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HYPERINVARIANT SUBSPACES AND QUASINILPOTENT OPERATORS

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Abstract. For a bounded linear operator on Hilbert space we define a sequence of the so-called weakly extremal vectors. We study the properties of weakly extremal vectors and show that the orthogonality equation is valid for weakly extremal vectors. Also we show that any quasinilpotent operator T has an hypernoncyclic vector, and so T has a nontrivial hyperinvariant subspace.

Keywords: Hyperinvariant subspace, extremal vector, quasinilpotent operator.
MSC(2010): Primary: 47A15; Secondary: 30C70; 47B07.

1. **Introduction**

Throughout this paper, let H denote a separable (complex) Hilbert space, and $B(H)$ the C^*-algebra of all bounded linear operators on H. For $W \subset B(H)$ we denote by W' the set of all operators which commute with elements of W and set $W'' = (W')'$. Let us recall that a subspace M of H is called a nontrivial hyperinvariant subspace for T if $\{0\} \neq M \neq H$ and it is an invariant subspace for every operator $S \in \{T\}'$. We use the matrix representation for bounded linear operators on a separable Hilbert space i.e., if $T \in B(H)$ and $\{e_n\}$ is an orthonormal basis for a separable Hilbert space H, then an infinite matrix (a_{ij}) represents T when $Tx = \sum_i(\sum_j a_{ij}x_j)e_i$ for all $x = \sum_{i=1}^{\infty} x_ie_i \in H$. In this case, we have $\sum_i |a_{ij}|^2 \leq c$ and $\sum_j |a_{ij}|^2 \leq c$ for some $c > 0$. For more details see [7]. An operator $Q \in B(H)$ is called quasinilpotent if $\sigma(T) = \{0\}$, where

$$\sigma(T) = \{\lambda \in \mathbb{C}; T - \lambda \text{ is not invertible}\}.$$

If T is a quasinilpotent operator then we have $\|T^n\|^{\frac{1}{n}} \to 0$. Also an operator T is called quasiaffinity if T and T^* are injective. A vector x is called

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hypernoncyclic for an operator T if the subspace $\{Ax; A \in \{T\}'\}$, which is hyperinvariant subspace for T, is not dense in H.

Our study motivated by the following problem;

> *Does every quasinilpotent operator on Hilbert space have a non-trivial hyperinvariant subspace?*

This problem has been considered in several papers and solved in some special cases [1-6]. In [1], S. Ansari and P. Enflo introduced extremal vectors as a method of constructing hypernoncyclic vectors and hyperinvariant subspaces for certain operators. Let us give some basic notations of extremal vectors. Assume that $T \in B(H)$ has dense range. For a unit vector $x_0 \in H$ and $0 < \epsilon < 1$ define $\mathcal{F} = \{y; \|Ty - x_0\| \leq \epsilon\}$, then $\mathcal{F}$ is a nonempty, closed and convex subset of H. So there exists a unique minimal vector $y_0 \in \mathcal{F}$. In this case $\|Ty_0 - x_0\| = \epsilon$. In [1], by the method of extremal vectors, it is shown that for any compact operator K and normal operator N, some weak limit of the subsequence of minimal vectors is noncyclic for all operators commuting with K and N, this approach gives a new method for the existence of nontrivial hyperinvariant subspaces. In [3], by the extremal vectors method, it is shown that if A is compact, then either $\begin{pmatrix} A & * \\ 0 & B \end{pmatrix}$ or $\begin{pmatrix} A & 0 \\ * & B \end{pmatrix}$ has a nontrivial hyperinvariant subspace.

2. **Main results**

Assume that T is a linear bounded operator on the separable complex Hilbert space H and T has dense range. For any $x_0 \in H$ and $0 < \epsilon < \|x_0\|$, define

$$\mathcal{W} = \{y \in H : |\langle Ty - x_0, x_0\rangle| \leq \epsilon\|x_0\|^2\}.$$

Then it is easy to see that $\mathcal{W}$ is a closed nonempty and convex subset of H, so there exists a unique minimal vector $y_0 \in \mathcal{W}$ such that

$$\|y_0\| = \inf\{\|y\| : y \in \mathcal{W}\}.$$

In this case we have $|\langle Ty_0 - x_0, x_0\rangle| = \epsilon\|x_0\|^2$, we say y_0 is the weakly extremal vector of T with respect to x_0 and ϵ.

Remark 2.1. *The weakly extremal vectors are equal to extremal vectors in some situations. For example, let $T = \operatorname{diag}(\lambda_i)$ be a diagonalizable operator respect to the orthonormal basis $\{e_i\}$ and $x_0 = e_1$. If y_n (y'_n) are weakly extremal vectors (extremal vectors) of T^n, then*

$$y_n = y'_n = \frac{1-\epsilon}{\lambda_1^n} e_1.$$

Note that the extremal vectors are different from weakly extremal vectors, in general. For example, if $T = \operatorname{diag}(\frac{1}{n})$ and $x_0 = \frac{1}{\sqrt{2}}(e_1 + e_2)$ then the minimum *of* $\{\|y\| : \|Ty - x_0\| \leq \frac{1}{2}\}$ *happens on an elipse which* $(\frac{\sqrt{2}}{2}, \sqrt{2}-1)$ *and* $(\frac{\sqrt{2}}{2}, \sqrt{2}+1)$ *are it's vertices, i.e., we get the following nonlinear programming*

$$\inf z = u^2 + v^2$$

such that

$$\frac{(u - \frac{\sqrt{2}}{2})^2}{1} + \frac{(v - \sqrt{2})^2}{4} \leq \frac{1}{4},$$

which has the unique solution $y' = \frac{2\sqrt{2}}{5}e_1 + \frac{\sqrt{2}}{5}e_2$*. But to find the minimum of* $\{\|y\| : |\langle Ty - x_0, x_0\rangle| \leq \frac{1}{2}\}$*, we get the following nonlinear programming*

$$\inf z = u^2 + v^2$$

such that

$$2 \leq 2\sqrt{2}u + \sqrt{2}v \leq 6,$$

which has the unique solution $y = \frac{\sqrt{2}}{4}e_1 + \frac{\sqrt{2}}{2}e_2$*.*

In the following, we get the orthogonality equation for weakly extremal vectors. First, we give a useful lemma.

Lemma 2.2. *[5] Suppose that* u *and* v *are nonzero vectors in* H *such that for every* $w \in H$*,* $Re\langle u, w\rangle < 0$ *implies that* $Re\langle v, w\rangle > 0$*. Then there exists a negative number* γ_0 *such that* $v = \gamma_0 u$*.*

Theorem 2.3. *(Orthogonality equation) Let* y_0 *be the weakly extremal vector of* T *respect to a unit vector* x_0 *and* $0 < \epsilon < 1$*. Then there exists* $\delta > 0$ *such that*

$$y_0 = \frac{\delta}{1 + \delta\|Tx_0\|^2} T^* x_0.$$

Proof. Assume that for some $z \in H$, $\mathrm{Re}\,\langle Ty_0 - x_0, x_0\rangle\langle x_0, Tz\rangle < 0$. Then there exists $t_0 > 0$ such that, the function $u(t) = |\langle Ty_0 - x_0 + tTz, x_0\rangle|$ decreases on $[0, t_0]$. So for any t in $[0, t_0]$,

$$\epsilon = |\langle Ty_0 - x_0, x_0\rangle| \geq |\langle T(y_0 + tz) - x_0, x_0\rangle|. \tag{2.1}$$

Minimality of $\|y_0\|$ gives that $\|y_0\| \leq \|y_0 + tz\|$, so $\mathrm{Re}\langle y_0, z\rangle > 0$. Hence, by Lemma 2.2, there is some $\gamma_0 < 0$ such that

$$y_0 = \gamma_0\langle Ty_0 - x_0, x_0\rangle T^* x_0, \tag{2.2}$$

thus

$$\begin{aligned}\langle Ty_0 - x_0, x_0\rangle &= \langle \gamma_0\langle Ty_0 - x_0, x_0\rangle TT^* x_0 - x_0, x_0\rangle \\ &= \gamma_0\langle Ty_0 - x_0, x_0\rangle\langle TT^* x_0, x_0\rangle - 1,\end{aligned}$$

which shows that $\langle Ty_0 - x_0, x_0\rangle = \frac{-1}{1 - \gamma_0\|T^* x_0\|^2}$ is a negative real number. Again by (2.2), and if we put $\delta = -\gamma_0$, we get

$$y_0 = \frac{\delta}{1 + \delta\|T^* x_0\|^2} T^* x_0.$$

□

Corollary 2.4. *Let y_0 be the weakly extremal vector of T respect to a unit vector x_0 and $0 < \epsilon < 1$. Then*

$$y_0 = \frac{1-\epsilon}{\|T^* x_0\|^2} T^* x_0.$$

Proof. By the proof of Theorem 2.3 we can see that $\langle Ty_0 - x_0, x_0 \rangle = -\epsilon$, so $\delta = \frac{1-\epsilon}{\epsilon \|T^* x_0\|^2}$ we get the desired result. □

Corollary 2.5. *Let $x_0 \in H$ be a unit vector and $0 < \epsilon < 1$. Then the function $\epsilon \to y_\epsilon$ is analytic, where y_ϵ is weakly extremal vector respect to x_0 and ϵ.*

Note that $|\langle Ty - x_0, x_0 \rangle| \leq \|Ty - x_0\|$ so the norm of weakly extremal vector is less than the norm of extremal vector. It is shown that in [1], if T is a quasinilpotent operator and y_n is extremal vector of T^n, then $\frac{\|y_{n_k - 1}\|}{\|y_{n_k}\|}$ converges to zero for some subsequence n_k. We show that the last assertion is valid for weakly extremal vectors of T^n.

Lemma 2.6. *If T is a quasinilpotent operator, $x_0 \in H, \|x_0\| = 1$, $0 < \epsilon < 1$ and y_n is weakly extremal vector for T^n, then $\lim_{k\to\infty} \frac{\|y_{n_k - 1}\|}{\|y_{n_k}\|} = 0$, for some subsequence y_{n_k}.*

Proof. If $\lim_{n\to\infty} \frac{\|y_{n_k - 1}\|}{\|y_{n_k}\|} \neq 0$ for any subsequences of y_n, then for some $t > 0$,

$$\frac{\|y_{n-1}\|}{\|y_n\|} \geq t > 0, \qquad n \geq 2.$$

So we get

$$\|y_1\| \geq t\|y_2\| \geq \cdots \geq t^{n-1}\|y_n\|,$$

on the other hand by use of the minimality of $\|y_1\|$ and $|\langle T^n y_n - x_0, x_0 \rangle| \leq \epsilon$, we see that $\|y_1\| \leq \|T^{n-1} y_n\|$, hence

$$t^{n-1}\|y_n\| \leq \|y_1\| \leq \|T^{n-1} y_n\| \leq \|T^{n-1}\|\|y_n\|.$$

But this is a contradiction. □

We study the stability of weakly extremal vectors. For this, we recall some definitions of stability of extremal vectors [3]. Suppose T has dense range. Then T is said to be strongly stable (for x_0), if there exist a unit vector x_0 in H and $0 < \epsilon < 1$ such that $T^n y_n \in \vee\{x_0\}$ for all $n \geq n_0$, where $y_n = y_n(x_0, \epsilon)$ are the extremal vectors for T^n and $\vee A$ is the space that spanned by vectors of A. Now we define the weakly stability for operators.

Definition 2.7. *We say that an operator T with dense range is weakly stable (for x_0) if there exist a unit vector x_0 and $0 < \epsilon < 1$ such that $T^n y_n \in \vee\{x_0\}$ for all $n \geq n_0$, where $y_n = y_n(x_0, \epsilon)$ are the weak extremal vectors for T^n.*

In [3] the following theorem is given which characterizes strongly stable operators.

Theorem 2.8. *[3] Suppose T has dense range. Then T is strongly stable for x_0 if and only if the operators in the family $\{T^nT^{*n}\}_{n=n_0}^{\infty}$ (for some n_0) have a common eigenvector x_0.*

For any quasiaffinity operator such as T and nontrivial invariant subspace M of T with $T|_M = A$, in [2] the following question is asked:

Question 2.9. *Is the implication " T is strongly stable for $x_0 \in M \Longrightarrow A$ is strongly stable for x_0 " true?*

We show that if T is strongly stable for x_0 and y'_n is the extremal vector for T^n and y_n is the weak extremal vector for T^n, then $y'_n = y_n$. Also we show that Question 2.9 is equivalent to the following implication:
" T is weakly stable for $x_0 \in M \Longrightarrow A$ is weakly stable for x_0 "

Lemma 2.10. *Suppose T has dense range. Then T is weakly stable for x_0 if and only if the operators in the family $\{T^nT^{*n}\}_{n=n_0}^{\infty}$ (for some n_0) have a common eigenvector x_0.*

Proof. Suppose that T is weakly stable for some x_0. Then by Definition 2.7, there exist c_n such that $T^ny_n = c_nx_0$ for all $n \geq n_0$. Hence by Corollary 2.4 we get

$$\frac{1-\epsilon}{\|T^{*n}\|^2}T^nT^{*n}x_0 = T^ny_n = c_nx_0$$

which shows that the operators in the family $\{T^nT^{*n}\}_{n=n_0}^{\infty}$ (for some n_0) have a common eigenvector x_0. For the reverse assertion suppose that $T^nT^{*n}x_0 = c_nx_0$ for all $n \geq n_0$. Then by Corollary 2.4 we get

$$T^ny_n = \frac{1-\epsilon}{\|T^{*n}\|^2}T^nT^{*n}x_0 = \frac{1-\epsilon}{\|T^{*n}\|^2}c_nx_0 \in \vee\{x_0\}.$$

□

Now, we have the following corollary.

Corollary 2.11. *Suppose T has dense range. Then T is strongly stable for some $x_0 \in H$ if and only if T is weakly stable for x_0.*

Proof. By Lemma 2.10 and Theorem 2.8, the desired result is obtained. □

By the last corollary, Question 2.9 is equivalent to the following question:

Question 2.12. *Is implication " T is weakly stable for $x_0 \in M \Longrightarrow A$ is weakly stable for x_0 " true?*

Theorem 2.13. *Let T be a weakly stable for x_0 and y_n be the weak extremal vector for T^n. Then y_n is the extremal vector for T^n.*

Proof. By hypothesis we see that there exists c_n such that $c_n x_0 = T^n y_n$ so we get

$$-\epsilon = \langle T^n y_n - x_0, x_0\rangle = c_n - 1$$

which gives that $c_n = 1 - \epsilon$. Hence, by Corollary 2.4, $T^n T^{*n} x_0 = \|T^{*n} x_0\|^2 x_0$. Since extremal vectors are unique and $\|T^n y_n - x_0\| = |c_n - 1| = \epsilon$, therefore y_n is the extremal vector for T^n. □

Theorem 2.14. *Let T be a quasiaffinity quasinilpotent operator on H. If there exist a unite vector x_0 and $0 < \epsilon < 1$ such that the weakly extremal vectors $\{y_n\}$ for T^n have the following property*

$$\limsup_{n\geq 1} \|T^{n-1} y_n\| < \infty\,,$$

then T has a hypernoncyclic vector. In particular T has a nontrivial hyperinvariant subspace.

Proof. Suppose $X \in \{T\}'$ is arbitrary. Also suppose y_n is the extremal vectors for T^n and the subsequence $\{y_{n_k}\}$ is such as Lemma 2.6. For any k, let $X y_{n_k-1} = \alpha_k y_{n_k} + \gamma_k$, in which α_k are scalars and $\gamma_k \perp y_{n_k}$. Then

$$\|X y_{n_k-1}\|^2 = |\alpha_k|^2 \|y_{n_k}\|^2 + \|\gamma_k\|^2 \geq |\alpha_k|^2 \|y_{n_k}\|^2$$

so we get the following inequalities

$$\|X\|\|y_{n_k-1}\| \geq \|X y_{n_k-1}\| \geq |\alpha_k|\|y_{n_k}\|.$$

Hence by Lemma 2.6, we have $\lim_{k\to\infty} \alpha_k = 0$. By hypothesis there is a subsequence of $\{y_{n_k}\}$, which we show by $\{y_{n_k}\}$ again, such that $\{T^{n_k-2} y_{n_k-1}\}$ weakly converges to b_0. We claim that $b_0 \neq 0$. Indeed, if $\{T^{n_k-2} y_{n_k-1}\}$ weakly converges to zero then by Corollary 2.4,

$$\begin{aligned}-\epsilon = \langle T^{n_k-1} y_{n_k-1} - x_0, x_0\rangle &= \langle T^{n_k-1} y_{n_k-1}, x_0\rangle - 1\\ &= \langle T^{n_k-2} y_{n_k-1}, T^* x_0\rangle - 1 \to -1.\end{aligned}$$

This is a contradiction, since $0 < \epsilon < 1$. Let $b_{n_k} = T^{n_k-2} y_{n_k-1} - x_0$ then by Theorem 2.3, we have

$$\begin{aligned}\langle T^2 X b_{n_k}, x_0\rangle &= \langle T^2 X (T^{n_k-2} y_{n_k-1} - x_0), x_0\rangle\\ &= \langle T^{n_k} X y_{n_k-1}, x_0\rangle - \langle T^2 X x_0, x_0\rangle\\ &= \langle T^{n_k}(\alpha_k y_{n_k} + \gamma_k), x_0\rangle - \langle T^2 X x_0, x_0\rangle\\ &= \alpha_k \langle T^{n_k} y_{n_k}, x_0\rangle + \langle \gamma_k, T^{*n_k} x_0\rangle - \langle T^2 X x_0, x_0\rangle\\ &= \alpha_k \langle T^{n_k} y_{n_k}, x_0\rangle - \langle T^2 X x_0, x_0\rangle \to -\langle T^2 X x_0, x_0\rangle.\end{aligned}$$

Therefore

$$\langle T^2 X (b_0 - x_0), x_0\rangle = -\langle T^2 X x_0, x_0\rangle\,,$$

so $\langle XT^2b_0, x_0\rangle = 0$. Since T is a quasiaffinity operator, we have $T^{*2}x_0 \neq 0$. Thus b_0 is a hypernoncyclic vector for T. If we put $M = \{XT^2b_0; X \in \{T\}'\}$, then M is a nontrivial hyperinvariant subspace of T. □

Corollary 2.15. *Let T be a quasinilpotent and quasiaffinity operator on H. If there exists a unit vector x_0 and $k > 0$ such that $\|T^nT^{*n}x_0| \leq k\|T^{*n}x_0\|^2$, then T has a hypernoncyclic vector. In particular, T has a nontrivial hyperinvariant subspace.*

Proof. By Corollary 2.4 and the hypothesis the sequence $\{T^ny_n\}$ is bounded, so we can by Lemma 2.6 choose a subsequence n_k and $b_0 \in H$ such that $T^{n_k-1}y_{n_k-1} \to b_0$ weakly, and $\frac{\|y_{n_k-1}\|}{\|y_{n_k}\|} \to 0$. It is easy to see that $b_0 \neq 0$. Let $X \in \{T\}'$ be arbitrary, then we write $Xy_{n_k-1} = \alpha_k y_{n_k} + \gamma_k$, where $\gamma_k \perp y_{n_k}$. By the same way as in the proof of Theorem 2.14 we get $\alpha_k \to 0$. Hence

$$\langle XT^{n_k}y_{n_k-1}, x_0\rangle = \alpha_k\langle T^{n_k}y_{n_k}, x_0\rangle \to 0.$$

On the other hand $\langle XT^{n_k}y_{n_k-1}, x_0\rangle \to \langle Xb_0, T^*x_0\rangle$. Therefore $\langle Xb_0, T^*x_0\rangle = 0$. If we set $M = \{Xb_0; X \in \{T\}'\}$, then M is a nontrivial hyperinvariant subspace of T. □

Corollary 2.16. *Let $T \in B(H)$ has dense range. If there exists a unit vector x_0 such that T is weakly stable for x_0, then T has a nontrivial hyperinvariant subspace.*

Proof. In the proof of Theorem 2.13, we see that $\|T^nT^{*n}x_0\| = \|T^{*n}x_0\|^2$. Hence by Corollary 2.15 we get the desired result. □

Finally, we show that any quasinilpotent operator has a nontrivial invariant subspace. We use the model of quasinilpotent operators due to Foias and Pearcy [4]. In the same paper the authors give a quasinilpotent operator K_α where $\alpha = \{\alpha_n\}$ is a positive sequence decreasing to zero and define $K_\alpha \in B(\oplus H)$ by

$$K_\alpha = \begin{pmatrix} 0 & \alpha_1 1_H & 0 & \\ 0 & 0 & \alpha_2 1_H & \\ & & \ddots & \ddots \end{pmatrix}.$$

The following theorem is the model theory for quasinilpotent operators.

Theorem 2.17. *[4] If T is a quasinilpotent operator, then there exist a decreasing sequence $\alpha = \{\alpha_n\}$ of nonnegative numbers converging to zero, an invariant subspace M of the operator K_α, and an invertible operator $S : H \to M$ such that $STS^{-1} = K_\alpha|_M$.*

We recall that the subspace M in Theorem 2.17 is of the form

$$\vee\{(b, w_1Tb, w_2T^2b, \ldots) : b \in H\} \subset \oplus H$$

where $w_k = \frac{1}{\alpha_1\alpha_2\ldots\alpha_k}$.

Theorem 2.18. *Let T be a quasinilpotent operator, then T has a nontrivial hyperinvariant subspace.*

Proof. Let K be a backward weighted shift mentioned in Theorem 2.17 such that T is similar to $K|_M$. We show that K has a nontrivial hyperinvariant subspace and so T has a such subspace. Let $x_0 \in H$ be a unit vector and $0 < \epsilon < 1$. Define $\tilde{x} = (x_0, w_1 T x_0, w_2 T^2 x_0, \ldots) \in M$. By an elementary computations we get

$$(K|_M)^{*n}\tilde{x} = (0, 0, \ldots, 0, c_{n,0}x_0, c_{n,1}w_1 T x_0, c_{n,2}w_2 T^2 x_0, \ldots)$$

and

$$(K|_M)^n (K|_M)^{*n}\tilde{x} = (c_{n,0}^2 x_0, c_{n,1}^2 w_1 T x_0, c_{n,2}^2 w_2 T^2 x_0, \ldots)$$

where $c_{n,k} = \alpha_{k+1}\alpha_{k+2}\ldots\alpha_{n+k}$. Since $\{\alpha_k\}$ is a decreasing sequence, we get for any $k = 1, 2, \ldots$

$$\begin{aligned} c_{n,k}^2 &= (\alpha_{k+1}\alpha_{k+2}\ldots\alpha_{k+n})^2 \\ &\leq (\alpha_1\alpha_2\ldots\alpha_n)^2 \leq \sum_{i=1}^{\infty} c_{n,i}^2 w_i^2 \|T^i x_0\|^2 = \|(K|M)^{*n}\tilde{x}\|^2. \end{aligned}$$

This yields that

$$\begin{aligned} \|(K|_M)^n (K|_M)^{*n}\tilde{x}\| &= \left[\sum_{i=1}^{\infty} c_{n,i}^4 w_i^2 \|T^i x_0\|^2\right]^{\frac{1}{2}} \\ &\leq \left[\left(\sum_{i=1}^{\infty} c_{n,i}^2 w_i^2 \|T^i x_0\|^2\right)^2\right]^{\frac{1}{2}} = \|(K|_M)^{*n}\tilde{x}\|^2. \end{aligned}$$

Hence by Corollary 2.15 the result is concluded. □

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