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SPACELIKE HYPERSURFACES WITH CONSTANT S OR K IN DE SITTER SPACE OR ANTI-DE SITTER SPACE

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Abstract. Let M^n be an $n(n \geq 3)$-dimensional complete connected and oriented spacelike hypersurface in a de Sitter space or an anti-de Sitter space, S and K be the squared norm of the second fundamental form and Gauss-Kronecker curvature of M^n. If S or K is constant, nonzero and M^n has two distinct principal curvatures one of which is simple, we obtain some characterizations of the Riemannian products: $S^{n-1}(a) \times H^1(\sqrt{a^2-1})$, or $H^{n-1}(a) \times H^1(\sqrt{1-a^2})$.
Keywords: spacelike hypersurface, mean curvature, second fundamental form, Gauss-Kronecker curvature, principal curvature.
MSC(2010): Primary: 53C42; Secondary: 53A10.

1. **Introduction**

By an $(n+1)$-dimensional Lorentzian space form $M_1^{n+1}(c)$ we mean a de Sitter space $S_1^{n+1}(c)$, a Minkowski space R_1^{n+1} or an anti-de Sitter space $H_1^{n+1}(c)$, according to $c > 0$, $c = 0$ or $c < 0$, respectively. That is, a Lorentzian space form $M_1^{n+1}(c)$ is a complete simply connected $(n+1)$-dimensional Lorentzian manifold with constant curvature c. A hypersurface in a Lorentzian manifold is said to be spacelike if the induced metric on the hypersurface is positive definite. Denote by (h_{ij}) the second fundamental form, by $H = \frac{1}{n}\sum_{i=1}^n h_{ii}$ the mean curvature and by $S = \sum_{i,j=1}^n h_{ij}^2$ the squared norm of the second fundamental form of M^n. The function $K = \det(h_{ij})$ is called the Gauss-Kronecker curvature of M^n. We choose $e_1, \ldots, e_n$ such that $h_{ij} = \lambda_i \delta_{ij}$. Then we see that $K = \det(h_{ij}) = \lambda_1 \lambda_2 \cdots \lambda_n$. We notice that if M^n has constant mean curvature or constant scalar curvature in $M_1^{n+1}(c)$, there are many important characteristic results for such spacelike hypersurfaces, see [2–6]. Since H, S

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and K are the important rigidity invariants under the isometric immersion, we may naturally ask the following questions:

(1) if S is constant, nonzero and H satisfies some pinching conditions (related to S), can we obtain any characteristic results?

(2) if K is constant, nonzero and H or S satisfies some pinching conditions (related to K), can we also obtain any characteristic results?

In this note, we try to give some answers to the above questions. Let $S^k(a)$ and $H^k(a)$ denote k-dimensional sphere and k-dimensional hyperbolic surface with radius $\frac{1}{a}$, $S_1^k(a)$ and $H_1^k(a)$ denote k-dimensional de Sitter sphere and k-dimensional anti-de Sitter sphere with radius $\frac{1}{a}$, where a is a constant parametric. Firstly, we introduce the well-known standard models of complete spacelike hypersurfaces with constant S or K in $S_1^{n+1}(1)$ or $H_1^{n+1}(-1)$.

Example 1.1. Spacelike hypersurface $x : S^k(a) \times H^{n-k}(\sqrt{a^2-1}) \to S_1^{n+1}(1)$, $1 \le k \le n-1$. Let $x = (x_1, x_2) \in S^k(a) \times H^{n-k}(\sqrt{a^2-1}) \subset R_1^{k+1} \times R_1^{n-k+1}$, $\langle x_1, x_1\rangle = a^2$, $\langle x_2, x_2\rangle = -(a^2-1)$, $e_{n+1} = (-\frac{\sqrt{a^2-1}}{a}x_1, -\frac{a}{\sqrt{a^2-1}}x_2)$ be the unit normal vector of x such that $\langle e_{n+1}, e_{n+1}\rangle = -1$. By a direct calculation, we know that x has two distinct principal curvatures $\frac{\sqrt{a^2-1}}{a}$ and $\frac{a}{\sqrt{a^2-1}}$ with multiplicities k and $n-k$, respectively. We easily see that x has constant squared norm of the second fundamental form $S = k\frac{a^2-1}{a^2} + (n-k)\frac{a^2}{a^2-1}$ and $a^2 = \frac{2k}{2k-(S\pm\sqrt{S^2-4k(n-k)})}$. Denote by H the mean curvature of $S^{n-1}(a) \times H^1(\sqrt{a^2-1})$, if $a^2 = \frac{2(n-1)}{2(n-1)-(S\mp\sqrt{S^2-4(n-1)})}$, then

$$H = \frac{\sqrt{n-1}(S \mp \sqrt{S^2-4(n-1)}+2)}{n\sqrt{2(S \mp \sqrt{S^2-4(n-1)})}}.$$

We see that the Gauss-Kronecker curvature of $S^{n-1}(a) \times H^1(\sqrt{a^2-1})$ is $K = (\frac{\sqrt{a^2-1}}{a})^{n-1}\frac{a}{\sqrt{a^2-1}}$. Thus, the mean curvature and the squared norm of the second fundamental form of $S^{n-1}(a) \times H^1(\sqrt{a^2-1})$ is

$$H = \frac{1}{n}\{(n-1)K^{\frac{1}{n-2}} - K^{-\frac{1}{n-2}}\},$$
$$S = (n-1)K^{\frac{2}{n-2}} + K^{-\frac{2}{n-2}},$$

where $a^2 = 1/(1-K^{\frac{2}{n-2}})$.

If $\lambda = \frac{\sqrt{a^2-1}}{a} \ge \kappa > \sqrt{\frac{S}{n}}$, we have $a^2 \ge \frac{1}{1-\kappa^2}$, $\kappa < 1$ and

$$S = (n-1)\frac{a^2-1}{a^2} + \frac{a^2}{a^2-1} < n\kappa^2. \tag{1.1}$$

By a direct and simple calculation, we see that (1.1) holds if and only if $\alpha(\kappa) < a^2 < \beta(\kappa)$, where

$$\alpha(\kappa) = \frac{2(n-1)}{2(n-1) + \sqrt{n^2\kappa^4 - 4(n-1)} - n\kappa^2}, \tag{1.2}$$

$$\beta(\kappa) = \frac{2(n-1)}{2(n-1) - \sqrt{n^2\kappa^4 - 4(n-1)} - n\kappa^2}, \tag{1.3}$$

and $\kappa^2 > \frac{2\sqrt{n-1}}{n}$.

It can be easily checked that $\alpha(\kappa) < \frac{1}{1-\kappa^2}$. Thus, we conclude that $\frac{1}{1-\kappa^2} \leq a^2 < \beta(\kappa)$ and $1 > \kappa^2 > \frac{2\sqrt{n-1}}{n}$.

If $\lambda = \frac{\sqrt{a^2-1}}{a} \leq \kappa' < \sqrt{\frac{S}{n}}$, we have $a^2 \geq \frac{1}{1-\kappa'^2}$ if $\kappa' > 1$; $a^2 \leq \frac{1}{1-\kappa'^2}$ if $\kappa' < 1$ and

$$S = (n-1)\frac{a^2-1}{a^2} + \frac{a^2}{a^2-1} > n\kappa'^2. \tag{1.4}$$

By a direct and simple calculation, we see that if $\kappa' > 1$, (1.4) holds if and only if $\alpha(\kappa') < a^2 < \beta(\kappa')$, if $\kappa' < 1$; (1.4) holds if and only if $a^2 < \alpha(\kappa')$ or $a^2 > \beta(\kappa')$, where

$$\alpha(\kappa') = \frac{2(n-1)}{2(n-1) + \sqrt{n^2\kappa'^4 - 4(n-1)} - n\kappa'^2}, \tag{1.5}$$

$$\beta(\kappa') = \frac{2(n-1)}{2(n-1) - \sqrt{n^2\kappa'^4 - 4(n-1)} - n\kappa'^2}, \tag{1.6}$$

and $\kappa'^2 > \frac{2\sqrt{n-1}}{n}$. Thus, we conclude that $\alpha(\kappa') < a^2 < \beta(\kappa')$ if $\kappa' > 1$; $a^2 < \alpha(\kappa')$ or $\beta(\kappa') < a^2 \leq \frac{1}{1-\kappa'^2}$ if $\frac{2\sqrt{n-1}}{n} < \kappa'^2 < 1$.

Example 1.2. Spacelike hypersurface $x : H^k(a) \times H^{n-k}(\sqrt{1-a^2}) \to H_1^{n+1}(-1)$, $1 \leq k \leq n-1$. Let $x = (x_1, x_2) \in H^k(a) \times H^{n-k}(\sqrt{1-a^2}) \subset R_1^{k+1} \times R_1^{n-k+1}$, $\langle x_1, x_1 \rangle = -a^2$, $\langle x_2, x_2 \rangle = -(1-a^2)$ and $e_{n+1} = (-\frac{\sqrt{1-a^2}}{a}x_1, \frac{a}{\sqrt{1-a^2}}x_2)$ be the unit normal vector of x, $\langle e_{n+1}, e_{n+1} \rangle = -1$. By a direct calculation, we know that x has two distinct principal curvatures $\frac{\sqrt{1-a^2}}{a}$ and $-\frac{a}{\sqrt{1-a^2}}$ with multiplicities k and $n-k$, respectively. We easily see that x has constant squared norm of the second fundamental form $S = k\frac{1-a^2}{a^2} + (n-k)\frac{a^2}{1-a^2}$ and $a^2 = \frac{2k}{2k+S\pm\sqrt{S^2-4k(n-k)}}$. Denote by H the mean curvature of $H^{n-1}(a) \times H^1(\sqrt{1-a^2})$, if $a^2 = \frac{2(n-1)}{2(n-1)+S\mp\sqrt{S^2-4(n-1)}}$, then

$$H = \frac{\sqrt{n-1}(S \mp \sqrt{S^2-4(n-1)} - 2)}{n\sqrt{2(S \mp \sqrt{S^2-4(n-1)})}}.$$

If $\lambda = \frac{\sqrt{1-a^2}}{a} \geq \kappa > \sqrt{\frac{S}{n}}$, we have $a^2 \leq \frac{1}{\kappa^2+1}$ and $S = (n-1)\frac{1-a^2}{a^2} + \frac{a^2}{1-a^2} < n\kappa^2$. Thus, we conclude that $\gamma(\kappa) < a^2 \leq \frac{1}{1+\kappa^2}$, $\kappa \neq 1$ and $\kappa^2 > \frac{2\sqrt{n-1}}{n}$, where

$$\gamma(\kappa) = \frac{2(n-1)}{n\kappa^2 + \sqrt{n^2\kappa^4 - 4(n-1)} + 2(n-1)}, \tag{1.7}$$

$$\delta(\kappa) = \frac{2(n-1)}{n\kappa^2 - \sqrt{n^2\kappa^4 - 4(n-1)} + 2(n-1)}. \tag{1.8}$$

If $\lambda = \frac{\sqrt{1-a^2}}{a} \leq \kappa' < \sqrt{\frac{S}{n}}$, we have $a^2 \geq \frac{1}{1+\kappa'^2}$ and $S = (n-1)\frac{1-a^2}{a^2} + \frac{a^2}{1-a^2} > n\kappa'^2$. Thus, we conclude that $a^2 > \delta(\kappa')$, $\kappa'^2 \geq \frac{2\sqrt{n-1}}{n}$, or $\frac{1}{1+\kappa'^2} \leq a^2 < \gamma(\kappa')$, $\frac{2\sqrt{n-1}}{n} \leq \kappa'^2 < 1$, where

$$\gamma(\kappa') = \frac{2(n-1)}{n\kappa'^2 + \sqrt{n^2\kappa'^4 - 4(n-1)} + 2(n-1)}, \tag{1.9}$$

$$\delta(\kappa') = \frac{2(n-1)}{n\kappa'^2 - \sqrt{n^2\kappa'^4 - 4(n-1)} + 2(n-1)}. \tag{1.10}$$

We shall prove the following:

Theorem 1.3. *Let M^n be an $n(n \geq 3)$-dimensional complete connected and oriented spacelike hypersurface in a de Sitter space $S_1^{n+1}(1)$ with nonzero constant S and two distinct nonzero principal curvatures λ and μ of multiplicities $n-1$ and 1.*

(1) *If λ is bounded from below by a positive constant $\kappa > \sqrt{\frac{S}{n}}$ and*

$$H \geq \frac{\sqrt{n-1}(S + \sqrt{S^2 - 4(n-1)} + 2)}{n\sqrt{2(S + \sqrt{S^2 - 4(n-1)})}},$$

then M^n is isometric to the Riemannian product $S^{n-1}(a) \times H^1(\sqrt{a^2-1})$, $a^2 = \frac{2(n-1)}{2(n-1)-(S+\sqrt{S^2-4(n-1)})}$, $\frac{1}{1-\kappa^2} \leq a^2 < \beta(\kappa)$ and $1 > \kappa^2 > \frac{2\sqrt{n-1}}{n}$;

(2) *If λ is bounded from above by a positive constant $\kappa' < \sqrt{\frac{S}{n}}$ and*

$$H \leq \frac{\sqrt{n-1}(S - \sqrt{S^2 - 4(n-1)} + 2)}{n\sqrt{2(S - \sqrt{S^2 - 4(n-1)})}},$$

then M^n is isometric to the Riemannian product $S^{n-1}(a) \times H^1(\sqrt{a^2-1})$, $a^2 = \frac{2(n-1)}{2(n-1)-(S-\sqrt{S^2-4(n-1)})}$, $\alpha(\kappa') < a^2 < \beta(\kappa')$ and $\kappa' > 1$; $a^2 < \alpha(\kappa')$ or $\beta(\kappa') < a^2 \leq \frac{1}{1-\kappa'^2}$ and $\frac{2\sqrt{n-1}}{n} < \kappa'^2 < 1$.

Theorem 1.4. *Let M^n be an $n(n \geq 3)$-dimensional complete connected and oriented spacelike hypersurface in an anti-de Sitter space $H_1^{n+1}(-1)$ with nonzero* *constant* S *and two distinct nonzero principal curvatures* λ *and* μ *of multiplicities* $n-1$ *and* 1.

(1) *If* λ *is bounded from below by a positive constant* $\kappa > \sqrt{\frac{S}{n}}$ *and*

$$H \geq \frac{\sqrt{n-1}(S + \sqrt{S^2 - 4(n-1)} - 2)}{n\sqrt{2(S + \sqrt{S^2 - 4(n-1)})}},$$

then M^n *is isometric to the Riemannian product* $H^{n-1}(a) \times H^1(\sqrt{1-a^2})$, $a^2 = \frac{2(n-1)}{2(n-1)+S+\sqrt{S^2-4(n-1)}}$, $\gamma(\kappa) < a^2 \leq \frac{1}{1+\kappa^2}$, $\kappa > 1$;

(2) *If* λ *is bounded from above by a positive constant* $\kappa' < \sqrt{\frac{S}{n}}$ *and*

$$H \leq \frac{\sqrt{n-1}(S - \sqrt{S^2 - 4(n-1)} - 2)}{n\sqrt{2(S - \sqrt{S^2 - 4(n-1)})}},$$

then M^n *is isometric to the Riemannian product* $H^{n-1}(a) \times H^1(\sqrt{1-a^2})$, $a^2 = \frac{2(n-1)}{2(n-1)+S-\sqrt{S^2-4(n-1)}}$, $a^2 > \delta(\kappa')$, $\kappa'^2 \geq \frac{2\sqrt{n-1}}{n}$, *or* $\frac{1}{1+\kappa'^2} \leq a^2 < \gamma(\kappa')$, $\frac{2\sqrt{n-1}}{n} \leq \kappa'^2 < 1$.

Theorem 1.5. *Let* M^n *be an* $n(n \geq 3)$*-dimensional complete connected and oriented spacelike hypersurface in a de Sitter space* $S_1^{n+1}(1)$ *with nonzero constant Gauss-Kronecker curvature* K *and two distinct principal curvatures* λ *and* μ *of multiplicities* $n-1$ *and* 1. *If* $0 < K < 1$ *and*

$$H \leq \frac{1}{n}\{(n-1)K^{\frac{1}{n-2}} - K^{-\frac{1}{n-2}}\},$$

or

$$H \geq \frac{1}{n}\{(n-1)K^{\frac{1}{n-2}} - K^{-\frac{1}{n-2}}\},$$

then M^n *is isometric to one of the Riemannian products:* $S^{n-1}(a) \times H^1(\sqrt{a^2-1})$, $a^2 = 1/(1-K^{\frac{2}{n-2}})$.

Theorem 1.6. *Let* M^n *be an* $n(n \geq 3)$*-dimensional complete connected and oriented spacelike hypersurface in a de Sitter space* $S_1^{n+1}(1)$ *with nonzero constant Gauss-Kronecker curvature* K *and two distinct principal curvatures* λ *and* μ *of multiplicities* $n-1$ *and* 1. *If* $0 < K < 1$ *and*

$$S \leq (n-1)K^{\frac{2}{n-2}} + K^{-\frac{2}{n-2}},$$

or

$$S \geq (n-1)K^{\frac{2}{n-2}} + K^{-\frac{2}{n-2}},$$

then M^n *is isometric to one of the Riemannian products:* $S^{n-1}(a) \times H^1(\sqrt{a^2-1})$, $a^2 = 1/(1-K^{\frac{2}{n-2}})$.

Remark 1.7. We notice that, in Lemma 3.1, in order to ensure that the positive function ϖ (see (3.18)) is bounded, the condition (in Theorem 1.3 and Theorem 1.4) that λ is bounded from below by a positive constant $\kappa > \sqrt{\frac{S}{n}}$ or is bounded from above by a positive constant $\kappa' < \sqrt{\frac{S}{n}}$ is necessary. We also notice that, in the proof of Lemma 4.1 and Lemma 4.2, the condition $0 < K < 1$ (in Theorem 1.5 and Theorem 1.6) is necessary.

Remark 1.8. If $c = -1$, from the proof of Lemma 4.1, we can not know whether the positive function $\varpi = |\lambda^n - K|^{-\frac{1}{n}}$ is bounded or not. Thus, the similar results as Theorem 1.5 and Theorem 1.6 in an anti-de Sitter space $H_1^{n+1}(-1)$ may be not held.

2. **Preliminaries**

Let M^n be an n-dimensional spacelike hypersurface in an (n+1)-dimensional Lorentzian space form $M_1^{n+1}(c)$ with constant sectional curvature c. We choose a local field of semi-Riemannian orthonormal frames $\{e_1, \dots, e_{n+1}\}$ in $M_1^{n+1}(c)$ such that at each point of M^n, $\{e_1, \dots, e_n\}$ span the tangent space of M^n and form an orthonormal frame there. We use the following convention on the range of indices:

$$1 \leq A, B, C, \dots \leq n+1; \quad 1 \leq i, j, k, \dots \leq n.$$

Let $\{\omega_1, \dots, \omega_{n+1}\}$ be the dual frame field so that the semi-Riemannian metric of $M_1^{n+1}(c)$ is given by $d\bar{s}^2 = \sum\limits_i \omega_i^2 - \omega_{n+1}^2 = \sum\limits_A \epsilon_A \omega_A^2$, where $\epsilon_i = 1$ and $\epsilon_{n+1} = -1$.

The structure equations of $M_1^{n+1}(c)$ are given by

$$d\omega_A = \sum_B \epsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} = \sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} + \Omega_{AB}, \tag{2.2}$$

where

$$\Omega_{AB} = -\frac{1}{2} \sum_{C,D} K_{ABCD} \omega_C \wedge \omega_D, \tag{2.3}$$

$$K_{ABCD} = \epsilon_A \epsilon_B c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{2.4}$$

Restrict these forms to M^n, we have

$$\omega_{n+1} = 0. \tag{2.5}$$

Cartan's Lemma implies that

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{2.6}$$

The structure equations of M^n are

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0, \tag{2.7}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.8}$$

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.9}$$

where R_{ijkl} are the components of the curvature tensor of M^n and

$$h = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j \tag{2.10}$$

is the second fundamental form of M^n.

From the above equation, we have

$$n(n-1)(R-c) = S - n^2H^2, \tag{2.11}$$

where $n(n-1)R$ is the scalar curvature of M^n, H is the mean curvature, and $S = \sum_{i,j} h_{ij}^2$ is the squared norm of the second fundamental form of M^n.

We choose $e_1, \ldots, e_n$ such that $h_{ij} = \lambda_i\delta_{ij}$. From (2.6) we have

$$\omega_{n+1i} = \lambda_i\omega_i, \quad i = 1, 2, \ldots, n. \tag{2.12}$$

From the curvature forms of $M_1^{n+1}(c)$,

$$\begin{aligned} \Omega_{ni} =& -\frac{1}{2}\sum_{C,D} K_{niCD}\omega_C \wedge \omega_D \\ =& \frac{1}{2}\sum_{C,D} c(\delta_{nC}\delta_{iD} - \delta_{nD}\delta_{iC})\omega_C \wedge \omega_D = c\omega_n \wedge \omega_i. \end{aligned} \tag{2.13}$$

Since the covariant derivative of the second fundamental form h_{ij} of M^n is defined by

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{ik}\omega_{kj} + \sum_k h_{kj}\omega_{ki},$$

we have

$$\sum_k h_{ijk}\omega_k = \delta_{ji}d\lambda_j + (\lambda_i - \lambda_j)\omega_{ij}.$$

Putting

$$\psi_{ij} = (\lambda_i - \lambda_j)\omega_{ij},$$

we have $\psi_{ij} = \psi_{ji}$ and

$$\psi_{ij} + \delta_{ij} d\lambda_j = \sum_k h_{ijk}\omega_k, \tag{2.14}$$

where h_{ijk} satisfy

$$h_{ijk} = h_{jik} = h_{ikj}. \tag{2.15}$$

We state a Proposition which can be proved by making use of the similar method due to Otsuki [7].

Proposition 2.1. *Let M^n be a spacelike hypersurface in an (n+1)-dimensional Lorentzian space form $M_1^{n+1}(c)$ such that the multiplicities of the principal curvatures are constant. Then the distribution of the space of the principal vectors corresponding to each principal curvature is completely integrable. In particular, if the multiplicity of a principal curvature is greater than 1, then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of the principal vectors.*

3. Proofs of Theorem 1.3 and Theorem 1.4

Let M^n be an n-dimensional complete spacelike hypersurface with nonzero constant squared norm of the second fundamental form and two distinct nonzero principal curvatures λ and μ of multiplicities $n-1$ and 1. By changing the orientation for M^n and renumbering $e_1, \ldots, e_n$ if necessary, we may assume that $\lambda > 0$. Thus, we have that

$$S = (n-1)\lambda^2 + \mu^2, \tag{3.1}$$

$$\mu = \pm\sqrt{S - (n-1)\lambda^2}, \tag{3.2}$$

and

$$0 \neq \lambda - \mu = \lambda \mp \sqrt{S - (n-1)\lambda^2}. \tag{3.3}$$

We denote the integral submanifold through $x \in M^n$ corresponding to λ by $D(x)$. Putting

$$d\lambda = \sum_{k=1}^{n} \lambda_{,k}\, \omega_k, \qquad d\mu = \sum_{k=1}^{n} \mu_{,k}\, \omega_k. \tag{3.4}$$

From Proposition 2.1, we have

$$\lambda_{,1} = \lambda_{,2} = \cdots = \lambda_{,n-1} = 0 \quad \text{on} \quad D(x). \tag{3.5}$$

From (3.2), we have

$$d\mu = \mp\frac{(n-1)\lambda}{\sqrt{S - (n-1)\lambda^2}} d\lambda. \tag{3.6}$$

Thus, we also have

$$\mu_{,1} = \mu_{,2} = \cdots = \mu_{,n-1} = 0 \quad \text{on} \quad D(x). \tag{3.7}$$

In this case, we may consider locally λ as a function of the arc length s of the integral curve of the principal vector field e_n corresponding to the principal curvature μ. From (2.14) and (3.5), we have for $1 \le j \le n-1$,

$$\lambda_{,n}\,\omega_n = \sum_{i=1}^{n} \lambda_{,i}\,\omega_i = d\lambda = d\lambda_j = \sum_{k=1}^{n} h_{jjk}\omega_k = \sum_{k=1}^{n-1} h_{jjk}\omega_k + h_{jjn}\omega_n. \tag{3.8}$$

Therefore, we have

$$h_{jjk} = 0, \quad 1 \le k \le n-1, \quad \text{and} \quad h_{jjn} = \lambda_{,n}\,. \tag{3.9}$$

By (2.14) and (3.7), we have

$$\mu_{,n}\,\omega_n = \sum_{i=1}^{n} \mu_{,i}\,\omega_i = d\mu = d\lambda_n = \sum_{k=1}^{n} h_{nnk}\omega_k = \sum_{k=1}^{n-1} h_{nnk}\omega_k + h_{nnn}\omega_n. \tag{3.10}$$

Hence, we obtain

$$h_{nnk} = 0, \quad 1 \le k \le n-1, \quad \text{and} \quad h_{nnn} = \mu_{,n}\,. \tag{3.11}$$

From (3.6), we get

$$h_{nnn} = \mu_{,n} = \mp \frac{(n-1)\lambda}{\sqrt{S-(n-1)\lambda^2}}\lambda_{,n}\,. \tag{3.12}$$

From the definition of ψ_{ij}, if $i \neq j$, we have $\psi_{ij} = 0$ for $1 \le i \le n-1$ and $1 \le j \le n-1$. Therefore, from (2.14), if $i \neq j$ and $1 \le i \le n-1$ and $1 \le j \le n-1$ we have

$$h_{ijk} = 0, \quad \text{for any} \quad k. \tag{3.13}$$

By (2.14), (3.9), (3.11) and (3.13), for $j < n$, we get

$$\begin{aligned} \psi_{jn} &= \sum_{k=1}^{n} h_{jnk}\omega_k \\ &= h_{jjn}\omega_j + h_{jnn}\omega_n = \lambda_{,n}\,\omega_j. \end{aligned} \tag{3.14}$$

From $\psi_{ij} = (\lambda_i - \lambda_j)\omega_{ij}$, (3.3) and (3.14), for $j < n$, we have

$$\omega_{jn} = \frac{\psi_{jn}}{\lambda-\mu} = \frac{\lambda_{,n}}{\lambda-\mu}\omega_j = \frac{\lambda_{,n}}{\lambda \mp \sqrt{S-(n-1)\lambda^2}}\omega_j. \tag{3.15}$$

Thus, from the structure equations of M^n we have $d\omega_n = \sum_{k=1}^{n-1} \omega_{kn} \wedge \omega_k + \omega_{nn} \wedge \omega_n = 0$. Therefore, we may put $\omega_n = ds$. By (3.5), we get $d\lambda = \lambda_{,n}\,ds$,

$\lambda_{,n} = \frac{d\lambda}{ds}$. Thus, we have

$$\omega_{jn} = \frac{\frac{d\lambda}{ds}}{\lambda \mp \sqrt{S-(n-1)\lambda^2}}\omega_j = \frac{d\{\ln|\sqrt{\frac{n-1}{S}}(\lambda \mp \sqrt{S-(n-1)\lambda^2})e^{\mp\sqrt{n-1}\arcsin\sqrt{\frac{n-1}{S}}\lambda}|^{\frac{1}{n}}\}}{ds}\omega_j. \tag{3.16}$$

From (3.16) and the structure equations of $M_1^{n+1}(c)$, for $j < n$, we have

$$\begin{aligned}
d\omega_{jn} &= \sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_{kn} + \omega_{jn}\wedge\omega_{nn} - \omega_{jn+1}\wedge\omega_{n+1n} + \Omega_{jn}\\
&= \sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_{kn} - \omega_{jn+1}\wedge\omega_{n+1n} - c\omega_j\wedge\omega_n\\
&= \frac{d\{\ln|\sqrt{\frac{n-1}{S}}(\lambda \mp \sqrt{S-(n-1)\lambda^2})e^{\mp\sqrt{n-1}\arcsin\sqrt{\frac{n-1}{S}}\lambda}|^{\frac{1}{n}}\}}{ds}\sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_k\\
&\quad - (c-\lambda\mu)\omega_j\wedge ds.
\end{aligned}$$

Differentiating (3.16), we have

$$\begin{aligned}
d\omega_{jn} &= \frac{d^2\{\ln|\sqrt{\frac{n-1}{S}}(\lambda \mp \sqrt{S-(n-1)\lambda^2})e^{\mp\sqrt{n-1}\arcsin\sqrt{\frac{n-1}{S}}\lambda}|^{\frac{1}{n}}\}}{ds^2}ds\wedge\omega_j\\
&\quad + \frac{d\{\ln|\sqrt{\frac{n-1}{S}}(\lambda \mp \sqrt{S-(n-1)\lambda^2})e^{\mp\sqrt{n-1}\arcsin\sqrt{\frac{n-1}{S}}\lambda}|^{\frac{1}{n}}\}}{ds}d\omega_j\\
&= \frac{d^2\{\ln|\sqrt{\frac{n-1}{S}}(\lambda \mp \sqrt{S-(n-1)\lambda^2})e^{\mp\sqrt{n-1}\arcsin\sqrt{\frac{n-1}{S}}\lambda}|^{\frac{1}{n}}\}}{ds^2}ds\wedge\omega_j
\end{aligned}$$

$$\begin{aligned}
&+ \frac{d\{\ln|\sqrt{\frac{n-1}{S}}(\lambda \mp \sqrt{S-(n-1)\lambda^2})e^{\mp\sqrt{n-1}\arcsin\sqrt{\frac{n-1}{S}}\lambda}|^{\frac{1}{n}}\}}{ds}\sum_{k=1}^{n}\omega_{jk}\wedge\omega_k\\
=&\Big\{-\frac{d^2\{\ln|\sqrt{\frac{n-1}{S}}(\lambda \mp \sqrt{S-(n-1)\lambda^2})e^{\mp\sqrt{n-1}\arcsin\sqrt{\frac{n-1}{S}}\lambda}|^{\frac{1}{n}}\}}{ds^2}\\
&+\Big[\frac{d\{\ln|\sqrt{\frac{n-1}{S}}(\lambda \mp \sqrt{S-(n-1)\lambda^2})e^{\mp\sqrt{n-1}\arcsin\sqrt{\frac{n-1}{S}}\lambda}|^{\frac{1}{n}}\}}{ds}\Big]^2\Big\}\omega_j\wedge ds\\
&+ \frac{d\{\ln|\sqrt{\frac{n-1}{S}}(\lambda \mp \sqrt{S-(n-1)\lambda^2})e^{\mp\sqrt{n-1}\arcsin\sqrt{\frac{n-1}{S}}\lambda}|^{\frac{1}{n}}\}}{ds}\sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_k.
\end{aligned}$$

From the previous two equalities, we have

$$\begin{aligned}(3.17)\qquad &\frac{d^2\{\ln|\sqrt{\frac{n-1}{S}}(\lambda\mp\sqrt{S-(n-1)\lambda^2})e^{\mp\sqrt{n-1}\arcsin\sqrt{\frac{n-1}{S}}\lambda}|^{\frac{1}{n}}\}}{ds^2}\\ &-\Big\{\frac{d\{\ln|\sqrt{\frac{n-1}{S}}(\lambda\mp\sqrt{S-(n-1)\lambda^2})e^{\mp\sqrt{n-1}\arcsin\sqrt{\frac{n-1}{S}}\lambda}|^{\frac{1}{n}}\}}{ds}\Big\}^2\\ &-(c-\lambda\mu)=0.\end{aligned}$$

Putting

$$(3.18)\qquad \varpi=|\sqrt{\frac{n-1}{S}}(\lambda\mp\sqrt{S-(n-1)\lambda^2})e^{\mp\sqrt{n-1}\arcsin\sqrt{\frac{n-1}{S}}\lambda}|^{-\frac{1}{n}},$$

from (3.17), we obtain

$$\frac{d^2\varpi}{ds^2}+\varpi(c-\lambda\mu)=0.$$

By (3.2), we have

$$(3.19)\qquad \frac{d^2\varpi}{ds^2}+\varpi(c\mp\lambda\sqrt{S-(n-1)\lambda^2})=0.$$

On the other hand, from (3.16), we have $\nabla_{e_n}e_n=\sum_{i=1}^n\omega_{ni}(e_n)e_i=0$. By the definition of geodesic, we know that any integral curve of the principal vector field corresponding to the principal curvature μ is a geodesic. Thus, we see that $\varpi(s)$ is a function defined in $(-\infty,+\infty)$ since M^n is complete and any integral curve of the principal vector field corresponding to μ is a geodesic.

We can prove the following Lemmas:

Lemma 3.1. *If λ is bounded from below by a positive constant $\kappa>\sqrt{\frac{S}{n}}$ or is bounded from above by a positive constant $\kappa'<\sqrt{\frac{S}{n}}$, then ϖ is bounded.*

Proof. Since $S-(n-1)\lambda^2=\mu^2>0$, we have $\lambda<\sqrt{\frac{S}{n-1}}$. Putting $\theta=\arcsin\sqrt{\frac{n-1}{S}}\lambda$, we have $|\sin\theta|=\sqrt{\frac{n-1}{S}}|\lambda|<1$,thus $|\theta|=|\arcsin\sqrt{\frac{n-1}{S}}\lambda|<\frac{\pi}{2}$.

If $\lambda\geq\kappa>\sqrt{\frac{S}{n}}$, we have $\lambda-\sqrt{S-(n-1)\lambda^2}\geq\kappa-\sqrt{S-(n-1)\kappa^2}>\sqrt{\frac{S}{n}}-\sqrt{S-(n-1)\frac{S}{n}}=0$. Thus, we see that

$$\begin{aligned}&|\sqrt{\frac{n-1}{S}}(\lambda-\sqrt{S-(n-1)\lambda^2})e^{-\sqrt{n-1}\arcsin\sqrt{\frac{n-1}{S}}\lambda}|\\ &>\sqrt{\frac{n-1}{S}}|\lambda-\sqrt{S-(n-1)\lambda^2}|e^{-\frac{\pi}{2}\sqrt{n-1}}\\ &\geq\sqrt{\frac{n-1}{S}}|\kappa-\sqrt{S-(n-1)\kappa^2}|e^{-\frac{\pi}{2}\sqrt{n-1}}>0,\end{aligned}$$

and $0<\varpi<\left(\sqrt{\frac{n-1}{S}}|\kappa-\sqrt{S-(n-1)\kappa^2}|e^{-\frac{\pi}{2}\sqrt{n-1}}\right)^{-\frac{1}{n}}$.

On the other hand

$$\begin{aligned}&|\sqrt{\frac{n-1}{S}}(\lambda+\sqrt{S-(n-1)\lambda^2})e^{\sqrt{n-1}\arcsin\sqrt{\frac{n-1}{S}}\lambda}|\\&>\sqrt{\frac{n-1}{S}}|\lambda+\sqrt{S-(n-1)\lambda^2}|e^{-\frac{\pi}{2}\sqrt{n-1}}\\&\geq\sqrt{\frac{n-1}{S}}\kappa e^{-\frac{\pi}{2}\sqrt{n-1}}>\sqrt{\frac{n-1}{n}}e^{-\frac{\pi}{2}\sqrt{n-1}}>0.\end{aligned}$$

Thus $0<\varpi<\left(\sqrt{\frac{n-1}{n}}e^{-\frac{\pi}{2}\sqrt{n-1}}\right)^{-\frac{1}{n}}$.

If $\lambda\leq\kappa'<\sqrt{\frac{S}{n}}$, by reasoning as above we see that ϖ is also bounded □

Lemma 3.2. (1) *Let*

$$P_1(t)=1-\sqrt{t}\sqrt{S-(n-1)t},$$

and $t_0=\frac{S}{2(n-1)}$.

If $S<2\sqrt{n-1}$, *then* $P_1(t)>0$; *if* $S\geq2\sqrt{n-1}$, *then* $P_1(t)$ *has two positive real roots* $t_1=\frac{S-\sqrt{S^2-4(n-1)}}{2(n-1)}$, $t_2=\frac{S+\sqrt{S^2-4(n-1)}}{2(n-1)}$ *and* $t_1\leq t_0\leq t_2$.

(*i*) *if* $t\geq t_0$, *then* $t\geq t_2$ *holds if and only if* $P_1(t)\geq0$ *and* $t\leq t_2$ *holds if and only if* $P_1(t)\leq0$;

(*ii*) *if* $t\leq t_0$, *then* $t\leq t_1$ *holds if and only if* $P_1(t)\geq0$ *and* $t\geq t_1$ *holds if and only if* $P_1(t)\leq0$.

(2) *Let*

$$P_2(t)=-1+\sqrt{t}\sqrt{S-(n-1)t},$$

and $t_0=\frac{S}{2(n-1)}$.

If $S<2\sqrt{n-1}$, *then* $P_2(t)<0$; *if* $S\geq2\sqrt{n-1}$, *then* $P_2(t)$ *has two positive real roots* $t_1=\frac{S-\sqrt{S^2-4(n-1)}}{2(n-1)}$, $t_2=\frac{S+\sqrt{S^2-4(n-1)}}{2(n-1)}$ *and* $t_1\leq t_0\leq t_2$.

(*i*) *if* $t\geq t_0$, *then* $t\geq t_2$ *holds if and only if* $P_2(t)\leq0$ *and* $t\leq t_2$ *holds if and only if* $P_2(t)\geq0$;

(*ii*) *if* $t\leq t_0$, *then* $t\leq t_1$ *holds if and only if* $P_2(t)\leq0$ *and* $t\geq t_1$ *holds if and only if* $P_2(t)\geq0$.

(3) *Let*

$$H_1(t)=(n-1)\sqrt{t}-\sqrt{S-(n-1)t}.$$

Then

(*i*) $t\geq t_2$ *holds if and only if* $H_1(t)\geq H_1(t_2)$ *and* $t\leq t_2$ *holds if and only if* $H_1(t)\leq H_1(t_2)$;

(*ii*) $t\geq t_1$ *holds if and only if* $H_1(t)\geq H_1(t_1)$ *and* $t\leq t_1$ *holds if and only if* $H_1(t)\leq H_1(t_1)$.

(4) *Let*

$$H_2(t) = (n-1)\sqrt{t} + \sqrt{S-(n-1)t}.$$

and $t_0' = \frac{S}{n}$. *Then*

(*i*) *if* $t \geq t_0'$, *when* $t_2 \geq t_0'$, *then* $t \geq t_2$ *holds if and only if* $H_2(t) \leq H_2(t_2)$ *and* $t \leq t_2$ *holds if and only if* $H_2(t) \geq H_2(t_2)$;

(*ii*) *if* $t \leq t_0'$, *when* $t_2 \leq t_0'$, *then* $t \geq t_2$ *holds if and only if* $H_2(t) \geq H_2(t_2)$ *and* $t \leq t_2$ *holds if and only if* $H_2(t) \leq H_2(t_2)$. *In addition,* $t \geq t_1$ *holds if and only if* $H_2(t) \geq H_2(t_1)$ *and* $t \leq t_1$ *holds if and only if* $H_2(t) \leq H_2(t_1)$.

Proof. (1) We have

$$\frac{dP_1(t)}{dt} = \frac{2(n-1)t - S}{2\sqrt{t}\sqrt{S-(n-1)t}},$$

it follows that the solution of $\frac{dP_1(t)}{dt} = 0$ is $t_0 = \frac{S}{2(n-1)}$. Therefore, we know that $t \leq t_0$ if and only if $P_1(t)$ is a decreasing function, $t \geq t_0$ if and only if $P_1(t)$ is an increasing function and $P_1(t)$ obtain its minimum at $t_0 = \frac{S}{2(n-1)}$ and $P_1(t_0) = 1 - \frac{S}{2\sqrt{n-1}}$.

If $S < 2\sqrt{n-1}$, we have $P_1(t) \geq P_1(t_0) > 0$;

If $S \geq 2\sqrt{n-1}$, then $P_1(t)$ has two positive real roots $t_1 = \frac{S-\sqrt{S^2-4(n-1)}}{2(n-1)}$, $t_2 = \frac{S+\sqrt{S^2-4(n-1)}}{2(n-1)}$. We easily see that $t_1 \leq t_0 \leq t_2$.

(*i*) if $t \geq t_0$, since $P_1(t)$ is an increasing function, we have $t \geq t_2$ holds if and only if $P_1(t) \geq P_1(t_2) = 0$ and $t \leq t_2$ holds if and only if $P_1(t) \leq P_1(t_2) = 0$.

(*ii*) if $t \leq t_0$, since $P_1(t)$ is a decreasing function, we have $t \leq t_1$ holds if and only if $P_1(t) \geq P_1(t_1) = 0$ and $t \geq t_1$ holds if and only if $P_1(t) \leq P_1(t_1) = 0$.

(2) By the same method, (2) of Lemma 3.3 follows.

(3) We have

$$\frac{dH_1(t)}{dt} = \frac{n-1}{2}\left(\frac{1}{\sqrt{t}} + \frac{1}{\sqrt{S-(n-1)t}}\right) > 0,$$

it follows that $H_1(t)$ is an increasing function, we conclude.

(4) We have

$$\frac{dH_2(t)}{dt} = \frac{n-1}{2}\left(\frac{1}{\sqrt{t}} - \frac{1}{\sqrt{S-(n-1)t}}\right),$$

it follows that the solution of $\frac{dH_2(t)}{dt} = 0$ is $t_0' = \frac{S}{n}$. Therefore, we know that $t \leq t_0'$ if and only if $H_2(t)$ is an increasing function, $t \geq t_0'$ if and only if $H_2(t)$ is a decreasing function and $H_2(t)$ obtain its maximum at $t_0' = \frac{S}{n}$ and $H_2(t_0') = \sqrt{nS}$. We see that (*i*) and (*ii*) of (4) follows by the monotonicity of $H_2(t)$. $\square$

Proof of Theorem 1.3 . Putting $t = \lambda^2 (> 0)$, since $c = 1$, from (3.19), we have

$$\frac{d^2\varpi}{ds^2} + \varpi(1 - \sqrt{t}\sqrt{S-(n-1)t}) = 0, \tag{3.20}$$

or

$$\frac{d^2\varpi}{ds^2} + \varpi(1 + \sqrt{t}\sqrt{S-(n-1)t}) = 0. \tag{3.21}$$

(1) If $\lambda \geq \kappa > \sqrt{\frac{S}{n}}$, we have $t > \frac{S}{n} > t_0 = \frac{S}{2(n-1)}$. We may easily see that (3.21) does not hold. In fact, since $1 + \sqrt{t}\sqrt{S-(n-1)t} > 0$, from (3.21), we have $\frac{d^2\varpi}{ds^2} < 0$. This implies that $\frac{d\varpi(s)}{ds}$ is a strictly monotone decreasing function of s and thus it has at most one zero point for $s \in (-\infty, +\infty)$. If $\frac{d\varpi(s)}{ds}$ has no zero point in $(-\infty, +\infty)$, then $\varpi(s)$ is a monotone function of s in $(-\infty, +\infty)$. If $\frac{d\varpi(s)}{ds}$ has exactly one zero point s_0 in $(-\infty, +\infty)$, then $\varpi(s)$ is a monotone function of s in both $(-\infty, s_0]$ and $[s_0, +\infty)$.

On the other hand, from Lemma 3.1, we know that $\varpi(s)$ is bounded. Since $\varpi(s)$ is bounded and monotonic when s tends to infinity, we know that both $\lim_{s\to-\infty}\varpi(s)$ and $\lim_{s\to+\infty}\varpi(s)$ exist and then we get

$$\lim_{s\to-\infty}\frac{d\varpi(s)}{ds} = \lim_{s\to+\infty}\frac{d\varpi(s)}{ds} = 0. \tag{3.22}$$

This is impossible because $\frac{d\varpi(s)}{ds}$ is a strictly monotone increasing function of s. Therefore, we conclude that only (3.20) holds, that is

$$\frac{d^2\varpi}{ds^2} + \varpi P_1(t) = 0. \tag{3.23}$$

If $S < 2\sqrt{n-1}$, by Lemma 3.2, we have $P_1(t) > 0$. From (3.23), we have $\frac{d^2\varpi}{ds^2} < 0$. This implies that $\frac{d\varpi(s)}{ds}$ is a strictly monotone decreasing function of s. By the same arguments as above, we know that this is impossible.

If $S \geq 2\sqrt{n-1}$, since $t > t_0$, from Lemma 3.2 and (3.23), we see that if

$$H \geq \frac{\sqrt{n-1}(S + \sqrt{S^2 - 4(n-1)} - 2)}{n\sqrt{2(S + \sqrt{S^2-4(n-1)})}},$$

that is, $H = H_1(t) \geq H_1(t_2)$ holds if and only if $t \geq t_2$ if and only if $P_1(t) \geq 0$ and if and only if $\frac{d^2\varpi}{ds^2} \leq 0$. Thus $\frac{d\varpi}{ds}$ is a monotonic function of $s \in (-\infty, +\infty)$. Therefore, as observed by Wei [8], $\varpi(s)$ must be monotonic when s tends to infinity. From Lemma 3.1, we know that the positive function $\varpi(s)$ is bounded. Since $\varpi(s)$ is bounded and monotonic when s tends to infinity, we know that both $\lim_{s\to-\infty}\varpi(s)$ and $\lim_{s\to+\infty}\varpi(s)$ exist and (3.22) holds. From the monotonicity of $\frac{d\varpi(s)}{ds}$, we have $\frac{d\varpi(s)}{ds} \equiv 0$ and $\varpi(s) = constant$. Combining $\varpi = |\sqrt{\frac{n-1}{S}}(\lambda \mp \sqrt{S-(n-1)\lambda^2})e^{\mp\sqrt{n-1}\arcsin\sqrt{\frac{n-1}{S}}\lambda}|^{-\frac{1}{n}}$ and (3.2), we conclude that λ and μ are constant, that is, M^n is isoparametric. By the congruence Theorem of Abe, Koike and Yamaguchi (see Theorem 5.1 of [1]) and Example 1.1, we conclude that M^n is isometric to the Riemannian product $S^{n-1}(a) \times H^1(\sqrt{a^2-1})$, $a^2 = \frac{2(n-1)}{2(n-1)-(S+\sqrt{S^2-4(n-1)})}$, $\frac{1}{1-\kappa^2} \le a^2 < \beta(\kappa)$ and $1 > \kappa^2 > \frac{2\sqrt{n-1}}{n}$.

(2) If $\lambda \le \kappa' < \sqrt{\frac{S}{n}}$, we have $t < \frac{S}{n} = t'_0$. From the arguments in (1), we know that only (3.20) holds.

If $S < 2\sqrt{n-1}$, by the same arguments in (1), we know that this is impossible.

If $S \ge 2\sqrt{n-1}$, from Lemma 3.2, we see that if

$$H \le \frac{\sqrt{n-1}(S - \sqrt{S^2-4(n-1)} + 2)}{n\sqrt{2(S-\sqrt{S^2-4(n-1)})}},$$

that is, $H = H_1(t) \le H_1(t_1)$ holds if and only if $t \le t_1$. Since $t_1 \le t_0$, from Lemma 3.2 and (3.23), we have that $t \le t_1$ if and only if $P_1(t) \ge 0$ and if and only if $\frac{d^2\varpi}{ds^2} \le 0$. Thus $\frac{d\varpi}{ds}$ is a monotonic function of $s \in (-\infty, +\infty)$. By the same arguments as in the proof of (1) and Example 1.1, we conclude that M^n is isometric to the Riemannian product $S^{n-1}(a) \times H^1(\sqrt{a^2-1})$, $a^2 = \frac{2(n-1)}{2(n-1)-(S-\sqrt{S^2-4(n-1)})}$, $\alpha(\kappa') < a^2 < \beta(\kappa')$ and $\kappa' > 1$; $a^2 < \alpha(\kappa')$ or $\beta(\kappa') < a^2 \le \frac{1}{1-\kappa'^2}$ and $\frac{2\sqrt{n-1}}{n} < \kappa'^2 < 1$. □

Proof of Theorem 1.4. Putting $t = \lambda^2 (> 0)$, since $c = -1$, from (3.19), we have

$$\frac{d^2\varpi}{ds^2} + \varpi(-1 - \sqrt{t}\sqrt{S-(n-1)t}) = 0, \tag{3.24}$$

or

$$\frac{d^2\varpi}{ds^2} + \varpi(-1 + \sqrt{t}\sqrt{S-(n-1)t}) = 0. \tag{3.25}$$

(1) If $\lambda \ge \kappa > \sqrt{\frac{S}{n}}$, we have $t > \frac{S}{n} > t_0 = \frac{S}{2(n-1)}$. By reasoning as in the proof of Theorem 1.3, we see that (3.24) does not hold. Thus it follows that only (3.25) holds, that is

$$\frac{d^2\varpi}{ds^2} + \varpi P_2(t) = 0. \tag{3.26}$$

If $S < 2\sqrt{n-1}$, by Lemma 3.2, we have $P_2(t) < 0$. From (3.26), we have $\frac{d^2\varpi}{ds^2} > 0$. By the same arguments as in the proof of Theorem 1.3, we know that this is impossible.

If $S \ge 2\sqrt{n-1}$, we consider two cases $S \ge n$ and $2\sqrt{n-1} \le S < n$.

If $S \geq n$, we easily check that $t_2 \geq t_0'$. Since $t_0' = \frac{S}{n} > t_0$, we have $t_2 \geq t_0' > t_0$. Since $t > t_0' > t_0$, from Lemma 3.2 and (3.26), we see that if

$$H \geq \frac{\sqrt{n-1}(S + \sqrt{S^2 - 4(n-1)} - 2)}{n\sqrt{2(S + \sqrt{S^2 - 4(n-1)})}},$$

that is, $H_2(t) \geq H_2(t_2)$ holds if and only if $t \leq t_2$ if and only if $P_2(t) \geq 0$ and if and only if $\frac{d^2\varpi}{ds^2} \leq 0$. Thus $\frac{d\varpi}{ds}$ is a monotonic function of $s \in (-\infty, +\infty)$. Since $n\kappa^2 > S \geq n$, that is $\kappa > 1$, by the same arguments as in the proof of Theorem 1.3 and Example 1.2, we conclude that M^n is isometric to the Riemannian product $H^{n-1}(a) \times H^1(\sqrt{1-a^2})$, $a^2 = \frac{2(n-1)}{2(n-1)+S+\sqrt{S^2-4(n-1)}}$, $\gamma(\kappa) < a^2 \leq \frac{1}{1+\kappa^2}$, $\kappa > 1$.

If $2\sqrt{n-1} \leq S < n$, we easily check that $t_2 < t_0'$. Since $t_2 \geq t_0$, we have $t_0' > t_2 \geq t_0$. Thus $t > t_2 \geq t_0$. From Lemma 3.2 and (3.26), we have $P_2(t) < 0$ and $\frac{d^2\varpi}{ds^2} > 0$. By the same arguments as in the proof of Theorem 1.3, we know that this is impossible. Thus, the case $2\sqrt{n-1} \leq S < n$ does not occur.

(2) If $\lambda \leq \kappa' < \sqrt{\frac{S}{n}}$, we have $t < \frac{S}{n} = t_0'$. By reasoning as in the proof of Theorem 1.3, we see that only (3.26) holds.

If $S < 2\sqrt{n-1}$, by the same arguments in the proof of Theorem 1.3, we know that this is impossible.

If $S \geq 2\sqrt{n-1}$, we may easily check that $t_1 \leq t_0'$. Since $t < t_0'$, from Lemma 3.2, we see that if

$$H \leq \frac{\sqrt{n-1}(S - \sqrt{S^2 - 4(n-1)} - 2)}{n\sqrt{2(S - \sqrt{S^2 - 4(n-1)})}},$$

that is, $H_2(t) \leq H_2(t_1)$ holds if and only if $t \leq t_1$. Since $t_1 \leq t_0$, by Lemma 3.2 and (3.26), we have that $t \leq t_1$ if and only if $P_2(t) \leq 0$ and if and only if $\frac{d^2\varpi}{ds^2} \geq 0$. Thus $\frac{d\varpi}{ds}$ is a monotonic function of $s \in (-\infty, +\infty)$. By the same arguments as in the proof of Theorem 1.3 and Example 1.2, we conclude that M^n is isometric to the Riemannian product $H^{n-1}(a) \times H^1(\sqrt{1-a^2})$, $a^2 = \frac{2(n-1)}{2(n-1)+S-\sqrt{S^2-4(n-1)}}$, $a^2 > \delta(\kappa')$, $\kappa'^2 \geq \frac{2\sqrt{n-1}}{n}$, or $\frac{1}{1+\kappa'^2} \leq a^2 < \gamma(\kappa')$, $\frac{2\sqrt{n-1}}{n} \leq \kappa'^2 < 1$. $\square$

4. Proofs of Theorem 1.5 and Theorem 1.6

Let M^n be an n-dimensional complete spacelike hypersurface with nonzero constant Gauss-Kronecker curvature K and two distinct principal curvatures λ and μ of multiplicities $n-1$ and 1. We have that

$$K = \lambda^{n-1}\mu. \tag{4.1}$$

From $K \neq 0$, we conclude that $\lambda \neq 0$. By changing the orientation for M^n and renumbering $e_1, \ldots, e_n$ if necessary, we may assume that $\lambda > 0$. Thus

$$\mu = \frac{K}{\lambda^{n-1}}, \tag{4.2}$$

$$0 \neq \lambda - \mu = \frac{\lambda^n - K}{\lambda^{n-1}}. \tag{4.3}$$

Denote by $D(x)$ the integral submanifold through $x \in M^n$ corresponding to λ. By Proposition 2.1, we have

$$\lambda_{,1} = \lambda_{,2} = \cdots = \lambda_{,n-1} = 0 \quad \text{on} \quad D(x). \tag{4.4}$$

From (4.2), we have

$$d\mu = -\frac{(n-1)K}{\lambda^n} d\lambda. \tag{4.5}$$

Thus

$$\mu_{,1} = \mu_{,2} = \cdots = \mu_{,n-1} = 0 \quad \text{on} \quad D(x). \tag{4.6}$$

From (2.14) and (4.4)–(4.6), by the same arguments as in section 3, we have

$$h_{jjk} = 0, \quad 1 \leq k \leq n-1, \quad \text{and} \quad h_{jjn} = \lambda_{,n}\,. \tag{4.7}$$

$$h_{nnk} = 0, \quad 1 \leq k \leq n-1, \quad \text{and} \quad h_{nnn} = \mu_{,n}\,. \tag{4.8}$$

$$h_{nnn} = \mu_{,n} = -\frac{(n-1)K}{\lambda^n}\lambda_{,n}\,, \quad h_{ijk} = 0, \quad \text{for any} \quad k. \tag{4.9}$$

By (2.14), (4.7)–(4.9), for $j < n$, we get

$$\begin{aligned} \psi_{jn} =& \sum_{k=1}^{n} h_{jnk}\omega_k \\ =& h_{jjn}\omega_j + Q_{jnn}\omega_n = \lambda_{,n}\,\omega_j. \end{aligned} \tag{4.10}$$

From $\psi_{ij} = (\lambda_i - \lambda_j)\omega_{ij}$, (4.3) and (4.10), for $j < n$, we have

$$\omega_{jn} = \frac{\psi_{jn}}{\lambda - \mu} = \frac{\lambda_{,n}}{\lambda - \mu}\omega_j = \frac{\lambda^{n-1}\lambda_{,n}}{\lambda^n - K}\omega_j. \tag{4.11}$$

From the structure equations of M^n we have $d\omega_n = 0$. Thus, we may put $\omega_n = ds$. By (4.4), we get $d\lambda = \lambda_{,n}\,ds$, $\lambda_{,n} = \frac{d\lambda}{ds}$. Thus, we have

$$\omega_{jn} = \frac{\lambda^{n-1}\frac{d\lambda}{ds}}{\lambda^n - K}\omega_j = \frac{d(\log|\lambda^n - K|^{\frac{1}{n}})}{ds}\omega_j. \tag{4.12}$$

From (4.12) and the structure equations of $M_1^{n+1}(c)$, for $j < n$, we have

$$\begin{aligned} d\omega_{jn} &= \sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_{kn} + \omega_{jn}\wedge\omega_{nn} - \omega_{jn+1}\wedge\omega_{n+1n} + \Omega_{jn} \\ &= \sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_{kn} - \omega_{jn+1}\wedge\omega_{n+1n} - c\omega_j\wedge\omega_n \\ &= \frac{d(\log|\lambda^n - K|^{\frac{1}{n}})}{ds}\sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_k - (c-\lambda\mu)\omega_j\wedge ds. \end{aligned}$$

Differentiating (4.12), we have

$$\begin{aligned} d\omega_{jn} =& \frac{d^2(\log|\lambda^n - K|^{\frac{1}{n}})}{ds^2}ds\wedge\omega_j + \frac{d(\log|\lambda^n - K|^{\frac{1}{n}})}{ds}d\omega_j \\ =& \frac{d^2(\log|\lambda^n - K|^{\frac{1}{n}})}{ds^2}ds\wedge\omega_j + \frac{d(\log|\lambda^n - K|^{\frac{1}{n}})}{ds}\sum_{k=1}^{n}\omega_{jk}\wedge\omega_k \\ =& \Big\{-\frac{d^2(\log|\lambda^n - K|^{\frac{1}{n}})}{ds^2} + \Big[\frac{d(\log|\lambda^n - K|^{\frac{1}{n}})}{ds}\Big]^2\Big\}\omega_j\wedge ds \\ &+ \frac{d(\log|\lambda^n - K|^{\frac{1}{n}})}{ds}\sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_k. \end{aligned}$$

From the previous two equalities, we have

$$\frac{d^2(\log|\lambda^n - K|^{\frac{1}{n}})}{ds^2} - \Big\{\frac{d(\log|\lambda^n - K|^{\frac{1}{n}})}{ds}\Big\}^2 - (c-\lambda\mu) = 0. \tag{4.13}$$

If we define $\varpi = |\lambda^n - K|^{-\frac{1}{n}}$, from (4.13) we obtain

$$\frac{d^2\varpi}{ds^2} + \varpi(c-\lambda\mu) = 0. \tag{4.14}$$

From (4.12), we have $\nabla_{e_n}e_n = \sum_{i=1}^n \omega_{ni}(e_n)e_i = 0$. By the same arguments as in section 3, we see that $\varpi(s)$ is a function defined in $(-\infty, +\infty)$.

We can prove the following Lemmas:

Lemma 4.1. *If $c = 1$ and $K < 1$, then the positive function ϖ is bounded from above.*

Proof. From (4.3), we know that $\lambda^n - K \neq 0$. Thus (4.2) and (4.14) imply that

$$\frac{d^2\varpi}{ds^2} + \varpi\frac{c\lambda^{n-2} - K}{\lambda^{n-2}} = 0, \tag{4.15}$$

that is

$$\frac{d^2\varpi}{ds^2} + \varpi\Big[c - K(K \pm \varpi^{-n})^{\frac{2}{n}-1}\Big] = 0. \tag{4.16}$$

Multiplying (4.16) by $2\frac{d\varpi}{ds}$ and integrating, we get

$$\left(\frac{d\varpi}{ds}\right)^2 + c\varpi^2 - \varpi^2(K \pm \varpi^{-n})^{\frac{2}{n}} = C,$$

where C is a constant. Thus, we have

$$c - (K \pm \varpi^{-n})^{\frac{2}{n}} \leq \frac{C}{\varpi^2}. \tag{4.17}$$

If the positive function ϖ is not bounded from above, that is, $\varpi \to +\infty$. From (3.17), we have that $c - K^{\frac{2}{n}} \leq 0$, a contradiction with the assumption. □

Lemma 4.2. (1) *Let*

$$S(t) = \frac{1}{t^{2(n-1)/n}}\{(n-1)t^2 + K^2\}, \quad t > 0,$$

$t_0 = K^{\frac{n}{n-2}}$, $K > 0$. *If* $t \leq K$ *and* $t_0 \leq K$, *then* $t \leq t_0$ *holds if and only if* $S(t) \geq (n-1)t_0^{2/n} + t_0^{-2/n}$ *and* $t \geq t_0$ *holds if and only if* $S(t) \leq (n-1)t_0^{2/n} + t_0^{-2/n}$.

(2) *Let*

$$H(t) = \frac{1}{nt^{(n-1)/n}}\{(n-1)t + K\}, \quad t > 0.$$

$t_0 = K^{\frac{n}{n-2}}$, $K > 0$. *If* $t \leq K$ *and* $t_0 \leq K$, *then* $t \leq t_0$ *holds if and only if* $H(t) \geq \frac{1}{n}\{(n-1)t_0^{1/n} - t_0^{-1/n}\}$ *and* $t \geq t_0$ *holds if and only if* $H(t) \leq \frac{1}{n}\{(n-1)t_0^{1/n} - t_0^{-1/n}\}$.

Proof. (1) We have

$$\frac{dS(t)}{dt} = \frac{2(n-1)t^{(2-3n)/n}}{n}(t^2 - K^2),$$

it follows that $t \leq K$ if and only if $S(t)$ is a decreasing function, $t \geq K$ if and only if $S(t)$ is an increasing function.

If $t_0 \leq K$, since $t \leq K$ if and only if $S(t)$ is a decreasing function, we infer that if $t \leq K$, then $t \leq t_0$ holds if and only if

$$\begin{aligned} S(t) \geq S(t_0) &= \frac{1}{t_0^{2(n-1)/n}}\{(n-1)t_0^2 + K^2\} \\ &= \frac{1}{t_0^{2(n-1)/n}}\left\{(n-1)t_0^2 + \left[\left(t_0^{\frac{n-2}{n}} - K\right) - t_0^{\frac{n-2}{n}}\right]^2\right\} \\ &= \frac{1}{t_0^{2(n-1)/n}}\left\{(n-1)t_0^2 + \left[-t_0^{\frac{n-2}{n}}\right]^2\right\} = (n-1)t_0^{2/n} + t_0^{-2/n}, \end{aligned}$$

By the same reason, the rest of (1) follows.

(2) Since $H(t)$ is a decreasing function if $t \leq K$ and an increasing function if $t \geq K$, it follows the result of (2). □

Proof of Theorem 1.5. Putting $t = \lambda^n (> 0)$ and $P_K(t) = t^{\frac{n-2}{n}} - K$, from (4.15), we have

$$\frac{d^2\varpi}{ds^2} + \varpi\frac{P_K(t)}{t^{\frac{n-2}{n}}} = 0. \tag{4.18}$$

Since we assume that $0 < K < 1$, we see that $t_0 < K$, where $t_0 = K^{\frac{n}{n-2}}$. We consider two cases $t \geq K$ and $t \leq K$.

If $t \geq K$, we have $t > t_0$. Thus, $P_K(t) > P_K(t_0) = 0$. From (4.18), we have $\frac{d^2\varpi}{ds^2} < 0$. This implies that $\frac{d\varpi(s)}{ds}$ is a strictly monotone decreasing function of s and thus it has at most one zero point for $s \in (-\infty, +\infty)$. By the same arguments as in the proof of Theorem 1.3, we know that this is impossible.

If $t \leq K$, since $t_0 < K$, from (2) of Lemma 4.2 and (4.18), we see that

$$H \leq \frac{1}{n}\{(n-1)K^{\frac{1}{n-2}} - K^{-\frac{1}{n-2}}\} = \frac{1}{n}\{(n-1)t_0^{1/n} - t_0^{-1/n}\}$$

holds if and only if $t \geq t_0$ if and only if $P_K(t) \geq 0$ and if and only if $\frac{d^2\varpi}{ds^2} \leq 0$. Also

$$H \geq \frac{1}{n}\{(n-1)K^{\frac{1}{n-2}} - K^{-\frac{1}{n-2}}\} = \frac{1}{n}\{(n-1)t_0^{1/n} - t_0^{-1/n}\}$$

holds if and only if $t \leq t_0$ if and only if $P_K(t) \leq 0$ and if and only if $\frac{d^2\varpi}{ds^2} \geq 0$. Thus $\frac{d\varpi}{ds}$ is a monotonic function of $s \in (-\infty, +\infty)$. By use of the same method as in the proof Theorem 1.3, we know that M^n is isometric to the Riemannian product $S^{n-1}(a) \times H^1(\sqrt{a^2-1})$, $a^2 = 1/(1 - K^{\frac{2}{n-2}})$. □

Proof of Theorem 1.6. Putting $t = \lambda^n (> 0)$ and $P_K(t) = t^{\frac{n-2}{n}} - K$, we see that (4.18) holds. Since $0 < K < 1$, we have $t_0 < K$, where $t_0 = K^{\frac{n}{n-2}}$. We also consider two cases $t \geq K$ and $t \leq K$.

If $t \geq K$, we have $t > t_0$. Thus, $P_K(t) > P_K(t_0) = 0$. From (4.18), we have $\frac{d^2\varpi}{ds^2} < 0$. By the same arguments as in the proof of Theorem 1.3, we know that this is impossible.

If $t \leq K$, since $t_0 < K$, from (1) of Lemma 4.2 and (4.18), we see that

$$S \leq (n-1)K^{\frac{2}{n-2}} + K^{-\frac{2}{n-2}} = (n-1)t_0^{2/n} + t_0^{-2/n}$$

holds if and only if $t \geq t_0$ if and only if $P_K(t) \geq 0$ and if and only if $\frac{d^2\varpi}{ds^2} \leq 0$. Also

$$S \geq (n-1)K^{\frac{2}{n-2}} + K^{-\frac{2}{n-2}} = (n-1)t_0^{2/n} + t_0^{-2/n}$$

holds if and only if $t \leq t_0$ if and only if $P_K(t) \leq 0$ and if and only if $\frac{d^2\varpi}{ds^2} \geq 0$. Thus $\frac{d\varpi}{ds}$ is a monotonic function of $s \in (-\infty, +\infty)$. By the same arguments as in the proof of Theorem 1.3, we know that M^n is isometric to the Riemannian product $S^{n-1}(a) \times H^1(\sqrt{a^2-1})$, $a^2 = 1/(1 - K^{\frac{2}{n-2}})$. □

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