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PSEUDO-ALMOST VALUATION RINGS

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Abstract. The aim of this paper is to generalize the notion of pseudo-almost valuation domains to arbitrary commutative rings. It is shown that the classes of chained rings and pseudo-valuation rings are properly contained in the class of pseudo-almost valuation rings; also the class of pseudo-almost valuation rings is properly contained in the class of quasi-local rings with linearly ordered prime ideals.
Keywords: Strongly prime ideal; Pseudo-almost valuation domain; Pseudo-valuation ring.
MSC(2010): Primary: 13A18; Secondary: 13G05, 13F30, 13F05, 13A15.

1. Introduction

Throughout this paper, R will be a commutative ring with identity. In [8], Hedstrom and Houston introduced a class of integral domains which is closely related to the class of valuation domains. An integral domain R with quotient field K is called a *pseudo-valuation domain* when each prime ideal P of R is a *strongly prime ideal*, in the sense that for every $x, y \in K$, if $xy \in P$, then $x \in P$ or $y \in P$. An interesting survey article on pseudo-valuation domains is [5].

In [3], the study of pseudo-valuation domains was generalized to arbitrary rings (possibly with nonzero zero-divisors), in the following way:
A prime ideal P of a ring R is said to be *strongly prime,* if aP and bR are comparable (under inclusion) for all $a, b \in R$. A ring R is called a *pseudo-valuation ring*, if each prime ideal of R is strongly prime. A pseudo-valuation ring is necessarily quasi-local ([3, Lemma 1]). Also, an integral domain is a pseudo-valuation ring if and only if it is a pseudo-valuation domain, by ([1, Proposition 3.1]), ([2, Proposition 4.2]) and ([4, Proposition 3]). Recall from [6] that for an integral domain R with quotient field K, a prime ideal P of R is called a *pseudo-strongly prime ideal*, if whenever $x, y \in K$ and $xyP \subseteq P$, there is a positive integer $m \geq 1$ such that either $x^m \in R$ or $y^m P \subseteq P$. If every prime

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ideal P of R is a pseudo-strongly prime ideal, then R is called a *pseudo-almost valuation domain*.

In this paper, we define a prime ideal P of a ring R to be a *pseudo-strongly prime ideal*, if for every $a, b \in R$, there is a positive integer $m \geq 1$ such that $a^m R \subseteq b^m R$ or $b^m P \subseteq a^m P$. We show that if R is an integral domain, then our definition is equivalent to the original definition of a pseudo-strongly prime as introduced by Badawi in [6]. If every maximal ideal P of R is a pseudo-strongly prime ideal, then R is called a *pseudo-almost valuation ring* (PAVR). We show that the classes of chained rings and pseudo-valuation rings are properly contained in the class of pseudo-almost valuation rings; also the class of pseudo-almost valuation rings is properly contained in the class of quasi-local rings with linearly ordered prime ideals.

In the second section, we prove in Proposition 2.3 that every pseudo-strongly prime ideal is comparable to each prime ideal of R. Also, we show that in Noetherian quasi-local rings every strongly prime ideal is a pseudo-strongly prime ideal.

In the third section, we show in Proposition 3.6 that every pseudo-valuation ring is a PAVR and we give in Proposition 3.4 a characterization of PAVRs. We prove that every PAVR is a quasi-local ring with linearly ordered prime ideals. Also, we show that every pseudo-almost valuation domain is a Goldie ring with $Gdim = 1$.

Furthermore, we consider the idealization construction $R(+)B = D$ arising from a ring R and an $R-$module B. For instance, if R is an integral domain and D is a PAVR then R is a pseudo-almost valuation domain. We then have the following implication, non of which is reversible:

$$\begin{array}{c}\text{chained ring}\\ \downarrow\\ \text{pseudo-valuation ring}\\ \downarrow\\ \text{pseudo-almost valuation ring}\\ \downarrow\\ \text{quasi-local ring with linearly ordered prime ideals}\end{array}$$

Our work generalize the work of Badawi on integral domains in [6]. We close this introduction by noting the following result:

Proposition 1.1. *Let R be a quasi-local ring with maximal ideal M. Then M is a strongly prime ideal, if whenever $a, b \in R$ and $aR \not\subseteq bR$, then $bM \subseteq aM$.*

Proof. Let $a, b \in R$ such that $aR \not\subseteq bM$. If $aR \not\subseteq bR$ then $bM \subseteq aM \subseteq aR$. Now, we assume that $aR \subseteq bR$. Since $aR \not\subseteq bM$, there is $r \in R\backslash M$ such that $a = rb$. Since r is a unit element of R, we have $b = r^{-1}a$ and so $bR \subseteq aR$.

Hence $aR = bR$ and consequently $bM \subseteq bR = aR$. Therefore M is a strongly prime ideal of R. □

2. **Pseudo-strongly prime ideal**

We start with our definition of a pseudo-strongly prime ideal.

Definition 2.1. *A prime ideal P of a ring R is called a pseudo-strongly prime ideal if for every $a, b \in R$, there is a positive integer $m \geq 1$ such that $a^m R \subseteq b^m R$ or $b^m P \subseteq a^m P$.*

In the following Proposition, we show that, if R is an integral domain, the above definition is equivalent to the definition of pseudo-strongly prime ideal introduced in [6].

Proposition 2.2. *Let R be an integral domain with quotient field K. If P is a prime ideal of R then the following conditions are equivalent:*

(1) *For every $x, y \in K$, if $xyP \subseteq P$ then there is a positive integer $m \geq 1$ such that $x^m \in R$ or $y^m P \subseteq P$.*
(2) *For every $a, b \in R$, there is a positive integer $m \geq 1$ such that $a^m R \subseteq b^m R$ or $b^m P \subseteq a^m P$.*

Proof. (1) $\Longrightarrow$ (2). Let $0 \neq a, b \in R$. Set $x = a/b$ and $y = b/a$. Since $xyP \subseteq P$, by hypothesis there is an $m \geq 1$ such that $x^m \in R$ or $y^m P \subseteq P$. Hence $a^m R \subseteq b^m R$ or $b^m P \subseteq a^m P$.
(2) $\Longrightarrow$ (1). Let $x, y \in K$ and $xyP \subseteq P$. Suppose that for every $m \geq 1$, $x^m \notin R$. If $x = a/b$ then $a^m R \nsubseteq b^m R$ for every $m \geq 1$. Hence there is $m \geq 1$ such that $b^m P \subseteq a^m P$. Thus $x^{-m} P = (b/a)^m P \subseteq P$. Since $x^m y^m P \subseteq P$ and $x^{-m} P \subseteq P$, we have $y^m P = x^{-m}(x^m y^m P) \subseteq x^{-m} P \subseteq P$. □

Proposition 2.3. *Let P be a pseudo-strongly prime ideal of R. Then the following statements are hold:*

(1) *P is comparable to each prime ideal.*
(2) *If Q_1 and Q_2 are two prime ideals of R contained in P, then Q_1 and Q_2 are comparable.*

Proof. (1). Suppose that Q is a prime ideal of R such that $Q \nsubseteq P$. Let $q \in Q \backslash P$ and $p \in P$. For every $n \geq 1$, $q^n R \nsubseteq p^n R$, because $q \notin P$. Since P is a pseudo-strongly prime ideal, there is $n \geq 1$ such that $p^n P \subseteq q^n P$. Therefore $P \subseteq Q$.
(2). Suppose that Q_1 and Q_2 are two distinct prime ideals of R such that $Q_1 \nsubseteq Q_2$. Then there is an element $a \in Q_1 \backslash Q_2$. Let $b \in Q_2$. Since P is a pseudo-strongly prime ideal, there is an $n \geq 1$ such that $a^n R \subseteq b^n R$ or $b^n P \subseteq a^n P$. If $a^n R \subseteq b^n R$ then $a \in Q_2$, which is a contradiction. Thus $b^n P \subseteq a^n P$. Since $b \in P$, we have $b^{n+1} \in a^n P \subseteq Q_1$ and so $b \in Q_1$. Therefore $Q_2 \subseteq Q_1$. □

Proposition 2.4. *Let P be a pseudo-strongly prime ideal of R. Then for every $p \in P$ and $r \in R \backslash P$ there is an $n \geq 1$ such that $p^n \in r^n P$.*

Proof. Let $p \in P$ and $r \in R \backslash P$. Since P is a pseudo-strongly prime ideal, there is an $n \geq 1$ such that $r^{2n} R \subseteq p^n R$ or $p^n P \subseteq r^{2n} P$. If $r^{2n} R \subseteq p^n R$ then $r \in P$, which is a contradiction. Thus $p^n P \subseteq r^{2n} P$. Hence $p^{2n} = p^n p^n \in r^{2n} P$. Therefore $p^m \in r^m P$ where $m = 2n$. □

Proposition 2.5. *Let P be a pseudo-strongly prime ideal of R. Suppose that P contains a prime ideal Q of R. Then for every $q \in Q$ and $p \in P \backslash Q$, there is an $n \geq 1$ such that $q^n \in p^n Q$.*

Proof. Let $q \in Q$ and $p \in P \backslash Q$. There is an $n \geq 1$ such that $q^n R \subseteq p^n R$ or $p^n P \subseteq q^n P$. If $p^n P \subseteq q^n P$ then $p \in Q$, which is a contradiction. Hence $q^n R \subseteq p^n R$. Thus $q^n = a p^n$ for some $a \in R$. Since $q^n \in Q$ and $p^n \notin Q$, we have $a \in Q$. Therefore $q^n \in p^n Q$. □

Proposition 2.6. *Let R be a Noetherian quasi-local ring. Then every strongly prime ideal of R is a pseudo-strongly prime ideal.*

Proof. Let P be a strongly prime ideal of R. Suppose that $a, b \in R$ such that $a^n R \nsubseteq b^n R$ for every $n \geq 1$. Thus $aR \nsubseteq bP$. Since P is a strongly prime ideal, we have $bP \subseteq aR$. If $a \notin P$, then $bP \subseteq aP$. Now, we assume that $a \in P$ and $bP \nsubseteq aP$. Then there is $p \in P$ such that $pb \notin aP$. Since $bP \subseteq aR$, there exists $r_0 \in R \backslash P$ such that $pb = ar_0$. Since P is a strongly prime ideal of R, we have $P \subseteq r_0 R$. Hence there is $r_1 \in P$ such that $a = r_1 r_0$. Thus we have $pb = ar_0 = r_1 r_0^2$. Hence $pb \in r_0^2 R$. Similarly, there exists $r_2 \in P$ such that $r_1 = r_2 r_0$ and so $pb = r_2 r_0^3$. Hence $pb \in r_0^3 R$. Proceeding in the same way, we get $pb \in r_0^n R$ for every $n \geq 1$. Thus $pb \in \bigcap_{n=1}^{\infty} r_0^n R$. Now, $r_0 \in M = Jac(R)$, because if $r_0 \notin M$ then r_0 is a unit element of R and so $aR \subseteq bR$, which is a contradiction. Since R is a Noetherian ring, $\bigcap_{n=1}^{\infty} r_0^n R = 0$, by the Krull intersection Theorem. Hence $pb = 0$, which is a contradiction. Therefore $bP \subseteq aP$. Thus P is a pseudo-strongly prime ideal of R. □

An element of R is called *regular*, if it is not a zero-divisor. A *regular ideal* of R is one that contains a regular element. Also, a ring R is called a *Marot ring,* if each regular ideal of R is generated by its set of regular elements.

Proposition 2.7. *Let P be a pseudo-strongly prime ideal of a Marot ring R. Then every regular prime ideal $Q \subseteq P$ of R is a pseudo-strongly prime.*

Proof. Let $a, b \in R$ such that $a^n R \nsubseteq b^n R$ for every $n \geq 1$. Since P is pseudo-strongly prime, there is $n \geq 1$ such that $b^n P \subseteq a^n P$. Thus $b^n Q \subseteq b^n P \subseteq a^n P$. Let q be a regular element of Q. Hence there is a $p \in P$ such that $qb^n = pa^n$.

If $p \notin Q$ then there is $m \geq 1$ such that $q^m \in p^m Q$, by Proposition 2.5. Thus there is $q' \in Q$ such that $q^m = q'p^m$. But $qb^n = pa^n$, and $q^m b^{mn} = p^m a^{mn}$. Thus $p^m q' b^{mn} = p^m a^{mn}$. Since q is regular, p^m is regular. Hence $q'b^{mn} = a^{mn}$ and consequently $a^{mn}R \subseteq b^{mn}R$, which is a contradiction. Hence $p \in Q$ and so $qb^n \in a^n Q$. Since Q is a regular ideal of the Marot ring R, Q is generated by its set of regular elements. Hence $b^n Q \subseteq a^n Q$. Therefore Q is a pseudo-strongly prime. □

3. Pseudo-almost valuation rings

Definition 3.1. *Let R be a commutative ring. If every maximal ideal P of R is a pseudo-strongly prime ideal, then R is called a pseudo-almost valuation ring (PAVR).*

Let R be an integral domain and M be a maximal ideal of R. If M is a pseudo-strongly prime ideal and P is a prime ideal of R contained in M then P is a pseudo-strongly prime ideal of R, by Proposition 2.7. Namely, an integral domain is a pseudo-almost valuation ring if and only if it is a pseudo-almost valuation domain.

Now, let R be a ring and M be a maximal ideal of R. If M is a pseudo-strongly prime ideal, then M is comparable to each prime ideal of R, by Proposition 2.3(1), and so R is a quasi-local ring. Therefore

Proposition 3.2. *A ring R is a PAVR if and only if some maximal ideal of R is a pseudo-strongly prime ideal.*

If R is a pseudo-almost valuation ring, then the set of all prime ideals of R is linearly ordered, by Proposition 2.3(2). Thus we have the following result

Corollary 3.3. *Let R be a PAVR. Then the prime ideals of R are linearly ordered. In particular, R is quasi-local.*

The following Theorem gives a characterization of pseudo-almost valuation rings.

Theorem 3.4. *A commutative ring R is a PAVR if and only if for every $a, b \in R$, there is an $n \geq 1$ such that $a^n R \subseteq b^n R$ or $b^n d \in a^n R$ for every non-unit $d \in R$.*

Proof. Suppose that R is a PAVR. Then R is quasi-local, by Corollary 3.3. Let M be the maximal ideal of R and $a, b \in R$. If $a^n R \nsubseteq b^n R$ for every $n \geq 1$, then there is $n \geq 1$ such that $b^n M \subseteq a^n M$, because M is a pseudo-strongly prime ideal. Thus $b^n d \in a^n R$ for every non-unit $d \in R$.

Conversely, suppose that for every $a, b \in R$, there is an $n \geq 1$ such that $a^n R \subseteq b^n R$ or $b^n d \in a^n R$ for every non-unit $d \in R$. First, we show that R is quasi-local. Let a and b be nonzero non-unit elements of R. We assume that $b \nmid a^n$ for every $n \geq 1$, then $a^n R \nsubseteq b^n R$ for every $n \geq 1$. Hence there is an $n \geq 1$

such that $b^n d \in a^n R$ for every non-unit $d \in R$. In particular, $b^{n+1} \in a^n R \subseteq aR$. Thus $a \mid b^{n+1}$. Therefore the set of all prime ideals of R is linearly ordered, by ([4, Theorem 1]), and so R is quasi-local. Let M be the maximal ideal of R. Suppose that $a, b \in R$ and $a^n R \not\subseteq b^n R$ for every $n \geq 1$. Thus there is an $n \geq 1$ such that $b^n d \in a^n R$ for every non-unit $d \in R$. Hence $b^n M \subseteq a^n R$. For every $d \in M$, there is an $r \in R$ such that $b^n d = a^n r$. If $r \notin M$ then r is a unit element of R. Thus $a^n R \subseteq b^n R$, which is a contradiction. Hence $b^n M \subseteq a^n M$. Therefore M is a pseudo-strongly prime ideal of R and so R is a PAVR. □

We recall that a ring R is called a *chained ring,* if the set of all ideals of R is linearly ordered by inclusion. Then the above Theorem implies that

Corollary 3.5. *Every chained ring is a PAVR.*

Proposition 3.6. *Let R be a pseudo-valuation ring. Then R is a PAVR.*

Proof. Let M be the maximal ideal of R and $a, b \in R$. Since M is a strongly prime ideal, $bM \subseteq aR$ or $aR \subseteq bM$. We assume that $a^n R \not\subseteq b^n R$ for every $n \geq 1$. Then $aR \not\subseteq bM$ and so $bM \subseteq aR$. If $bM \not\subseteq aM$ then there are elements $d \in M$ and $r \in R \backslash M$ such that $bd = ra$. Since r is a unit element of R, we have $aR \subseteq bR$, which is a contradiction. Hence $bM \subseteq aM$. Thus M is a pseudo-strongly prime ideal of R. Therefore R is a PAVR. □

Definition 3.7. *A commutative ring R is called root closed, if whenever $a, b \in R$ and $a^n R \subseteq b^n R$ for some $n \geq 1$, then $aR \subseteq bR$.*

Theorem 3.8. *Let R be a root closed PAVR. Then R is a pseudo valuation ring.*

Proof. Let M be the maximal ideal of R. By ([3, Theorem 2]), it is enough to show that M is a strongly prime ideal. Let $a, b \in R$ such that $aR \not\subseteq bR$. Since R is root closed, $a^n R \not\subseteq b^n R$ for every $n \geq 1$. Hence $b^n M \subseteq a^n M$ for some $n \geq 1$, because M is a pseudo-strongly prime ideal of R. Let $c \in M$. Then $c^n b^n \in a^n M$ and so $(cb)^n R \subseteq a^n R$. Since R is root closed, we have $cbR \subseteq aR$. If $cb \notin aM$ then there is $r \in R \backslash M$ such that $cb = ra$. Since R is a quasi-local ring, r is a unit element of R. Thus $a = r^{-1} cb$. Hence $aR \subseteq bR$, which is a contradiction. Therefore $bM \subseteq aM$. Thus M is a strongly prime ideal, by Proposition 1.1. □

Proposition 3.9. *Let P be a pseudo-strongly prime ideal of R. Then R_P is a PAVR.*

Proof. Suppose that $x, y \in R_P$ and $x^n R_P \not\subseteq y^n R_P$ for every $n \geq 1$. Then $x = a/s$ and $y = b/t$ for some $a, b \in R$ and $s, t \in R \backslash P$. If $a^n R \subseteq b^n R$ for some $n \geq 1$ then there is $r \in R$ such that $a^n = rb^n$. Thus

$$x^n = a^n/s^n = (rb^n)/s^n = (rt^n b^n)/(s^n t^n) = ((rt^n)/s^n)\,(b/t)^n = zy^n,$$

where $z = (rt^n)/s^n \in R_P$, and so $x^n R_P \subseteq y^n R_P$, which is a contradiction. Since P is a pseudo-strongly prime ideal of R, there is $n \geq 1$ such that $b^n P \subseteq a^n P$. Thus $(b/t)^n\, PR_P \subseteq (a/s)^n\, PR_P$. Hence PR_P is a pseudo-strongly prime ideal of R_P and consequently R_P is a PAVR. □

Proposition 3.10. *Let R be a PAVR and I be an ideal of R. Then R/I is a PAVR. In particular, if P is a prime ideal of R then R/P is a pseudo-almost valuation domain.*

Proof. Suppose that M is the maximal ideal of R and so M/I is the maximal ideal of R. Let $x, y \in D = R/I$. Then $x = a + I$ and $y = b + I$ for some $a, b \in R$. Since M is a pseudo-strongly prime ideal of R, there is $n \geq 1$ such that $a^n R \subseteq b^n R$ or $b^n M \subseteq a^n M$. If $a^n R \subseteq b^n R$ then $(a^n + I)D \subseteq (b^n + I)D$. Thus $x^n D \subseteq y^n D$. If $b^n M \subseteq a^n M$ then $(b^n + I)M/I \subseteq (a^n + I)M/I$. Thus $y^n(M/I) \subseteq x^n(M/I)$. Therefore M/I is a pseudo-strongly prime ideal of D. Hence D is a PAVR. □

In view of the above Proposition, if D is a pseudo-almost valuation domain and I is a non-prime ideal of D then D/I is a pseudo-almost valuation ring with zero divisor.

Proposition 3.11. *If R is a Noetherian PAVR, then R has Krull dimension ≤ 1.*

Proof. Let M be the maximal ideal of R and P be a minimal prime ideal of R. Hence R/P is a pseudo-almost valuation domain, by Proposition 3.10, and so $\dim R/P \leq 1$, by ([6, Proposition 2.22]). We know that $\dim R/P = ht(M/P) = htM = \dim R$. Therefore $\dim R \leq 1$. □

Let I be an ideal of a ring R. We say that I is an *essential ideal* of R, if $I \cap J \neq 0$, for all nonzero ideals J of R. A nonzero ring R is said to be a *uniform ring*, if each nonzero ideal of R is an essential ideal. A ring R is said to have finite Goldie dimension if it contains no infinite direct sum of nonzero ideals, and R has Goldie dimension n, $Gdim(R) = n$, if n is the largest finite number of ideals of R, forming a direct sum. For instance, the Goldie dimension of every uniform ring is equal to 1. Also, R is called a *Goldie ring*, if R has finite Goldie dimension and a.c.c on annihilators.

Proposition 3.12. *Let R be an integral domain. If the set of all prime ideals of R is linearly ordered, then R is uniform and so $Gdim(R) = 1$.*

Proof. Suppose that I and J are two nonzero ideals of R. Then $Rad(I)$ and $Rad(J)$ are comparable. We assume that $Rad(I) \subseteq Rad(J)$ and $0 \neq a \in I$. Then $a^n \in J$ for some $n \geq 1$. If $I \cap J = 0$ then $a^n = 0$ and so $a = 0$, which is a contradiction. Therefore $I \cap J \neq 0$. Hence each nonzero ideal of R is an essential ideal. □

Corollary 3.13. *Every pseudo-almost valuation domain is a Goldie ring with Goldie dimension equal to 1.*

The following results is on the idealization construction $R(+)B$ arising from a ring R and an $R-$module B as in ([9, Chapter VI]). For a ring R and R-module B, we consider commutative ring $R(+)B$. We recall that if R is an integral domain and B is an R-module, then B is said to be *divisible*, if for every nonzero element $r \in R$ and $b \in B$, there exists $f \in B$ such that $rf = b$. First, we show that the reverse of part 2 of Theorem 3.1 in [7] holds.

Proposition 3.14. *Let R be an integral domain and B be an R-module. Set $D = R(+)B$. Then*

(1) *If D is a pseudo-valuation ring, then R is a pseudo-valuation domain.*
(2) *R is a pseudo-valuation domain and B is a divisible $R-$module if and only if D is a pseudo-valuation ring.*

Proof. (1). ([7, Theorem 3.1]).
(2). ($\Rightarrow$) ([7, Theorem 3.1]).

Conversely, suppose that $D = R(+)B$ is a pseudo-valuation ring. Then R is a pseudo-valuation domain, by 1. Now, we show that B is a divisible $R-$module. If M is the maximal ideal of R then $M(+)B$ is the maximal ideal of D, by ([9, Theorem 25.1]). Let $0 \neq r \in R$ and $b \in B$. Set $x := (r, 0)$ and $y := (0, b)$. We have $x(M(+)B) \subseteq yD$ or $yD \subseteq x(M(+)B)$, because $M(+)B$ is a strongly prime ideal of D. If $x(M(+)B) \subseteq yD$ then $x = 0$, which is a contradiction. Thus $yD \subseteq x(M(+)B)$. Hence there is $c \in M$ and $a \in B$ such that $y = x(c, a)$ and so $b = ra$. Therefore B is a divisible $R-$module. □

Proposition 3.15. *Let R be an integral domain and B be an R-module. Set $D = R(+)B$. Then*

(1) *If D is a PAVR, then R is a pseudo-almost valuation domain.*
(2) *If R is a pseudo-almost valuation domain and B is a divisible $R-$module, then D is a PAVR.*

Proof. (1). Suppose that D is a PAVR. Then D is a quasi-local ring. By ([9, Theorem 25.1]), R is also quasi-local ring and $M(+)B$ is the maximal ideal of D where M is the maximal ideal of R. Let $r, s \in R$ such that $r^nR \nsubseteq s^nR$ for every $n \geq 1$. If $x = (r, 0)$ and $y = (s, 0)$, then $x^nD \nsubseteq y^nD$. Since $M(+)B$ is pseudo-strongly prime, there is $n \geq 1$ such that $y^n(M(+)B) \subseteq x^n(M(+)B)$. Hence for every $m \in M$, there exist $m' \in M$ and $d \in B$ such that $(m, 0)(s^n, 0) = (m', d)(r^n, 0)$. It follows that $s^nm = r^nm'$ and so $s^nM \subseteq r^nM$. Thus M is a pseudo-strongly prime ideal of R. Therefore R is a PAVR.
(2). Suppose that R is a pseudo-almost valuation domain with the maximal ideal M and B is a divisible $R-$module. By ([9, Theorem 25.1]), $M(+)B$ is the maximal ideal of D. Let $x, y \in D$ such that $x^nD \nsubseteq y^nD$ for every $n \geq 1$. Then $x = (r, a)$ and $y = (s, b)$ for some $r, s \in R$ and $a, b \in B$. If $r^nR \subseteq s^nR$ for some

$n \geq 1$, then there is $t \in R$ such that $r^n = s^n t$. Set $c = nr^{n-1}a - nts^{n-1}b$. Since B is a divisible $R-$module, there is $d \in B$ such that $s^n d = c$. Thus $x^n = (t, d)y^n$ and so $x^n D \subseteq y^n D$, which is a contradiction. Since M is a pseudo-strongly prime ideal of R, there is $n \geq 1$ such that $s^n M \subseteq r^n M$. Now, let $m \in M$ and $c \in B$. Then there is $m' \in M$ such that $s^n m = r^n m'$. Also, there exists $d \in B$ such that $s^n c + nms^{n-1}b - nm'r^{n-1}a = r^n d$, because B is a divisible $R-$module. Thus $y^n(m, c) = x^n(m', d)$ and so $y^n(M(+)B) \subseteq x^n(M(+)B)$. Hence $M(+)B$ is a pseudo-strongly prime ideal of D. Therefore D is a PAVR. □

In the following example, we give a quasi-local ring with linearly ordered prime ideals that is not a PAVR.

Example 3.16. *Let $R = \mathbf{C} + \mathbf{C}X^2 + X^4\mathbf{C}[[X]] = \mathbf{C}[[X^2, X^5]]$, where $\mathbf{C}$ is the field of complex numbers. Then R is a quasi-local domain with linearly ordered prime ideals that is not a pseudo-almost valuation domain, by ([6, Example 3.4]). Then for every $R-$module B, the ring $D = R(+)B$ is quasi-local with linearly ordered prime ideals, by ([9, Theorem 25.1]), that is not a PAVR, by Proposition 3.15.*

Example 3.17. *Let F be a field and $X_1, ..., X_n, ...$ be an infinite set of indeterminates over F and $R_\infty = F[X_1, ..., X_n, ...]$. Suppose that I is an ideal of R_∞ generated by the set $\{X_i^i \mid i \in \mathbf{N}\}$. Then R_∞/I is a PAVR with only prime ideal $(X_1, X_2, ...)/(X_1, X_2^2, ...)$ that is not a pseudo-valuation ring.*

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