Title:
Pseudo-almost valuation rings

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PSEUDO-ALMOST VALUATION RINGS

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(Communicated by Mohammad Taghi Dibaei)

Abstract. The aim of this paper is to generalize the notion of pseudo-almost valuation domains to arbitrary commutative rings. It is shown that the classes of chained rings and pseudo-valuation rings are properly contained in the class of pseudo-almost valuation rings; also the class of pseudo-almost valuation rings is properly contained in the class of quasi-local rings with linearly ordered prime ideals.

Keywords: Strongly prime ideal; Pseudo-almost valuation domain; Pseudo-valuation ring.


1. Introduction

Throughout this paper, $R$ will be a commutative ring with identity. In [8], Hedstrom and Houston introduced a class of integral domains which is closely related to the class of valuation domains. An integral domain $R$ with quotient field $K$ is called a pseudo-valuation domain when each prime ideal $P$ of $R$ is a strongly prime ideal, in the sense that for every $x, y \in K$, if $xy \in P$, then $x \in P$ or $y \in P$. An interesting survey article on pseudo-valuation domains is [5].

In [3], the study of pseudo-valuation domains was generalized to arbitrary rings (possibly with nonzero zero-divisors), in the following way:

A prime ideal $P$ of a ring $R$ is said to be strongly prime, if $aP$ and $bR$ are comparable (under inclusion) for all $a, b \in R$. A ring $R$ is called a pseudo-valuation ring, if each prime ideal of $R$ is strongly prime. A pseudo-valuation ring is necessarily quasi-local ([3, Lemma 1]). Also, an integral domain is a pseudo-valuation ring if and only if it is a pseudo-valuation domain, by ([1, Proposition 3.1]), ([2, Proposition 4.2]) and ([4, Proposition 3]). Recall from [6] that for an integral domain $R$ with quotient field $K$, a prime ideal $P$ of $R$ is called a pseudo-strongly prime ideal, if whenever $x, y \in K$ and $xyP \subseteq P$, there is a positive integer $m \geq 1$ such that either $x^m \in R$ or $y^mP \subseteq P$. If every prime
ideal $P$ of $R$ is a pseudo-strongly prime ideal, then $R$ is called a pseudo-almost valuation domain.

In this paper, we define a prime ideal $P$ of a ring $R$ to be a pseudo-strongly prime ideal, if for every $a, b \in R$, there is a positive integer $m \geq 1$ such that $a^m R \subseteq b^m R$ or $b^m P \subseteq a^m P$. We show that if $R$ is an integral domain, then our definition is equivalent to the original definition of a pseudo-strongly prime as introduced by Badawi in [6]. If every maximal ideal $P$ of $R$ is a pseudo-strongly prime ideal, then $R$ is called a pseudo-almost valuation ring (PAVR). We show that the classes of chained rings and pseudo-valuation rings are properly contained in the class of pseudo-almost valuation rings; also the class of pseudo-almost valuation rings is properly contained in the class of quasi-local rings with linearly ordered prime ideals.

In the second section, we prove in Proposition 2.3 that every pseudo-strongly prime ideal is comparable to each prime ideal of $R$. Also, we show that in Noetherian quasi-local rings every strongly prime ideal is a pseudo-strongly prime ideal.

In the third section, we show in Proposition 3.6 that every pseudo-valuation ring is a PAVR and we give in Proposition 3.4 a characterization of PAVRs. We prove that every PAVR is a quasi-local ring with linearly ordered prime ideals. Also, we show that every pseudo-almost valuation domain is a Goldie ring with $Gdim = 1$.

Furthermore, we consider the idealization construction $R(+)B = D$ arising from a ring $R$ and an $R$–module $B$. For instance, if $R$ is an integral domain and $D$ is a PAVR then $R$ is a pseudo-almost valuation domain. We then have the following implication, non of which is reversible:

\[
\text{chained ring} \quad \downarrow \quad \text{pseudo-valuation ring} \quad \downarrow \quad \text{pseudo-almost valuation ring} \quad \downarrow \quad \text{quasi-local ring with linearly ordered prime ideals}
\]

Our work generalize the work of Badawi on integral domains in [6]. We close this introduction by noting the following result:

**Proposition 1.1.** Let $R$ be a quasi-local ring with maximal ideal $M$. Then $M$ is a strongly prime ideal, if whenever $a, b \in R$ and $aR \not\subseteq bR$, then $bM \subseteq aM$.

**Proof.** Let $a, b \in R$ such that $aR \not\subseteq bM$. If $aR \not\subseteq bR$ then $bM \subseteq aM \subseteq aR$. Now, we assume that $aR \subseteq bR$. Since $aR \not\subseteq bM$, there is $r \in R \setminus M$ such that $a = rb$. Since $r$ is a unit element of $R$, we have $b = r^{-1}a$ and so $bR \subseteq aR$. 

Hence $aR = bR$ and consequently $bM \subseteq bR = aR$. Therefore $M$ is a strongly prime ideal of $R$. 

\[ \square \]

2. Pseudo-strongly prime ideal

We start with our definition of a pseudo-strongly prime ideal.

**Definition 2.1.** A prime ideal $P$ of a ring $R$ is called a pseudo-strongly prime ideal if for every $a, b \in R$, there is a positive integer $m \geq 1$ such that $a^m R \subseteq b^m P$ or $b^m P \subseteq a^m P$.

In the following Proposition, we show that, if $R$ is an integral domain, the above definition is equivalent to the definition of pseudo-strongly prime ideal introduced in [6].

**Proposition 2.2.** Let $R$ be an integral domain with quotient field $K$. If $P$ is a prime ideal of $R$ then the following conditions are equivalent:

1. For every $x, y \in K$, if $xyP \subseteq P$ then there is a positive integer $m \geq 1$ such that $x^m \in R$ or $y^m P \subseteq P$.
2. For every $a, b \in R$, there is a positive integer $m \geq 1$ such that $a^m R \subseteq b^m P$ or $b^m P \subseteq a^m P$.

**Proof.** (1) $\implies$ (2). Let $0 \neq a, b \in R$. Set $x = a/b$ and $y = b/a$. Since $xyP \subseteq P$, by hypothesis there is an $m \geq 1$ such that $x^m \in R$ or $y^m P \subseteq P$. Hence $a^m R \subseteq b^m R$ or $b^m P \subseteq a^m P$.

(2) $\implies$ (1). Let $x, y \in K$ and $xyP \subseteq P$. Suppose that for every $m \geq 1$, $x^m \notin R$. If $x = a/b$ then $a^m R \nsubseteq b^m R$ for every $m \geq 1$. Hence there is $m \geq 1$ such that $b^m P \subseteq a^m P$. Thus $x^{-m}P = (b/a)^m P \subseteq P$. Since $x^{m}y^{m}P \subseteq P$ and $x^{-m}P \subseteq P$, we have $y^{m}P = x^{-m}(x^{m}y^{m}P) \subseteq x^{-m}P \subseteq P$. 

\[ \square \]

**Proposition 2.3.** Let $P$ be a pseudo-strongly prime ideal of $R$. Then the following statements are hold:

1. $P$ is comparable to each prime ideal.
2. If $Q_1$ and $Q_2$ are two prime ideals of $R$ contained in $P$, then $Q_1$ and $Q_2$ are comparable.

**Proof.** (1). Suppose that $Q$ is a prime ideal of $R$ such that $Q \nsubseteq P$. Let $q \in Q \setminus P$ and $p \in P$. For every $n \geq 1$, $q^n R \nsubseteq p^n R$, because $q \notin P$. Since $P$ is a pseudo-strongly prime ideal, there is $n \geq 1$ such that $p^n P \subseteq q^n P$. Therefore $P \subseteq Q$.

(2). Suppose that $Q_1$ and $Q_2$ are two distinct prime ideals of $R$ such that $Q_1 \nsubseteq Q_2$. Then there is an element $a \in Q_1 \setminus Q_2$. Let $b \in Q_2$. Since $P$ is a pseudo-strongly prime ideal, there is an $n \geq 1$ such that $a^n R \subseteq b^n R$ or $b^n P \subseteq a^n P$. If $a^n R \subseteq b^n R$ then $a \in Q_2$, which is a contradiction. Thus $b^n P \subseteq a^n P$. Since $b \in P$, we have $b^{n+1} \in a^n P \subseteq Q_1$ and so $b \in Q_1$. Therefore $Q_2 \subseteq Q_1$. 

\[ \square \]
Proposition 2.4. Let \( P \) be a pseudo-strongly prime ideal of \( R \). Then for every \( p \in P \) and \( r \in R \setminus P \) there is an \( n \geq 1 \) such that \( p^n \in r^n P \).

Proof. Let \( p \in P \) and \( r \in R \setminus P \). Since \( P \) is a pseudo-strongly prime ideal, there is an \( n \geq 1 \) such that \( r^{2n} R \subseteq p^n R \) or \( p^n P \subseteq r^{2n} P \). If \( r^{2n} R \subseteq p^n R \) then \( r \in P \), which is a contradiction. Thus \( p^n P \subseteq r^{2n} P \). Hence \( p^{2n} = p^n p^n \in r^{2n} P \). Therefore \( p^n \in r^n P \) where \( m = 2n \).

Proposition 2.5. Let \( P \) be a pseudo-strongly prime ideal of \( R \). Suppose that \( P \) contains a prime ideal \( Q \) of \( R \). Then for every \( q \in Q \) and \( p \in P \setminus Q \), there is an \( n \geq 1 \) such that \( q^n \in p^n Q \).

Proof. Let \( q \in Q \) and \( p \in P \setminus Q \). There is an \( n \geq 1 \) such that \( q^n R \subseteq p^n R \) or \( p^n P \subseteq q^n P \). If \( p^n P \subseteq q^n P \) then \( p \in Q \), which is a contradiction. Hence \( q^n R \subseteq p^n R \). Thus \( q^n = ap^n \) for some \( a \in R \). Since \( q^n \in Q \) and \( p^n \notin Q \), we have \( a \in Q \). Therefore \( q^n \in p^n Q \).

Proposition 2.6. Let \( R \) be a Noetherian quasi-local ring. Then every strongly prime ideal of \( R \) is a pseudo-strongly prime ideal.

Proof. Let \( P \) be a strongly prime ideal of \( R \). Suppose that \( a, b \in R \) such that \( a^n R \not\subseteq b^n R \) for every \( n \geq 1 \). Thus \( aR \not\subseteq bP \). Since \( P \) is a strongly prime ideal, we have \( bP \subseteq aR \). If \( a \notin P \), then \( bP \subseteq aP \). Now, we assume that \( a \in P \) and \( bP \not\subseteq aP \). Then there is \( p \in P \) such that \( pb \notin aP \). Since \( bP \subseteq aR \), there exists \( r_0 \in R \setminus P \) such that \( pb = ar_0 \). Since \( P \) is a strongly prime ideal of \( R \), we have \( P \subseteq r_0 R \). Hence there is \( r_1 \in P \) such that \( a = r_1 r_0 \). Thus we have \( pb = ar_0 = r_1 r_0 ^2 \). Hence \( pb \in r_0 ^2 R \). Similarly, there exists \( r_2 \in P \) such that \( r_1 = r_2 r_0 \) and so \( pb = r_2 r_0 ^3 \). Hence \( pb \in r_0 ^3 R \). Proceeding in the same way, we get \( pb \in r_0 ^n R \) for every \( n \geq 1 \). Thus \( pb \in \bigcap \limits_{n=1}^{\infty} r_0 ^n R \). Now, \( r_0 \in M = Jac(R) \), because if \( r_0 \notin M \) then \( r_0 \) is a unit element of \( R \) and so \( aR \subseteq bR \), which is a contradiction. Since \( R \) is a Noetherian ring, \( \bigcap \limits_{n=1}^{\infty} r_0 ^n R = 0 \), by the Krull intersection Theorem. Hence \( pb = 0 \), which is a contradiction. Therefore \( bP \subseteq aP \). Thus \( P \) is a pseudo-strongly prime ideal of \( R \).

An element of \( R \) is called regular, if it is not a zero-divisor. A regular ideal of \( R \) is one that contains a regular element. Also, a ring \( R \) is called a Marot ring, if each regular ideal of \( R \) is generated by its set of regular elements.

Proposition 2.7. Let \( P \) be a pseudo-strongly prime ideal of a Marot ring \( R \). Then every regular prime ideal \( Q \subseteq P \) of \( R \) is a pseudo-strongly prime.

Proof. Let \( a, b \in R \) such that \( a^n R \not\subseteq b^n R \) for every \( n \geq 1 \). Since \( P \) is pseudo-strongly prime, there is \( n \geq 1 \) such that \( b^n P \subseteq a^n P \). Thus \( b^n Q \subseteq b^n P \subseteq a^n P \). Let \( q \) be a regular element of \( Q \). Hence there is a \( p \in P \) such that \( qb^n = pa^n \).
If \( p \notin Q \) then there is \( m \geq 1 \) such that \( q^m \in p^mQ \), by Proposition 2.5. Thus there is \( q' \in Q \) such that \( q^m = q'p^m \). But \( qb^n = pa^n \), and \( q'q^n = p^nq'a^{mn} \). Thus \( p^m q'q^n = p^nq'a^{mn} \). Since \( q \) is regular, \( p^m \) is regular. Hence \( q'q^n = q'^n \) and consequently \( a^{mn}R \subseteq b^{mn}R \), which is a contradiction. Hence \( p \in Q \) and so \( qb^n \in a^nQ \). Since \( Q \) is a regular ideal of the Marot ring \( R \), \( Q \) is generated by its set of regular elements. Hence \( b^nQ \subseteq a^nQ \). Therefore \( Q \) is a pseudo-strongly prime.

\[ \square \]

### 3. Pseudo-almost valuation rings

**Definition 3.1.** Let \( R \) be a commutative ring. If every maximal ideal \( P \) of \( R \) is a pseudo-strongly prime ideal, then \( R \) is called a pseudo-almost valuation ring (PAVR).

Let \( R \) be an integral domain and \( M \) be a maximal ideal of \( R \). If \( M \) is a pseudo-strongly prime ideal and \( P \) is a prime ideal of \( R \) contained in \( M \) then \( P \) is a pseudo-strongly prime ideal of \( R \), by Proposition 2.7. Namely, an integral domain is a pseudo-almost valuation ring if and only if it is a pseudo-almost valuation domain.

Now, let \( R \) be a ring and \( M \) be a maximal ideal of \( R \). If \( M \) is a pseudo-strongly prime ideal, then \( M \) is comparable to each prime ideal of \( R \), by Proposition 2.3(1), and so \( R \) is a quasi-local ring. Therefore

**Proposition 3.2.** A ring \( R \) is a PAVR if and only if some maximal ideal of \( R \) is a pseudo-strongly prime ideal.

If \( R \) is a pseudo-almost valuation ring, then the set of all prime ideals of \( R \) is linearly ordered, by Proposition 2.3(2). Thus we have the following result

**Corollary 3.3.** Let \( R \) be a PAVR. Then the prime ideals of \( R \) are linearly ordered. In particular, \( R \) is quasi-local.

The following Theorem gives a characterization of pseudo-almost valuation rings.

**Theorem 3.4.** A commutative ring \( R \) is a PAVR if and only if for every \( a, b \in R \), there is an \( n \geq 1 \) such that \( a^nR \subseteq b^nR \) or \( b^n \in a^nR \) for every non-unit \( d \in R \).

**Proof.** Suppose that \( R \) is a PAVR. Then \( R \) is quasi-local, by Corollary 3.3. Let \( M \) be the maximal ideal of \( R \) and \( a, b \in R \). If \( a^nR \nsubseteq b^nR \) for every \( n \geq 1 \), then there is \( n \geq 1 \) such that \( b^nM \nsubseteq a^nM \), because \( M \) is a pseudo-strongly prime ideal. Thus \( b^n \in a^nR \) for every non-unit \( d \in R \).

Conversely, suppose that for every \( a, b \in R \), there is an \( n \geq 1 \) such that \( a^nR \subseteq b^nR \) or \( b^n \in a^nR \) for every non-unit \( d \in R \). First, we show that \( R \) is quasi-local. Let \( a \) and \( b \) be nonzero non-unit elements of \( R \). We assume that \( b \neq a^n \) for every \( n \geq 1 \), then \( a^nR \nsubseteq b^nR \) for every \( n \geq 1 \). Hence there is an \( n \geq 1 \)
such that \( b^n d \in a^n R \) for every non-unit \( d \in R \). In particular, \( b^{n+1} a^n R \subseteq aR \). Thus \( a | b^{n+1} \). Therefore the set of all prime ideals of \( R \) is linearly ordered, by ([4, Theorem 1]), and so \( R \) is quasi-local. Let \( M \) be the maximal ideal of \( R \). Suppose that \( a, b \in R \) and \( a^n R \not\subseteq b^n R \) for every \( n \geq 1 \). Thus there is an \( n \geq 1 \) such that \( b^n d \in a^n R \) for every non-unit \( d \in R \). Hence \( b^n M \subseteq a^n R \). For every \( d \in M \), there is an \( r \in R \) such that \( b^n d = a^n r \). If \( r \notin M \) then \( r \) is a unit element of \( R \). Thus \( a^n R \subseteq b^n R \), which is a contradiction. Hence \( b^n M \subseteq a^n M \). Therefore \( M \) is a pseudo-strongly prime ideal of \( R \) and so \( R \) is a PAVR.

We recall that a ring \( R \) is called a \textit{chained ring}, if the set of all ideals of \( R \) is linearly ordered by inclusion. Then the above Theorem implies that

**Corollary 3.5.** Every chained ring is a PAVR.

**Proposition 3.6.** Let \( R \) be a pseudo-valuation ring. Then \( R \) is a PAVR.

**Proof.** Let \( M \) be the maximal ideal of \( R \) and \( a, b \in R \). Since \( M \) is a strongly prime ideal, \( bM \subseteq aR \) or \( aR \subseteq bM \). We assume that \( a^n R \not\subseteq b^n R \) for every \( n \geq 1 \). Then \( aR \not\subseteq bM \) and so \( bM \subseteq aR \). If \( bM \not\subseteq aM \) then there are elements \( d \in M \) and \( r \in R \setminus M \) such that \( bd = ra \). Since \( r \) is a unit element of \( R \), we have \( aR \subseteq bR \), which is a contradiction. Hence \( bM \subseteq aM \). Thus \( M \) is a pseudo-strongly prime ideal of \( R \). Therefore \( R \) is a PAVR.

**Definition 3.7.** A commutative ring \( R \) is called root closed, if whenever \( a, b \in R \) and \( a^n R \subseteq b^n R \) for some \( n \geq 1 \), then \( aR \subseteq bR \).

**Theorem 3.8.** Let \( R \) be a root closed PAVR. Then \( R \) is a pseudo valuation ring.

**Proof.** Let \( M \) be the maximal ideal of \( R \). By ([3, Theorem 2]), it is enough to show that \( M \) is a strongly prime ideal. Let \( a, b \in R \) such that \( aR \not\subseteq bR \). Since \( R \) is root closed, \( a^n R \not\subseteq b^n R \) for every \( n \geq 1 \). Hence \( b^n M \subseteq a^n M \) for some \( n \geq 1 \), because \( M \) is a pseudo-strongly prime ideal of \( R \). Let \( c \in M \). Then \( c^n b^n \in a^n M \) and so \( (cb)^n R \subseteq a^n R \). Since \( R \) is root closed, we have \( cb R \subseteq aR \). If \( cb \notin aM \) then there is \( r \in R \setminus M \) such that \( cb = ra \). Since \( R \) is a quasi-local ring, \( r \) is a unit element of \( R \). Thus \( a = r^{-1} cb \). Hence \( aR \subseteq bR \), which is a contradiction. Therefore \( bM \subseteq aM \). Thus \( M \) is a strongly prime ideal, by Proposition 1.1.

**Proposition 3.9.** Let \( P \) be a pseudo-strongly prime ideal of \( R \). Then \( R_P \) is a PAVR.

**Proof.** Suppose that \( x, y \in R_P \) and \( x^n R_P \not\subseteq y^n R_P \) for every \( n \geq 1 \). Then \( x = a/s \) and \( y = b/t \) for some \( a, b \in R \) and \( s, t \in R \setminus P \). If \( a^n R \not\subseteq b^n R \) for some \( n \geq 1 \) then there is \( r \in R \) such that \( a^n = rb^n \). Thus

\[
x^n = a^n/s^n = (rb^n)/s^n = (rt^n b^n)/(s^n t^n) = ((rt^n)/s^n) (b/t)^n = z y^n,
\]
where \( z = (rt^n)/s^n \in R_P \), and so \( x^nR_P \subseteq y^nR_P \), which is a contradiction. Since \( P \) is a pseudo-strongly prime ideal of \( R \), there is \( n \geq 1 \) such that \( b^nP \subseteq a^nP \). Thus \( (b/t)^nP R_P \subseteq (a/s)^nP R_P \). Hence \( PR_P \) is a pseudo-strongly prime ideal of \( R_P \) and consequently \( R_P \) is a PAVR.

**Proposition 3.10.** Let \( R \) be a PAVR and \( I \) be an ideal of \( R \). Then \( R/I \) is a PAVR. In particular, if \( P \) is a prime ideal of \( R \) then \( R/P \) is a pseudo-almost valuation domain.

**Proof.** Suppose that \( M \) is the maximal ideal of \( R \) and \( M/I \) is the maximal ideal of \( R/I \). Let \( x, y \in D = R/I \). Then \( x = a + I \) and \( y = b + I \) for some \( a, b \in R \). Since \( M \) is a pseudo-strongly prime ideal of \( R \), there is \( n \geq 1 \) such that \( a^nR \subseteq b^nR \) or \( b^nM \subseteq a^nM \). If \( a^nR \subseteq b^nR \) then \( (a^n + I)D \subseteq (b^n + I)D \). Thus \( x^nD \subseteq y^nD \). If \( b^nM \subseteq a^nM \) then \( (b^n + I)M/I \subseteq (a^n + I)M/I \). Hence \( y^n(M/I) \subseteq x^n(M/I) \). Therefore \( M/I \) is a pseudo-strongly prime ideal of \( D \). Hence \( D \) is a PAVR.

In view of the above Proposition, if \( D \) is a pseudo-almost valuation domain and \( I \) is a non-prime ideal of \( D \) then \( D/I \) is a pseudo-almost valuation ring with zero divisor.

**Proposition 3.11.** If \( R \) is a Noetherian PAVR, then \( R \) has Krull dimension \( \leq 1 \).

**Proof.** Let \( M \) be the maximal ideal of \( R \) and \( P \) be a minimal prime ideal of \( R \). Hence \( R/P \) is a pseudo-almost valuation domain, by Proposition 3.10, and so \( \dim R/P \leq 1 \), by ([6, Proposition 2.22]). We know that \( \dim R/P = \text{ht}(M/P) = \text{ht}M = \dim R \). Therefore \( \dim R \leq 1 \).

Let \( I \) be an ideal of a ring \( R \). We say that \( I \) is an essential ideal of \( R \), if \( I \cap J \neq 0 \), for all nonzero ideals \( J \) of \( R \). A nonzero ring \( R \) is said to be a uniform ring, if each nonzero ideal of \( R \) is an essential ideal. A ring \( R \) is said to have finite Goldie dimension if it contains no infinite direct sum of nonzero ideals, and \( R \) has Goldie dimension \( n \), \( \text{Gdim}(R) = n \), if \( n \) is the largest finite number of ideals of \( R \), forming a direct sum. For instance, the Goldie dimension of every uniform ring is equal to 1. Also, \( R \) is called a Goldie ring, if \( R \) has finite Goldie dimension and a.c.c on annihilators.

**Proposition 3.12.** Let \( R \) be an integral domain. If the set of all prime ideals of \( R \) is linearly ordered, then \( R \) is uniform and so \( \text{Gdim}(R) = 1 \).

**Proof.** Suppose that \( I \) and \( J \) are two nonzero ideals of \( R \). Then \( \text{Rad}(I) \) and \( \text{Rad}(J) \) are comparable. We assume that \( \text{Rad}(I) \subseteq \text{Rad}(J) \) and \( 0 \neq a \in I \). Then \( a^n \in J \) for some \( n \geq 1 \). If \( I \cap J = 0 \) then \( a^n = 0 \) and so \( a = 0 \), which is a contradiction. Therefore \( I \cap J \neq 0 \). Hence each nonzero ideal of \( R \) is an essential ideal.
Corollary 3.13. Every pseudo-almost valuation domain is a Goldie ring with Goldie dimension equal to 1.

The following results is on the idealization construction $R(+)B$ arising from a ring $R$ and an $R$–module $B$ as in ([9, Chapter VI]). For a ring $R$ and $R$-module $B$, we consider commutative ring $R(+)B$. We recall that if $R$ is an integral domain and $B$ is an $R$-module, then $B$ is said to be divisible, if for every nonzero element $r \in R$ and $b \in B$, there exists $f \in B$ such that $rf = b$. First, we show that the reverse of part 2 of Theorem 3.1 in [7] holds.

Proposition 3.14. Let $R$ be an integral domain and $B$ be an $R$-module. Set $D = R(+)B$. Then

1. If $D$ is a pseudo-valuation ring, then $R$ is a pseudo-valuation domain.
2. $R$ is a pseudo-valuation domain and $B$ is a divisible $R$–module if and only if $D$ is a pseudo-valuation ring.

Proof. (1). ([7, Theorem 3.1]). (2). $\Rightarrow$ ([7, Theorem 3.1]).

Conversely, suppose that $D = R(+)B$ is a pseudo-valuation ring. Then $R$ is a pseudo-valuation domain, by 1. Now, we show that $B$ is a divisible $R$–module. If $M$ is the maximal ideal of $R$ then $M(+)B$ is the maximal ideal of $D$, by ([9, Theorem 25.1]). Let $0 \neq r \in R$ and $b \in B$. Set $x := (r,0)$ and $y := (0,b)$. We have $x(M(+)B) \subseteq yD$ or $yD \subseteq x(M(+)B)$, because $M(+)B$ is a strongly prime ideal of $D$. If $x(M(+)B) \subseteq yD$ then $x = 0$, which is a contradiction. Thus $yD \subseteq x(M(+)B)$. Hence there is $c \in M$ and $a \in B$ such that $y = x(c,a)$ and so $b = ra$. Therefore $B$ is a divisible $R$–module.

Proposition 3.15. Let $R$ be an integral domain and $B$ be an $R$-module. Set $D = R(+)B$. Then

1. If $D$ is a PAVR, then $R$ is a pseudo-almost valuation domain.
2. If $R$ is a pseudo-almost valuation domain and $B$ is a divisible $R$–module, then $D$ is a PAVR.

Proof. (1). Suppose that $D$ is a PAVR. Then $D$ is a quasi-local ring. By ([9, Theorem 25.1]), $R$ is also quasi-local ring and $M(+)B$ is the maximal ideal of $D$ where $M$ is the maximal ideal of $R$. Let $r, s \in R$ such that $r^nR \nsubseteq s^nR$ for every $n \geq 1$. If $x = (r,0)$ and $y = (s,0)$, then $x^nD \nsubseteq y^nD$. Since $M(+)B$ is pseudo-strongly prime, there is $n \geq 1$ such that $y^n(M(+)B) \nsubseteq x^n(M(+)B)$. Hence for every $m \in M$, there exist $m' \in M$ and $d \in B$ such that $(m,0)(s^n,0) = (m',d)(r^n,0)$. It follows that $s^nm = r^nm'$ and so $s^nM \subseteq r^nM$. Thus $M$ is a pseudo-strongly prime ideal of $R$. Therefore $R$ is a PAVR.

(2). Suppose that $R$ is a pseudo-almost valuation domain with the maximal ideal $M$ and $B$ is a divisible $R$–module. By ([9, Theorem 25.1]), $M(+)B$ is the maximal ideal of $D$. Let $x, y \in D$ such that $x^nD \nsubseteq y^nD$ for every $n \geq 1$. Then $x = (r,a)$ and $y = (s,b)$ for some $r, s \in R$ and $a, b \in B$. If $r^nR \nsubseteq s^nR$ for some
n ≥ 1, then there is t ∈ R such that r^n = s^n a. Set c = nr^{n-1} - nts^{n-1} b. Since B is a divisible R-module, there is d ∈ B such that s^n d = c. Thus x^n = (t, d)y^n and so x^n D ⊆ y^n D, which is a contradiction. Since M is a pseudo-strongly prime ideal of R, there is n ≥ 1 such that s^n M ⊆ r^n M. Now, let m ∈ M and c ∈ B. Then there is m' ∈ M such that s^n m = r^n m'. Also, there exists d ∈ B such that s^n c + nms^n b - nm'r^n a = r^n d, because B is a divisible R-module. Thus x^n (m, c) = x^n (m', d) and so y^n (M(+) B) ⊆ x^n (M(+) B). Hence M(+) B is a pseudo-strongly prime ideal of D. Therefore D is a PAVR.

Example 3.16. Let R = C + C[X^2 + X^4C[[X]]] = C[[X^2, X^5]], where C is the field of complex numbers. Then R is a quasi-local domain with linearly ordered prime ideals that is not a pseudo-almost valuation domain, by ([6, Example 3.4]). Then for every R-module B, the ring D = R(+) B is quasi-local with linearly ordered prime ideals, by ([9, Theorem 25.1]), that is not a PAVR, by Proposition 3.15.

Example 3.17. Let F be a field and X_1, ..., X_n, ... be an infinite set of indeterminates over F and R_∞ = F[X_1, ..., X_n, ...]. Suppose that I is an ideal of R_∞ generated by the set \{X_i^j | i ∈ N\}. Then R_∞/I is a PAVR with only prime ideal (X_1, X_2, ...)/(X_1, X_2, ...) that is not a pseudo-valuation ring.

Acknowledgments

The authors would like to thank the referee for the valuable suggestions and comments.

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