, if $y_n = y_0$, then

$$(3.4) \quad u_n = td_n \neq \theta.$$

Substituting (3.4) into (3.2), one has

$$(3.5) \quad \theta \neq s_n(w_n + q_n - y_0) = -td_n \in (-D),$$

which contradicts $(x_0, y_0) \in \text{Min}(F, A)$.

Dividing $(t_n + s_n)$ by (3.3) one has

$$\left(\frac{t_n y_n}{t_n + s_n} + \frac{s_n w_n}{t_n + s_n}\right) + \left(\frac{t_n d_n}{t_n + s_n} + \frac{s_n q_n}{t_n + s_n}\right) - y_0 = \theta.$$

From F being convex map on convex set A , D a convex cone and $y_0 \in \text{ext}(F(A))$, we can assume that there exists $y \in F(A)$, $y \neq y_0$, $d \in D$ such that

$$y + d - y_0 = \theta.$$

This contradicts also $(x_0, y_0) \in \text{Min}(F, A)$. The proof is complete. \square

We note that the condition $D \subset (\text{int}(T(F(A) + D, y_0) \cup \{\theta\}))$ of Theorem 3.6 is easily fulfilled if the cone D is not necessarily closed. Let us consider the following example.

Example 3.7. Let $A = [0, 1] \subset \mathbb{R}$, $D = \text{int}\mathbb{R}_+^2 \cup \{(0, 0)\}$ and let $F : A \rightarrow 2^{\mathbb{R}^2}$ be defined by

$$F(x) = \begin{cases} (x; +\infty) \times (x; +\infty), & x \neq 0, \\ (0, 0), & x = 0. \end{cases}$$

It is easy to see $F(A) + D = D$ and $y_0 = (0, 0)$, so $D \subset (\text{int}(T(F(A) + D, y_0) \cup \{\theta\}))$.

Lemma 3.8 ([1], Proposition 3.1). *The following assertions hold: $PI(A) \subset \text{ext}(A) \cap \text{Min}(A)$.*

Where $\text{Min}(A)$, $PI(A)$ and $\text{ext}(A)$ are all minimal points, proper ideal points and extreme points of the set A , respectively.

Next, we establish optimality conditions for proper ideal efficiencies of (3.1).

Theorem 3.9. *Let A be a nonempty subset of X , $F : A \rightarrow 2^Y$ be D -convexlike and D -closed valued. If $(x_0, y_0) \in \text{gr}F$ and $y_0 \in \text{ext}(F(A) + D)$ such that $(F(A) + D - y_0)$ satisfies the property Λ near θ , then $(x_0, y_0) \in PI(F, A)$.*

Proof. Following the D -convex of F on subset A and Remark 2.8, we get that $F(A) + D$ is a convex subset of Y . Obviously, $F(A) + D$ is star-shaped at y_0 . Furthermore, F is D -closed valued on A , so $F(A) + D$ is a closed convex subset of Y . From Corollary 2.7 and proposition 2.1, we have

$$(3.6) \quad T(F(A) + D, y_0) = \text{cone}(F(A) + D - y_0),$$

and

$$(3.7) \quad F(A) - y_0 \subseteq F(A) + D - y_0 \subseteq T(F(A) + D, y_0).$$

(3.6) implies that $T(F(A) + D, y_0)$ is a closed convex cone and $D \subset T(F(A) + D, y_0)$. In the following, we will show that $T(F(A) + D, y_0)$ is a pointed cone.

Suppose to the contrary that there is $u \in T(F(A) + D, y_0) \cap (-T(F(A) + D, y_0))$ and $u \neq \theta$. Then there exists $t, s > 0$, $x_1, x_2 \in A$, $y_1 \in F(x_1)$, $y_2 \in F(x_2)$ and $d_1, d_2 \in D$ such that

$$(3.8) \quad u = t(y_1 + d_1 - y_0),$$

and

$$(3.9) \quad u = -s(y_2 + d_2 - y_0),$$

Obviously, $y_0 \neq y_1 + d_1 \neq y_2 + d_2$.

(3.8) and (3.9) yield

$$(3.10) \quad t(y_1 + d_1 - y_0) + s(y_2 + d_2 - y_0) = \theta.$$

Dividing (3.10) by $s + t$ we obtain

$$(3.11) \quad y_0 = \frac{t(y_1 + d_1)}{s + t} + \frac{s(y_2 + d_2)}{s + t},$$

which is contrary to $y_0 \in \text{ext}(F(A) + D)$. The proof is complete. \square

The following theorem illustrates a minimal point is a proper point under determined conditions.

Theorem 3.10. *Let A be a nonempty subset of X , $F : A \rightarrow 2^Y$ be D -convex and D -closed valued. If $(x_0, y_0) \in \text{Min}(F, A)$, $y_0 \in \text{ext}(F(A))$ such that $(F(A) + D - y_0)$ satisfies the property Λ near θ , then $(x_0, y_0) \in \text{PI}(F, A)$.*

Proof. By virtue of the proof of Theorem 3.8, we need only prove $T(F(A) + D, y_0)$ is a pointed cone. In fact, (3.10) can be rewritten as

$$(3.12) \quad \left(\frac{ty_1}{s+t} + \frac{sy_2}{s+t}\right) + \left(\frac{td_1}{s+t} + \frac{sd_2}{s+t}\right) - y_0 = \theta.$$

We can assume that $y_1 \neq y_0 \neq y_2$. Otherwise, if $y_1 = y_0$, then

$$(3.13) \quad u = td_1 \neq \theta.$$

Substituting (3.13) into (3.9), one has

$$(3.14) \quad \theta \neq s(y_2 + d_2 - y_0) = -td_1 \in (-D),$$

furthermore, there exists $d \in D$ and $d \neq \theta$ such that

$$(3.15) \quad y_2 - y_0 = -d \in (-D \setminus \{\theta\}),$$

which contradicts $(x_0, y_0) \in \text{Min}(F, A)$. By the same reason, $y_0 \neq y_2$. Furthermore, $y_0 \in \text{ext}(F(A))$ implies that

$$(3.16) \quad \frac{ty_1}{s+t} + \frac{sy_2}{s+t} \neq y_0.$$

Following the convexity of D and the D -convexlikeness of F on A , we have,

$$(3.17) \quad \left(\frac{ty_1}{s+t} + \frac{sy_2}{s+t}\right) + \left(\frac{td_1}{s+t} + \frac{sd_2}{s+t}\right) \in F(A) + D + D$$

$$(3.18) \quad = F(A) + D.$$

Taking into account (3.12–3.18), we conclude that there exist $\bar{y} \in F(A)$, $\bar{y} \neq y_0$, and $\bar{d} \in D$ such that $\bar{y} + \bar{d} - y_0 = \theta$, which contradicts $(x_0, y_0) \in \text{Min}(F, A)$. The proof is complete. \square

The following example satisfies all the conditions of Theorem 3.9.

Example 3.11. Let $A = [0, 1] \subset R$, $D = R_+^2$ and let $F : A \rightarrow 2^{R^2}$ be defined by

$$F(x) = \begin{cases} 0 \leq y \leq 2x, & 0 \leq x \leq \frac{1}{4}, \\ y \leq \sqrt{x}, & \frac{1}{4} \leq x \leq 1. \end{cases}$$

Taking any $\varepsilon \in (0, \frac{1}{4}]$, $(F(A) + D - y_0)$ obviously satisfies the property Λ near θ with $y_0 = (0, 0)$.

Theorem 3.12. Let A be a nonempty subset of X , $F : A \rightarrow 2^Y$ be D -convex and D -closed valued on A . If $(x_0, y_0) \in \text{PI}(F, A)$ such that $(F(A) + D - y_0)$ satisfies the property Λ near θ , then $T(F(A) + D, y_0)$ is a closed, convex and pointed cone.

Proof. We need only show that $T(F(A) + D, y_0)$ is a pointed cone. From the assumption $(x_0, y_0) \in \text{PI}(F, A)$ and Lemma 3.7, one has $y_0 \in \text{ext}(F(A))$ and $(x_0, y_0) \in \text{Min}(F, A)$. The following proof is similar to that of Theorem 3.9, so we omit it. \square

Corollary 3.13. Let A be a nonempty subset of X , $F : A \rightarrow 2^Y$ be D -convex and D -closed valued, $(x_0, y_0) \in \text{gr}F$ such that $(F(A) + D - y_0)$ satisfy the property Λ near θ . Then $(x_0, y_0) \in \text{PI}(F, A)$ if and only if $T(F(A) + D, y_0)$ is a closed, convex and pointed cone.

4. Conclusions

Under certain conditions, Proposition 2.3 states $\text{cone}(clA) = cl(\text{cone}A)$, for A a star-shaped set with $\text{int}A \neq \emptyset$, which is stronger than $\text{cone}(clA) \subseteq cl(\text{cone}A)$ in the literature. As a consequence, Corollary 2.7 furthermore shows that $T(A, x_0) = \text{cone}(A - x_0)$ (in general, $T(A, x_0) \supseteq \text{cone}(A - x_0)$, see [11]). Based on the Corollary 2.7 and by means of Bouligand tangent cone of image set of a set-valued map, we established necessary conditions (see, Theorem 3.4 and Theorem 3.6) for the existence of Pareto efficiency (only sufficient condition obtained in [9]) and both necessary and sufficient conditions of proper ideal efficiency (see, Theorem 3.9 and Theorem 3.10) for nonsmooth vector optimization problems with set-valued map are established, which are all new results.

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