ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 41 (2015), No. 4, pp. 825–834

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 41 (2015), No. 4, pp. 825–834
Online ISSN: 1735-8515

OPTIMAL RESULTS FOR A TIME-FRACTIONAL INVERSE DIFFUSION PROBLEM UNDER THE HÖLDER TYPE SOURCE CONDITION

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(Communicated by Davod Khojasteh Salkuyeh)

Abstract. In the present paper we consider a time-fractional inverse diffusion problem, where data is given at $x = 1$ and the solution is required in the interval $0 < x < 1$. This problem is typically ill-posed: the solution (if it exists) does not depend continuously on the data. We give the optimality analysis for this problem, and an optimal regularization method is also provided. Numerical examples show that this method works effectively.

Keywords: Ill-posed problem, time-fractional inverse diffusion problem, optimal, regularization, error estimate.
MSC(2010): Primary: 35R25; Secondary: 65M32, 65N20.

1. Introduction

Recently, more and more people are concerned about the problems of the fractional differential equation. The main reason is that it is frequently encountered in physics, chemistry, biological systems and so on. Moreover, time-fractional diffusion equation is often used to describe viscoelastic and viscoplastic flow [5] and anomalous diffusion (superdiffusion, non-Gaussian diffusion, subdiffusion) [1, 2, 8, 10], which might be inconsistent with the classical Brownian motion model. Therefore, it is very meaningful for studying this topic [4, 7, 9, 15].

In the past decades, studies on the problems of the fractional differential equation mainly focused on direct problem and boundary value problems. However, it is usually encountered into determine the temperature on the surface of a body, where the surface itself is inaccessible to measurements [11]. In this case, it is necessary to determine surface temperature distribution $0 < x < 1$ from a measured temperature history at a fixed location ($x = 1$) inside the body,

Article electronically published on August 16, 2015.
Received: 3 December 2013, Accepted: 4 May 2014.
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which leads to time-fractional diffusion inverse problem. In this paper we consider the following problem, which arises from many real applications [11, 16]:
Problem I

$$\begin{aligned}
&-u_x(x,t) = {}_0D_t^\alpha u(x,t), && x>0, t>0, 0<\alpha<1,\\
&u(x,0)=0, && x\geq 0,\\
&u(1,t)=f(t), && t\geq 0,\\
&\lim_{x\to\infty} u(x,t)=0, && t\geq 0,
\end{aligned}$$

where the time-fractional derivative ${}_0D_t^\alpha u(x,t)$ is the Caputo fractional derivative of order $\alpha(0<\alpha\leq 1)$, which is defined by [12]

$$(1.1)\qquad {}_0D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(x,s)}{\partial s}\frac{ds}{(t-s)^\alpha}, \qquad 0<\alpha<1,$$

$$(1.2)\qquad {}_0D_t^\alpha u(x,t) = \frac{\partial u(x,t)}{\partial t}, \qquad \alpha=1.$$

We want to determine the temperature for $0<x<1$. This problem is seriously ill-posed [3, 11, 13, 16], therefore, an effective and easy-to-use numerical method for solving such equation is needed. In [11, 13], Murio and Qian give space marching mollification algorithm and optimal regularization method only for $\alpha=\frac{1}{2}$, respectively. In [16], the authors give a new regularization method which bases on mollification the fraction differential equation, but the convergence estimate is not optimal in theory. The main purpose of this paper is to give an optimality result for this problem, and an optimal regularization method is also provided. In the last section, numerical examples show that this method works effectively.

2. Preliminary results

Consider an ill-posed operator equation

$$(2.1)\qquad Ax=y,$$

where $A : X\to Y$ is a linear bounded operator between infinite dimensional Hilbert spaces X and Y with non-closed range in Y. We assume that the noisy data $y_\delta\in Y$ satisfies

$$(2.2)\qquad \|y-y_\delta\|\leq\delta.$$

Any operator $R : Y\to X$ can be considered as a special method for approximately solving (2.1) and the approximate solution is denoted by Ry_δ.

Let $M\subset X$ be a bounded set. We introduce the worst case error $\Delta(\delta,R)$ [14] for identifying x from y_δ as:

$$(2.3)\qquad \Delta(\delta,R) := \sup\left\{\|Ry_\delta - x\| \mid x\in M, y_\delta\in Y, \|Ax-y_\delta\|\leq\delta\right\}.$$

The best possible error bound (or optimal error bound) is defined as the infimum over all mappings $R : Y \to X$ [14]:

$$\omega(\delta) := \inf_R \Delta(\delta, R). \tag{2.4}$$

Now we review an optimality result for the source set $M = M_{\varphi,E}$ which is given by [14]:

$$M_{\varphi,E} = \{x \in X \mid x = [\varphi(A^*A)]^{\frac{1}{2}} v, \|v\| \le E\}. \tag{2.5}$$

The operator function $\varphi(A^*A)$ is well defined via the representation

$$\varphi(A^*A) = \int_0^a \varphi(\lambda) dE_\lambda, \tag{2.6}$$

where $A^*A = \int_0^a \lambda dE_\lambda$ is the spectral decomposition of A^*A, $\{E_\lambda\}$ denotes the spectral family of the operator A^*A, and a is a constant such that $\|A^*A\| \le a$, with $a = \infty$ if A^*A is unbounded [6,14]. A parameter dependent regularization method $R = R_\delta$ is called *optimal* on the set $M_{\varphi,E}$, if $\Delta(\delta, R_\delta) = \omega(\delta)$ holds.

In order to derive an explicit (best possible) optimal error bound for the worst case error $\Delta(\delta, R)$ defined in (2.3), we assume that the function φ in (2.5) satisfies the following assumption:

Assumption 2.1. [14] *The function* $\varphi(\lambda)$ *in (2.5):* $(0, a] \to (0, \infty)$*, where a is a constant with $\|A^*A\| \le a$, is continuous and satisfies:*

(i) $\lim_{\lambda \to 0} \varphi(\lambda) = 0$*;*

(ii) φ is strictly monotonically increasing on $(0, a]$*;*

(iii) $\rho(\lambda) := \lambda \varphi^{-1}(\lambda) : (0, \varphi(a)] \to (0, a\varphi(a)]$ *is convex.*

Under Assumption 2.1, the next lemma gives a general formula for the optimal error bound.

Lemma 2.2. [14] *Let $M_{\varphi,E}$ be given by (2.5) and Assumption 2.1 be satisfied. Moreover, let $\frac{\delta^2}{E^2} \in \sigma(A^*A\varphi(A^*A))$, where $\sigma(A^*A)$ denotes the spectrum of operator A^*A. Then there holds*

$$\omega(\delta, E) = E\sqrt{\rho^{-1}(\frac{\delta^2}{E^2})}. \tag{2.7}$$

Now we give the optimality result for Problem (I) by using Lemma 2.2. Let $\hat{g}(\xi)$ denote the Fourier transform of the function $g(x)$, which is defined by

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} g(x) dx, \quad i = \sqrt{-1}. \tag{2.8}$$

The solution of Problem (I) is easily obtained as the following form:

$$\hat{u}(x, \xi) = e^{(i\xi)^\alpha (1-x)} \hat{f}(\xi) = e^{|\xi|^\alpha \cos(\frac{\alpha\pi}{2})(1-x)} e^{isgn(\xi)|\xi|^\alpha \sin(\frac{\alpha\pi}{2})(1-x)} \hat{f}(\xi), \tag{2.9}$$

and equivalently,

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\xi} e^{(i\xi)^\alpha (1-x)} \hat{f}(\xi) d\xi. \tag{2.10}$$

The ill-posedness of Problem (I) can be seen from (2.10), the details can be found in [11, 13, 16].

Assume the exact data $f(x)$ and the measured data $f_\delta(x)$ both belong to $L^2(\mathbb{R})$ and satisfy

$$\|f - f_\delta\| \le \delta, \tag{2.11}$$

where $\delta > 0$ is the noise level, and $\|\cdot\|$ denotes the norm in $L^2(\mathbb{R})$. Moreover, assume there holds the following *a priori* bound

$$\|u(0,t)\| \le E, \tag{2.12}$$

where E is a fixed positive constant.

The equality (2.9) can be viewed as the following operator equation

$$A\hat{u}(x,\xi) = \hat{f}(\xi), \tag{2.13}$$

where $A = e^{(i\xi)^\alpha (x-1)}$ is a multiplication operator. By a simple calculation, we know $A^* = e^{(-i\xi)^\alpha (x-1)}$ and $AA^* = A^*A = e^{2|\xi|^\alpha \cos(\frac{\alpha\pi}{2})(x-1)}$.

Note that *a priori* bound (2.12), the source set (2.5) for Problem (I) can be described as the following form:

$$\begin{aligned} M_{\varphi,E} = \{\hat{u}(x,\xi) \in L^2(\mathbb{R}) \,\big|\, \hat{u}(x,\xi) &= e^{-x(i\xi)^\alpha} \hat{u}(0,\xi) \\ &:= [\varphi(A^*A)]^{\frac{1}{2}} \hat{u}(0,\xi), \|\hat{u}(0,\xi)\| \le E\}. \end{aligned} \tag{2.14}$$

By denoting $B = e^{-x(i\xi)^\alpha}$, we can easily find $B^* = e^{-x(-i\xi)^\alpha}$ and $BB^* = B^*B = e^{-2x|\xi|^\alpha \cos(\frac{\alpha\pi}{2})}$. Due to the relation $\|BU\| = \|(B^*B)^{\frac{1}{2}}U\|$, we obtain that

$$\varphi(A^*A) = e^{-2x|\xi|^\alpha \cos(\frac{\alpha\pi}{2})} = \left(e^{2|\xi|^\alpha \cos(\frac{\alpha\pi}{2})(x-1)}\right)^{\frac{x}{1-x}}, \tag{2.15}$$

the function $\varphi(\lambda)$ in (2.14) is given by

$$\varphi(\lambda) = \lambda^{\frac{x}{1-x}}, \quad 0 < x < 1, \tag{2.16}$$

and $\varphi(\lambda) : (0,\infty) \to (0,\infty)$ is continuous.

We now verify the conditions in Assumption 2.1.

(i) It is obvious that $\lim\limits_{\lambda\to 0} \varphi(\lambda) = 0$*;*

(ii) From $\varphi'(\lambda) = \frac{x}{1-x}\lambda^{\frac{2x-1}{1-x}} > 0, \quad 0 < x < 1$*, we know* φ *is strictly monotonically increasing on* $(0,\infty)$*;*

(iii) From (2.16), we have $\varphi^{-1}(\lambda) = \lambda^{\frac{1-x}{x}}$ *and* $\rho(\lambda) = \lambda\varphi^{-1}(\lambda) = \lambda^{\frac{1}{x}}$*. By an elementary calculation, we get*

$$\rho''(\lambda) = \frac{1-x}{x^2} \lambda^{\frac{1-2x}{x}} > 0, \tag{2.17}$$

therefore $\rho(\lambda)$ is convex and

$$\rho^{-1}(\lambda) = \lambda^x. \tag{2.18}$$

Theorem 2.3. *Supposing conditions (2.11) and (2.12) hold, then the optimal error bound for Problem (I) is*

$$\omega(\delta, E) = E^{1-x}\delta^x. \tag{2.19}$$

Proof. According to Lemma 2.2 and (2.18), we know that for Problem (I) there holds

$$\omega(\delta, E) = E\sqrt{\rho^{-1}(\frac{\delta^2}{E^2})} = E\sqrt{(\frac{\delta^2}{E^2})^x} = E^{1-x}\delta^x. \tag{2.20}$$

3. The optimal regularization method

Define the filtering function $\kappa(x, \xi)$

(3.1)

$$\kappa(x,\xi) = \begin{cases} e^{|\xi|^\alpha \cos(\frac{\alpha\pi}{2})(1-x)} e^{isgn(\xi)|\xi|^\alpha \sin(\frac{\alpha\pi}{2})(1-x)}, & \beta(x) \geq e^{|\xi|^\alpha \cos(\frac{\alpha}{2})(1-x)}, \\ \beta(x) e^{isgn(\xi)|\xi|^\alpha \sin(\frac{\alpha\pi}{2})(1-x)}, & \beta(x) < e^{|\xi|^\alpha \cos(\frac{\alpha\pi}{2})(1-x)}, \end{cases}$$

with

$$\beta(x) = x(\frac{\delta}{E})^{x-1}. \tag{3.2}$$

Let the approximate solution $\upsilon(x,t)$ be defined by it's Fourier transform with respect to variable t:

$$\hat{\upsilon}(x,\xi) = \kappa(x,\xi)\hat{f}_\delta(\xi), \tag{3.3}$$

or equality $\upsilon(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{it\xi}\kappa(x,\xi)\hat{f}_\delta(\xi)d\xi$, where $\kappa(x,\xi)$ is given by (3.1)-(3.2).

Theorem 3.1. *Let $u(x,t)$ be the exact solution of Problem (I), and $\upsilon(x,t)$ be its regularization approximation defined by (3.3). Assume the conditions (2.11) and (2.12) be satisfied, then there holds the estimate*

$$\|u(x,\cdot) - \upsilon(x,\cdot)\| \leq E^{1-x}\delta^x, \quad for \quad 0 < x < 1. \tag{3.4}$$

Proof. Due to the Parseval formula and (2.9),(3.3),(2.11),(2.12), we have

$$\begin{aligned}
&\|u(x,t)-\upsilon(x,t)\| \\
=&\|\hat{u}(x,\xi)-\hat{\upsilon}(x,\xi)\| \\
=&\|e^{(i\xi)^{\alpha}(1-x)}\hat{f}(\xi)-\kappa(x,\xi)\hat{f}_{\delta}(\xi)\| \\
\le&\|e^{(i\xi)^{\alpha}(1-x)}\hat{f}(\xi)-\kappa(x,\xi)\hat{f}(\xi)\|+\|\kappa(x,\xi)\hat{f}(\xi)-\kappa(x,\xi)\hat{f}_{\delta}(\xi)\| \\
=&\|e^{(i\xi)^{\alpha}}\hat{f}(\xi)\left(e^{-x(i\xi)^{\alpha}}-\kappa(x,\xi)e^{-(i\xi)^{\alpha}}\right)\|+\|\kappa(x,\xi)(\hat{f}(\xi)-\hat{f}_{\delta}(\xi))\| \\
\le&E\sup_{\xi\in\mathbb{R}}|e^{-x(i\xi)^{\alpha}}-\kappa(x,\xi)e^{-(i\xi)^{\alpha}}|+\delta\sup_{\xi\in\mathbb{R}}|\kappa(x,\xi)|. \qquad (3.5)
\end{aligned}$$

From (3.1) and (3.2), we know

$$|\kappa(x,\xi)|\le\beta(x)\quad uniformly\ \ for\ \ \xi\in\mathbb{R}, \qquad (3.6)$$

and therefore

$$\begin{aligned}
&\|\hat{u}(x,\xi)-\hat{\upsilon}(x,\xi)\| \\
\le&E\sup_{\xi\in\mathbb{R}}|e^{-x(i\xi)^{\alpha}}-\kappa(x,\xi)e^{-(i\xi)^{\alpha}}|+\delta\beta(x) \\
=&E\sup_{\xi\in\mathbb{R}}|(e^{-x(i\xi)^{\alpha}}-\beta(x)e^{isgn(\xi)|\xi|^{\alpha}\sin(\frac{\alpha\pi}{2})(1-x)}e^{-(i\xi)^{\alpha}})|+\delta\beta(x) \\
=&E\sup_{\xi\in\mathbb{R}}\left(e^{-x|\xi|^{\alpha}\cos(\frac{\alpha\pi}{2})}-\beta(x)e^{-|\xi|^{\alpha}\cos(\frac{\alpha\pi}{2})}\right)+\delta\beta(x). \qquad (3.7)
\end{aligned}$$

Let $g(s):=e^{-xs}-\beta(x)e^{-s}$, where $s=|\xi|^{\alpha}\cos(\frac{\alpha\pi}{2})\ge 0$, we can easily get

$$g^{'}(s)=-xe^{-xs}+\beta(x)e^{-s}, \qquad (3.8)$$

and the maximum point of function $g(s)$ is $s_0=\frac{\ln(\frac{\beta(x)}{x})}{1-x}$, therefore

$$\begin{aligned}
&\sup_{s\ge 0}g(s)=\sup_{s\ge 0}\left(e^{-xs}-\beta(x)e^{-s}\right) \\
&=g(s_0)=\left(\frac{x}{\beta(x)}\right)^{\frac{x}{1-x}}-\beta(x)\left(\frac{x}{\beta(x)}\right)^{\frac{1}{1-x}}. \qquad (3.9)
\end{aligned}$$

Combining (3.9) with (3.7), we obtain

$$\|u(x,\cdot)-\upsilon(x,\cdot)\|\le E\left(\left(\frac{x}{\beta(x)}\right)^{\frac{x}{1-x}}-\beta(x)\left(\frac{x}{\beta(x)}\right)^{\frac{1}{1-x}}\right)+\delta\beta(x). \qquad (3.10)$$

Taking $\beta(x)$ given by (3.2) into the right-hand side of (3.10), the estimate (3.4) is obtained.

Due to Theorem 2.3, we know the result of Theorem 3.1 is just the optimal error bound of Problem I.

Remark 3.2. In general, the *a prior* bound E in (2.12) is unknown exactly. In the practical calculation, we can replace $\beta(x)$ in (3.2) by

$$\beta(x) = x\delta^{x-1}, \tag{3.11}$$

and we have

$$\|u(x,\cdot) - \upsilon(x,\cdot)\| \le E\delta^{x}, \quad 0 < x < 1, \tag{3.12}$$

where E is just a positive constant and it is not necessary to be known exactly.

4. **Numerical examples**

In this section a simple example is given to verify the validity of the regularization method proposed in this paper. We use the Fast Fourier and Inverse Fourier transform to complete our numerical experiments. In these numerical experiments we always take $\alpha = 0.1, 0.3$ for different points respectively.

Suppose the vector F represents samples from the function $f(t)$, then we can obtain the perturbed data through

$$F^{\epsilon} = F + \epsilon randn(size(F)), \tag{4.1}$$

where the function $''randn(\cdot)''$ generates arrays of random numbers whose elements are normally distributed with mean 0, variance $\sigma^2 = 1$. The error is given by

$$\delta = \|F^{\delta} - F\|_{l^2} := \sqrt{\frac{1}{M+1}\sum_{n=1}^{M+1} |F^{\epsilon}(n) - F(n)|^2}, \tag{4.2}$$

here we usually choose $M = 100$ and $U_k^{\delta}(x,t)$ to represent the discrete regularization solution of $u(x,t)$.

In the numerical experiments, we compute the approximation $U_k^{\delta}(x,t)$ according to Theorem 3.1, and take $E = \|f\|_{l^2(\mathbb{R})}$. The following numerical example is given by [16].

Example The function

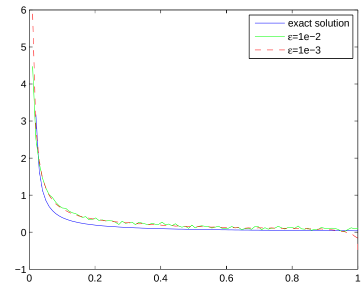
$$u(x,t) = \begin{cases} \dfrac{\alpha(x+1)}{t^{\alpha+1}} M_{\alpha}(\dfrac{x+1}{t^{\alpha}}), & t > 0, \quad x > 0, \\ 0, & t \le 0, \end{cases}$$

is the exact of Problem (I) with the data

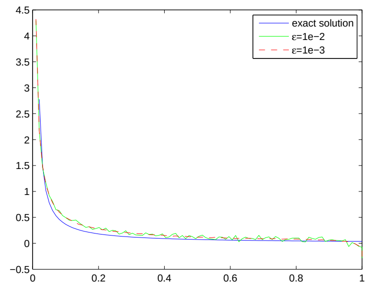
$$f(t) = \begin{cases} \dfrac{2\alpha}{t^{\alpha+1}} M_{\alpha}(\dfrac{2}{t^{\alpha}}), & t > 0, \quad x > 0, \\ 0, & t \le 0, \end{cases}$$

where $M_\alpha(z)$ denotes Mainardi's function

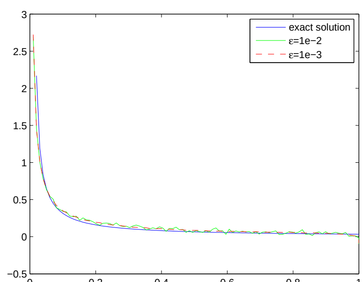
$$M_\alpha = \sum_{k=0}^{+\infty} \frac{(-z)^k}{k!\Gamma(1-\alpha-\alpha k)}, \quad 0<\alpha<1.$$



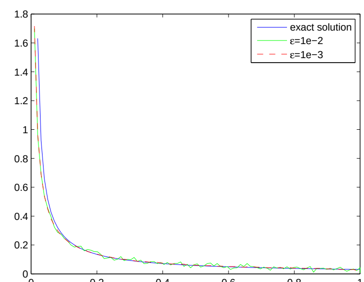
(a)



(b)

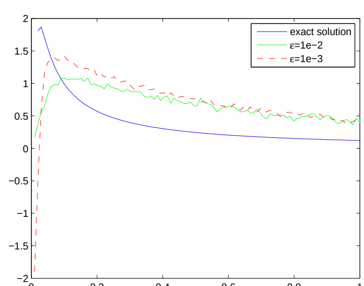


(c)

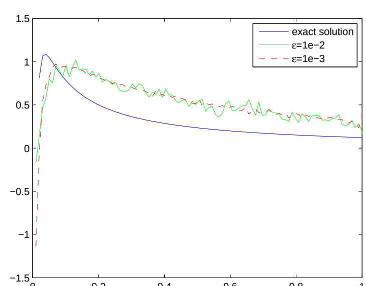


(d)

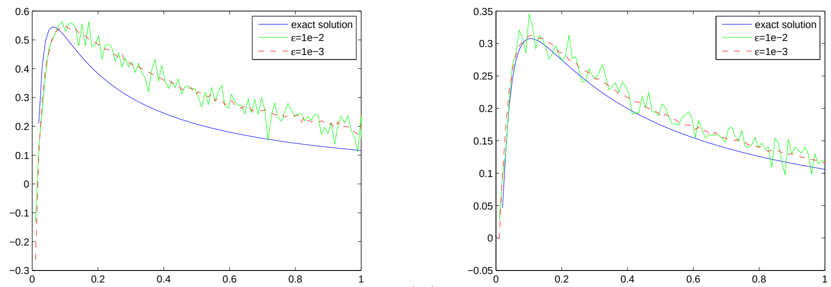
The exact solution and approximate solution for $\alpha = 0.1$,(a) $x = 0.1$, (b) $x = 0.3$, (c) $x = 0.6$, (d) $x = 0.9$.



(a)



(b)



(c) (d)

The exact solution and approximate solution for $\alpha = 0.3$,(a) $x = 0.1$, (b) $x = 0.3$, (c)$x = 0.6$, (d) $x = 0.9$.

Figure 1 shows the comparison of the exact solution and the approximate solution at the different points $x = 0.1$, 0.3, 0.6 and 0.9 for $\alpha = 0.1$ for different noise levels $\epsilon = 10^{-2}, 10^{-3}$ respectively. Figure 2 shows the comparison of the exact solution and the approximate solution at the different points and different noise levels for $\alpha = 0.3$ respectively. From these two figures, we can find that the smaller the ϵ is, the better the computed approximation is, and the smaller the x is, the worse the computed approximation is.

Although the optimal filtering regularization method is optimal in theory, its numerical effect may be not perfect. The reasons may be that new errors must appear in the computational process and we can approximately adjust the regularization parameter for the other methods to obtain a better result. While for the optimal filtering method, the parameter $\beta(x)$ is completely fixed and can not be corrected. Therefore, *a posteriori* rule should be used. We will consider this problem by combining the general convergence rate results for the discrepancy principle with our concrete problem in forthcoming work.

Acknowledgments

The project is supported by the NNSF of China (Nos. 11426117, 11401258, 11271163) and the Natural Science Foundation of Jiangsu Province of China for Young Scholar (No. BK20130118). The project is also supported by National Social Science Fund (No. 14BJY017). We would like to thank immensely the reviewers for their very constructive comments and suggestions which have helped us to improve the paper.

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