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Author(s):

M. Tahmasebi and S. Zamani

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WEAK DIFFERENTIABILITY OF SOLUTIONS TO SDES WITH SEMI-MONOTONE DRIFTS

M. TAHMASEBI* AND S. ZAMANI

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Abstract. In this work we prove Malliavin differentiability for the solution to an SDE with locally Lipschitz and semi-monotone drift. To prove this formula, we construct a sequence of SDEs with globally Lipschitz drifts and show that the p-moments of their Malliavin derivatives are uniformly bounded.
Keywords: Smoothness of density, stochastic differential equation, semi-monotone drift, Malliavin calculus.
MSC(2010): Primary: 60H07; Secondary: 60H10.

1. **Introduction**

Stochastic flows and weak derivatives in Wiener space are studied by various authors. In [10] Kusuoka and Stroock have shown that an SDE with coefficients which are C^{∞}-globally Lipschitz and have polynomial growth, has a strong Malliavin differentiable solution of any order. In recent years, there were attempts to generalize these results to SDEs with non-globally Lipschitz coefficients. For example, in [6,16] the authors studied the existence of a global stochastic flow for SDEs with unbounded and Hölder continuous drift and a nondegenerate diffusion coefficients. Zhang [17] considered the flow of stochastic transport equations with irregular coefficients . The SDEs with non-globally Lipschitz coefficients have many applications in Financial Mathematics. The interested reader could see [1,3,9,14].

The SDE we consider has both non-globally Lipschitz and semi-monotone drift coefficient. Such equations mostly come from finance and biology and also dynamical systems and are more challenging when considered on infinite dimensional spaces (see e.g. [2,7,18]).

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*Corresponding author.

In this paper, we consider an SDE with locally Lipschitz and monotone drift and globally Lipschitz diffusion. We prove the existence of a unique infinitely Malliavin differentiable strong solution to this SDE.

Since the drift of the SDE we consider is not globally Lipschitz, we will construct a sequence of SDEs with globally Lipschitz drifts whose solutions are Malliavin differentiable of any order. In this way we can apply the classical Malliavin calculus to the solutions. Then we can find also a uniform bound for the moments of all the the Malliavin derivatives of solutions. We will prove that the solutions to the constructed sequence of SDEs converge to the solution of the desired SDE. Then by the uniform boundedness of the moments of the mentioned solutions and the convergence result we are able to prove infinite Malliavin differentiability of the solution to the original SDE.

The organization of the paper is as follows. In section 2, we recall some basic results from Malliavin calculus that will be used in the paper. In section 3, we state the assumptions and prove the existence and uniqueness of the solution. In section 4, we construct our approximating SDEs with globally Lipschitz coefficients and prove the convergence of their solutions to the unique solution of the original SDE (3.1). In section 5, we will prove uniform boundedness of the Malliavin derivatives associated to the approximating processes, which results to the infinite weak differentiability of the solution to this SDE.

2. **Some basic results from Malliavin calculus**

Let Ω denote the Wiener space $C_0([0,T];\mathbb{R}^d)$. We furnish Ω with the $\|\ .\ \|_\infty$-norm making it a (separable) Banach space. Consider $(\Omega,\mathcal{F},P)$ a complete probability space, in which $\mathcal{F}$ is generated by the open sets of the Banach space, W_t is a d-dimensional Brownian motion, and $\mathcal{F}_t$ is the filtration generated by W_t.

Consider the Hilbert space $H := L^2([0,T];\mathbb{R}^d)$ and the space $L^p(\Omega;H)$, the set of H-valued random variable X such that $\mathbb{E}\Big[\|X\|_H^p\Big] < \infty$. We denote $L^p(\Omega) := L^p(\Omega;\mathbb{R}^d)$. Let $\{W(h), h \in H\}$ be a Gaussian process associated to the Hilbert space H in which $W(h) = \int_0^\infty h(t)dW_t$. We denote by $C^\infty_{pol}(\mathbb{R}^n)$ the set of all infinitely differentiable functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that f and all of its partial derivatives have polynomial growth. Let $\mathcal{S}$ be the class of all smooth random variables $F : \Omega \longrightarrow \mathbb{R}$ such that $F = f(W(h_1),...,W(h_n))$ for some f belonging to $C^\infty_{pol}(\mathbb{R}^n)$, and $h_1,...,h_n \in H$ for some $n \geq 1$.
The derivative of the smooth random variable $F \in \mathcal{S}$ is an H-valued random variable given by

$$D_tF = \Sigma_{i=1}^n \partial_i f(W(h_1),...,W(h_n))h_i(t).$$

The operator D is closable from $L^p(\Omega)$ to $L^p(\Omega;H)$, for every $p \geq 1$. We denote its domain by $\mathbb{D}^{1,p}$ which is exactly the closure of $\mathcal{S}$ with respect to $\|\ .\ \|_{1,p}$

where

$$\parallel F \parallel_{1,p} = \left[E|F|^p + \parallel DF \parallel^p_{L^p(\Omega;H)}\right]^{\frac{1}{p}}$$

(see [15]). One can also define the k-th order derivative of F as a random vector in $[0,T]^k \times \Omega$. We denote by $\mathbb{D}^{k,p}$ the completion of $\mathcal{S}$ with respect to the norm

$$\parallel F \parallel_{k,p} = \left[E|F|^p + \parallel D^{i_1,\cdots,i_k} F \parallel^p_{L^p(\Omega;H^{\otimes k})}\right]^{\frac{1}{p}},$$

and set $\mathbb{D}^\infty := \bigcap_{k,p} \mathbb{D}^{k,p}$.

Kusouka and Stroock have proved the following proposition [11, Theorem 1.9.].

Proposition 2.1. *Consider the following SDE.*

$$dX_t = B(X_t)dt + G(X_t)dW_t, \qquad X_0 = x_0,$$

where the coefficients B and G are globally Lipschitz functions and all of their derivatives have polynomial growth, then (3.1) *has a strong solution in $\mathbb{D}^\infty$ whose Malliavin derivative satisfies the following linear equations. For every $r \leq t$*

$$D_r X^i_t = G^i(X_r) + \int_r^t \nabla B^i(X_s) D_r X_s ds + \int_r^t \nabla G^i_l(Y_s) D_r X_s dW^l_s,$$

and for every $r > t$, $D_r X_t = 0$. Also it holds

$$\sup_{0\leq r\leq T} \mathbb{E}[\sup_{r\leq s\leq T} |D^j_r X^i|] < \infty.$$

In what follows use the upper index shows a specified row, and the subindex shows a specified column of a matrix.

3. **Existence and Uniquness of the solution**

Consider the following stochastic differential equation

$$dX_t = [b(X_t) + f(X_t)]dt + \sigma(X_t)dW_t, \qquad X_0 = x_0. \tag{3.1}$$

where $b, f : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ are measurable functions and $\sigma : \mathbb{R}^d \longrightarrow M_{d\times d}(\mathbb{R})$ is a measurable C^∞ function. We denote by $\mathcal{L}$ the second-order differential operator associated to SDE (3.1):

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^d (\sigma\sigma^*)^i_j(x)\partial_i\partial_j + \sum_{i=1}^d [b^i(x) + f^i(x)]\partial_i,$$

where $*$ denotes the transpose of matrix. Throughout the paper we assume that b, f and σ satisfy the following Hypothesis.

Hypothesis 3.1. • *The function b is an C^∞ uniformly monotone function, i.e., there exists a constant $K>0$ such that for every $x, y \in \mathbb{R}^d$,*

$$< b(y)-b(x), y-x > \leq -K|y-x|^2, \tag{3.2}$$

where $\langle .,. \rangle$ denotes the scalar product in $\mathbb{R}^d$. Furthermore, b is locally Lipschitz and all of its derivatives have polynomial growth, i.e., for each multi-index α with $|\alpha| = m$, there exist positive constants γ_m and q_m such that for each $x \in \mathbb{R}^d$

$$|\partial_\alpha b(x)|^2 \leq \gamma_m(1+|x|^{q_m}). \tag{3.3}$$

Set $\xi := \max_{m\geq 1} q_m < \infty$.

• *The functions f and σ are C^∞, globally Lipschitz with Lipschitz constant $k_1 > 0$, and all of their derivatives of any order are bounded. Furthermore f has linear growth, i.e., for every $x \in \mathbb{R}^d$,*

$$|f(x)| \leq k_1(1+|x|). \tag{3.4}$$

Hypothesis (3.1) yields to the following useful inequalities

$$\langle b(a)+f(a), a\rangle \vee |\sigma(a)|^2 \leq \alpha + \beta|a|^2 \qquad \forall a \in \mathbb{R}^d, \tag{3.5}$$

where

$$\alpha := \frac{1}{2}|b(0)|^2 + k_1^2 \vee 2|\sigma(0)|^2, \quad \text{and} \quad \beta := (-K+1+k_1^2) \vee 2k_1^2, \tag{3.6}$$

and

$$\langle \nabla b(x)y, y\rangle \leq -K|y|^2 \qquad \forall x, y \in \mathbb{R}^d. \tag{3.7}$$

It is well-known that by inequality (3.5), the SDE (3.1) has a strong solution $\{X_t\}$ (see e.g., [12] and [13]). The uniqueness of the solution is obtained by using Itô's formula and Gronwall's inequality (Lemma 3.2). We will show that this solution is in $\mathbb{D}^\infty$. To this end, we first show that $X_t \in L^P(\Omega)$ and does not blow up in finite time. Then we construct an almost everywhere convergent sequence of processes X_t^n whose limit is X_t and has uniformly bounded Malliavin derivatives of any order with respect to n.
For each $n \geq 1$, define the stopping time τ_n by

$$\tau_n := inf\{t \geq 0\ ;\ |X_t| \geq n^\xi\}.$$

Lemma 3.2. *For each $t \in [0,T]$ and every integer $p > 1$, the strong solution X_t to SDE (3.1) is unique, belongs to $L^p(\Omega)$ and does not blow up in finite time.*

Proof. To proceed, first we use Fatou's lemma to show that X_t belongs to $L^p(\Omega)$ and does not blow up. Then we prove the uniqueness of X_t.

By the definition of $\mathcal{L}$ and (3.5) , we have

$$\begin{aligned}
\mathcal{L}|X_t|^p &= p|X_t|^{p-2}\langle X_t, b(X_t)+f(X_t)\rangle + \frac{p}{2}|X_t|^{p-2}|\sigma(X_t)|^2 \\
&\quad + \frac{p(p-2)}{2}|X_t|^{p-4}|\langle X_t, \sigma(X_t)\rangle|^2 \\
&\leq p|X_t|^{p-2}\langle X_t, b(X_t)+f(X_t)\rangle + \frac{p(p-1)}{2}|X_t|^{p-2}|\sigma(X_t)|^2 \\
&\leq \frac{p(p+1)}{2}\beta|X_t|^p + \frac{p(p+1)}{2}\alpha|X_t|^{p-2} \\
&=: \beta_p|X_t|^p + \alpha_p|X_t|^{p-2}. \qquad (3.8)
\end{aligned}$$

Applying Itô's formula and using (3.8),

$$\frac{d}{dt}\mathbb{E}\Big[|X_{t\wedge\tau_n}|^p\Big] = \mathbb{E}\Big[\mathcal{L}|X_{t\wedge\tau_n}|^p\Big] \leq \beta_p\mathbb{E}\Big[|X_{t\wedge\tau_n}|^p\Big] + \alpha_p\mathbb{E}\Big[|X_{t\wedge\tau_n}|^{p-2}\Big]. \qquad (3.9)$$

Setting $p=2$ and using Gronwall's inequality, we have

$$\mathbb{E}\Big[|X_{t\wedge\tau_n}|^2\Big] \leq |x_0|^2\alpha_2 exp\{\beta_2 T\}. \qquad (3.10)$$

From (3.10) we can deduce the following inequality

$$(\frac{n}{2}-1)^{\frac{1}{q_0}}P\Big(t \geq \tau_n\Big) \leq |x_0|^2\alpha_2 exp\{\beta_2 T\}.$$

Letting n tend to ∞, $lim_{n\to\infty}\tau_n = \infty$ almost surely, which implies that X_t does not blow up in the finite time interval $[0,T]$. Also, let n tend to infinity in (3.10) and use Fatou's lemma, then

$$\mathbb{E}(|X_t|^2) \leq \mathbb{E}\big(\liminf_{n\to\infty}|X_{t\wedge\tau_n}|^2\big) \leq \liminf_{n\to\infty}\mathbb{E}\big(|X_{t\wedge\tau_n}|^2\big) \leq |x_0|^2\alpha_2 exp\{\beta_2 T\}.$$

Finally by (3.9) and induction on p we conclude that $X_t \in L^p(\Omega)$.

To prove uniqueness, we assume that the SDE (3.1) has two strong solutions X_t and Y_t. Since $X_t, Y_t \in L^2(\Omega)$, applying Itô's formula we have

$$\begin{aligned}
\frac{d}{dt}\mathbb{E}\Big[|X_t - Y_t|^2\Big] &= 2\mathbb{E}\Big[\langle X_t - Y_t, b(X_t) - b(Y_t)\rangle\Big] \\
&\quad + 2\mathbb{E}\Big[\langle X_t - Y_t, f(X_t) - f(Y_t)\rangle\Big] \\
&\quad + \mathbb{E}\Big[|\sigma(X_t) - \sigma(Y_t)|^2\Big]
\end{aligned}$$

From which by (3.2) and the Lipschitz property of σ and f we derive

$$\frac{d}{dt}\mathbb{E}\Big[|X_t - Y_t|^2\Big] \leq (-2K + 2k_1)\mathbb{E}\Big[|X_t - Y_t|^2\Big].$$

By Gronwall's inequality which is proved in [8, Lemma 1.1] we conclude that $\mathbb{E}\Big[|X_t - Y_t|^2\Big] = 0$. So that

$$P\Big(|X_t - Y_t| = 0 \quad \text{for all } t \in \mathbb{Q}\cap[0,T]\Big) = 1,$$

where $\mathbb{Q}$ denotes the set of rational numbers. Since $t \longrightarrow |X_t - Y_t|$ is continuous, then

$$P\Big(|X_t - Y_t| = 0 \quad \text{for all } t \in [0,T]\Big) = 1,$$

and uniqueness is proved. $\square$

4. **Approximation of the solution**

In this section we will show that there exists a sequence X_t^n converging to the unique strong solution X_t of the SDE (3.1), and the moments of DX_t^n are uniformly bounded with respect to n and t. This way we can use Lemma 1.2.3 in [15] to deduce the Malliavin differentiability of X_t and show that $X_t \in \mathbb{D}^\infty$. Here we show how to construct this sequence.

Lemma 4.1. *There exist smooth functions $\phi_n : \mathbb{R}^d \longrightarrow \mathbb{R}$ with compact support such that $\phi_n(x) = 1$ on $B_{n^\xi}(0)$, $\phi_n(x) = 0$ outside $B_{2n^\xi}(0)$ (ξ is defined in Hypothesis 3.1) and for each multi-index L with $|L| = l \geq 1$,*

$$\sup_{n\geq 1, x\in\mathbb{R}^d} \Big(\|\partial_{L}\phi_n\| + |b(x)\partial_L\phi_n(x)\rangle|\Big) \leq M_l \tag{4.1}$$

for some $M_l > 0$.

Proof. This proof is motivated by Berhanu in [4, Theorem 2.9]. Assume that $U = B_{r_1}(0)$ and $V = B_{r_2}(0)$ are two sets in $\mathbb{R}^d$ with distance $a := r_2 - r_1 > 0$. For $0 \leq \epsilon \leq a$, define $U_\epsilon = \{x; d(x,U) < \epsilon\}$. Then $U_\epsilon = \bigcup_{x\in U} B_\epsilon(x)$ and $U \subseteq U_\epsilon \subseteq V$. Fix ϵ such that $0 < 2\epsilon \leq a$ and let $h^\epsilon(x)$ be the characteristic function of U_ϵ. Let $\psi \in C_0^\infty(\mathbb{R}^d)$ with *supp* $\psi \subseteq B_1(0)$ and $\int \psi(x)dx = 1$. Set $\psi_\epsilon(x) = \frac{1}{\epsilon^d}\psi(\frac{x}{\epsilon})$. Consider now the convolution functions function $\psi_\epsilon \star h^\epsilon$ for $0 < 2\epsilon < d$. Since *supp* $\psi_\epsilon \subseteq B_\epsilon(0)$, then $\psi_\epsilon \star h^\epsilon = 1$ on U and $\psi_\epsilon \star h^\epsilon = 0$ outside $U_{2\epsilon}$. Note that for each multi-index α,

$$\begin{aligned}\partial_\alpha(\psi_\epsilon \star h^\epsilon)(x) &= \int \partial_\alpha(\psi_\epsilon(y))h^\epsilon(x-y)dy = \frac{1}{\epsilon^{d+|\alpha|}}\int (\partial_\alpha\psi)(\frac{y}{\epsilon})h^\epsilon(x-y)dy \\ &= \frac{1}{\epsilon^{|\alpha|}}\int (\partial_\alpha\psi)(z)h^\epsilon(x-\epsilon z)dz \leq \parallel \psi \parallel_\infty \frac{1}{\epsilon^{|\alpha|}} \end{aligned} \tag{4.2}$$

Now, for every $n \geq 1$ consider $U = B_{n^\xi}(0)$, $V = B_{2n^\xi}(0)$ and $\epsilon = n^\xi$. Then there exist functions $\phi_n(x) := \psi_\epsilon \star h^\epsilon(x)$ such that $\phi_n(x) = 1$ on U and $\phi_n(x) = 0$ outside V. Since *supp* $\phi_n(x) \subseteq B_{2n^\xi}(0)$, by (4.2) and (3.3) for each multi-index α with $|\alpha| = c \geq 1$, we have

$$|b(x)\partial_\alpha \phi_n(x)| \leq |b(x)\chi_{|x|\leq 2n^\xi}| \parallel \psi \parallel_\infty \frac{1}{n^{\xi|\alpha|}}$$
$$\leq \gamma_c(1+2^\xi n^\xi) \parallel \psi \parallel_\infty \frac{1}{n^{\xi|\alpha|}} \leq 2^{\xi+1}\gamma_c \parallel \psi \parallel_\infty,$$

and

$$|\partial_\alpha \phi_n(x)| \leq \parallel \psi \parallel_\infty .$$

□

Now, for ϕ_n (from Lemma 4.1), set

$$b_n(x) := \phi_n(x)b(x),$$

for every $x \in \mathbb{R}^d$ and $n > 0$. Then b_ns are globally Lipschitz and continuously differentiable. By (3.3) for each $x \in \mathbb{R}^d$ and each multi-index L with $|L| = l$, there exist positive constants Γ_l and p_l such that

$$|\partial_L b_n(x)|^2 \leq \Gamma_l(1+|x|^{p_l}). \tag{4.3}$$

Now by Proposition 2.1, for every $n \geq 1$, there exists a strong solution to the following SDE, which is unique, is in $\mathbb{D}^\infty$ and satisfies

$$X_t^n = x_0 + \int_0^t [b_n(X_s^n) + f(X_s^n)]ds + \int_0^t \sigma(X_s^n)dW_s. \tag{4.4}$$

We use $\mathcal{L}_n$ to show the infinitesimal operators associated to SDEs (4.4):

$$\mathcal{L}_n = \frac{1}{2}\sum_{i,j=1}^d (\sigma\sigma^*)_j^i(x)\partial_i\partial_j + \sum_{i=1}^d [b_n^i(x) + f^i(x)]\partial_i.$$

We will show that the sequence X_t^n converges to the unique strong solution X_t to SDE (3.1).

Lemma 4.2. *For each $t \in [0,T]$ and every integer $p > 1$, the sequence X_t^n converges to X_t in $L^p(\Omega)$.*

Proof. To proceed, first we prove the almost sure convergence of X_t^n to X_t. Then by showing the uniform integrability of X_t^n we will conclude.
Let X^{τ_n} denotes X stopped at τ_n. By the choice of $\phi_n(.)$, it follows that $X_t^{\tau_{2n}} = X_t^{\tau_n}$ for all $t \leq \tau_n$. So, for fixed $t \in [0,T]$, letting n tend to ∞, $lim_{n\to\infty} X_t^n = lim_{n\to\infty} X_t^{\tau_n} = X_t$ a.s.
Now, we are going to prove that the sequence X_t^n is uniformly integrable. In fact, we will show that for every integer $p > 1$,

$$\sup_{n\geq 1} \sup_{0\leq t\leq T} \mathbb{E}\Big[|X_t^n|^p\Big] \leq c_p. \tag{4.5}$$

By the definition of $\mathcal{L}_n$, we have

$$\begin{aligned}
\mathcal{L}_n|X_t^n - x_0|^p &= p|X_t^n - x_0|^{p-2}\langle X_t^n - x_0, b_n(X_t^n) + f(X_t^n)\rangle \\
&+ \frac{p}{2}|X_t^n - x_0|^{p-2}|\sigma(X_t^n)|^2 \\
&+ \frac{p(p-2)}{2}|X_t^n - x_0|^{p-4}|\langle X_t^n - x_0, \sigma(X_t^n)\rangle|^2 \\
&= p|X_t^n - x_0|^{p-2}\langle X_t^n - x_0, b_n(X_t^n) - b(x_0)\phi_n(X_t^n)\rangle \\
&+ p|X_t^n - x_0|^{p-2}\langle X_t^n - x_0, b(x_0)\phi_n(X_t^n) + f(X_t^n)\rangle \\
&+ \frac{p}{2}|X_t^n - x_0|^{p-2}|\sigma(X_t^n)|^2 \\
&+ \frac{p(p-2)}{2}|X_t^n - x_0|^{p-4}|\langle X_t^n - x_0, \sigma(X_t^n)\rangle|^2.
\end{aligned}$$

Using the inequality $-ac \leq a^2/2 + c^2/2$ for $a = K$ and $c = \phi_n(X_t^n)$ (note that $\phi_n(.) \leq 1$), by (3.2) and (3.5), we have

$$\begin{aligned}
\mathcal{L}_n|X_t^n - x_0|^p &\leq -Kp|X_t^n - x_0|^p\phi_n(X_t^n) \\
&+ p|X_t^n - x_0|^{p-2}\langle X_t^n - x_0, b(x_0)\phi_n(X_t^n) + f(X_t^n)\rangle\Big] \\
&+ \frac{p(p-1)}{2}|X_t^n - x_0|^{p-2}|\sigma(X_t^n)|^2 \\
&\leq \frac{K^2+1}{2}p|X_t^n - x_0|^p\phi_n(X_t^n) \\
&+ p|X_t^n - x_0|^{p-2}\Big[\frac{1}{2}|X_t^n - x_0|^2 + \Big(|b(x_0)|^2 + |f(X_t^n)|^2\Big)\Big] \\
&+ \frac{p(p-1)}{2}|X_t^n - x_0|^{p-2}|\sigma(X_t^n)|^2 \\
&\leq \alpha_p|X_t^n - x_0|^p + \beta_p|X_t^n - x_0|^{p-2}, \qquad (4.6)
\end{aligned}$$

for some constants α_p, $\beta_p > 0$. Using Itô's formula, we have

$$\begin{aligned}
\frac{d}{dt}\mathbb{E}\Big[|X_t^n - x_0|^p\Big] &= \mathbb{E}\Big[\mathcal{L}_n(|X_t^n - x_0|^p)\Big] \\
&\leq \alpha_p\mathbb{E}\Big[|X_t^n - x_0|^p\Big] + \beta_p\mathbb{E}\Big[|X_t^n - x_0|^{p-2}\Big] \\
&\leq \alpha_p\mathbb{E}\Big[|X_t^n - x_0|^p\Big] + \beta_p\Big(\mathbb{E}\Big[|X_t^n - x_0|^{p-1}\Big]\Big)^{1-\frac{1}{p-1}}.
\end{aligned}$$

Applying Gronwall's inequality for $p = 2$ and then using mathematical induction on p, (4.5) will be proved for every integer $p \geq 2$.

Now the almost sure convergence of X_t^n to X_t and inequality (4.5) complete the proof of lemma. □

We prove the uniform boundedness of the moments of DX_t^n in the next section.

5. **Weak differentiability in the Wiener space**

In this section, we will mainly use Lemma 1.2.3 from [15] to deduce Malliavin differentiability of the solution to (3.1). Then we show that $X_t \in \mathbb{D}^\infty$. Note that by Proposition 2.1, the solutions to SDEs (4.4) are in $\mathbb{D}^\infty$.

Lemma 5.1. *Assume that Hypothesis* 3.1 *holds, then the unique strong solution of SDE* (3.1) *is in* $\mathbb{D}^{1,p}$ *for every integer* $p > 1$*. Moreover, for* $r \le t$

$$
\begin{aligned}
D_r X_t^i = \sigma^i(X_r) &+ \int_r^t [\nabla b^i(X_s) + \nabla f^i(X_s)].D_r X_s ds \\
&+ \int_r^t \nabla \sigma_l^i(X_s).D_r X_s dW_s^l
\end{aligned}
$$

and for $r > t$*,* $D_r X_t^i = 0$*, where* $\sigma_l(X_s)$ *is the l-th column of* $\sigma(X_s)$ *and* $u.C$ *denotes the product* C^*u *of vector* u *and matrix* C*.*

Proof. By Proposition 2.1 we know that for every $r \le t$ and $1 \le i \le d$

$$
\begin{aligned}
D_r(X_t^n)^i = \sigma^i(X_r^n) &+ \int_r^t [\nabla b_n^i(X_s^n) + \nabla f^i(X_s^n)].D_r X_s^n ds \\
&+ \int_r^t \nabla \sigma_l^i(X_s^n).D_r X_s^n dW_s^l,
\end{aligned}
$$

and for every $r > t$, $D_r(X_t^n)^i = 0$.
Now by Lemma 1.2.3 in [15], it is sufficient to show that

$$
\sup_{n\ge 1} \sup_{0\le t\le T} \mathbb{E}\Big[\|DX_t^n\|_H^p\Big] \le c_p. \tag{5.1}
$$

To this end, note that for every $1 \le i \le d$ by Itô's formula

$$
\mathbb{E}\Big[|D_r(X_t^n)^i|^p\Big] = \mathbb{E}\Big[|\sigma^i(X_r^n)|^p\Big] + \mathbb{E}\Big[\int_r^t \mathcal{G}_n\Big(|D_r(X_s^n)^i|^p\Big) ds\Big] + \mathbb{E}\Big[M_t^n\Big], \tag{5.2}
$$

where

$$
\begin{aligned}
\mathcal{G}_n\Big(|D_r(X_s^n)^i|^p\Big) = p|D_r(X_s^n)^i|^{p-2} S_{i,s} &+ \frac{p}{2}|D_r(X_s^n)^i|^{p-4} U_{i,s} \\
&+ p|D_r(X_s^n)^i|^{p-2}\langle D_r(X_s^n)^i, \nabla f^i(X_s^n).D_r X_s^n\rangle
\end{aligned}
$$

in which

$$
\begin{aligned}
S_{i,s} &:= \langle D_r(X_s^n)^i, \nabla b_n^i(X_s^n).D_r X_s^n\rangle, \\
U_{i,s} &:= |D_r(X_s^n)^i|^2 |\nabla \sigma_l^i(X_s^n).D_r X_s^n|^2 \\
&\qquad + (p-2)|\langle D_r(X_s^n)^i, \nabla \sigma_l^i(X_s^n).D_r X_s^n\rangle|^2,
\end{aligned}
$$

and

$$
M_t^n := \int_r^t p|D_r(X_s^n)^i|^{p-2}\langle D_r(X_s^n)^i, \nabla \sigma_l^i(X_s^n).D_r X_s^n dW_s^l\rangle.
$$

Note that by Proposition 2.1, M_t^n is a local martingale and thus $\mathbb{E}[M_t^n] = 0$. Since σ and f have bounded derivatives, there exists some $\gamma > 0$ such that

$$\frac{p}{2}|D_r(X_s^n)^i|^{p-4}U_{i,s} \le \gamma\frac{p(p-1)}{2}|D_r(X_s^n)^i|^{p-2}|D_rX_s^n|^2, \tag{5.3}$$

and

$$\begin{aligned} p|D_r(X_s^n)^i|^{p-2}\langle D_r(X_s^n)^i, \nabla f^i(X_s^n).D_rX_s^n\rangle \le \\ \frac{p}{2}|D_r(X_s^n)^i|^p + \gamma\frac{p}{2}|D_r(X_s^n)^i|^{p-2}|D_rX_s^n|^2. \end{aligned} \tag{5.4}$$

By using (3.7) and (4.1) for every $0 \le t \le T$, we have

$$\begin{aligned} \sum_{i=1}^d S_{i,s} &= \sum_{j=1}^d < \nabla b_n(X_s^n)D_r^jX_s^n, D_r^jX_s^n > \\ &= \sum_{j=1}^d \phi_n(X_s^n)\langle \nabla b(X_s^n)D_r^jX_s^n, D_r^jX_s^n\rangle \\ &+ \sum_{j=1}^d \langle\langle b(X_s^n), \nabla\phi_n(X_s^n)\rangle D_r^jX_s^n, D_r^jX_s^n\rangle \\ &\le (-K\phi_n(X_s^n) + M_1)\sum_{j=1}^d |D_r^jX_s^n|^2 \le M_1\sum_{j=1}^d |D_r^jX_s^n|^2 \end{aligned} \tag{5.5}$$

where $D_r^jX_t^n$ is the j-th column of DX_t^n. As for every $Y = (Y^1, \cdots, Y^d) \in \mathbb{R}^d$ and for every $1 \le i \le d$

$$A: \quad |Y^i|^p \le |Y|^p, \qquad\qquad B: \quad |Y|^p \le d^{\frac{p}{2}-1}\sum_i^d |Y^i|^p. \tag{5.6}$$

Substituting (5.5), (5.3) and (5.4) in (5.2) and taking summation on i we derive:

$$\begin{aligned}
\mathbb{E}\Big[|D_r X_t^n|^p\Big] &\le d^{\frac{p}{2}-1}\sum_{i=1}^{d}\mathbb{E}\Big[|D_r(X_t^n)^i|^p\Big]\\
&\le d^{\frac{p}{2}-1}\sum_{i=1}^{d}\mathbb{E}\Big[|\sigma^i(X_r^n)|^p\Big]\\
&+ d^{\frac{p}{2}-1}pdM_1\sum_{i=1}^{d}\int_r^t \mathbb{E}\Big[|D_r(X_s^n)^i|^{p-2}|D_r X_s^n|^2\Big]ds\\
&+ d^{\frac{p}{2}-1}\sum_{i=1}^{d}\int_r^t \mathbb{E}\Big[\frac{p}{2}|D_r(X_s^n)^i|^p\Big]ds\\
&+ d^{\frac{p}{2}-1}\sum_{i=1}^{d}\gamma\frac{p}{2}\int_r^t \mathbb{E}\Big[|D_r(X_s^n)^i|^{p-2}|D_r X_s^n|^2\Big]ds\\
&+ d^{\frac{p}{2}-1}\sum_{i=1}^{d}\gamma\frac{p(p-1)}{2}\int_r^t \mathbb{E}\Big[|D_r(X_s^n)^i|^{p-2}|D_r X_s^n|^2\Big]ds
\end{aligned}$$

Now by part A of (5.6), we can find a constant $\alpha'_p > 0$ such that

$$\mathbb{E}\Big[|D_r X_t^n|^p\Big] \le d^{\frac{p}{2}-1}\sum_{i=1}^{d}\mathbb{E}\Big[|\sigma^i(X_r^n)|^p\Big] + \alpha'_p\int_r^t \mathbb{E}\Big[|D_r X_s^n|^p\Big]ds.$$

Using Gronwall's inequality, we have

$$\mathbb{E}\Big[|D_r X_t^n|^p\Big] \le d^{\frac{p}{2}-1}\sum_{i=1}^{d}\mathbb{E}\Big[|\sigma^i(X_r^n)|^p\Big]exp\{\alpha'_p T\}.$$

From which by the Lipschitz property of σ and inequality (4.5) the result follows. $\square$

Here we are going to prove higher order differentiability of X_t. To avoid complexity, we will only show the second order differentiability. Higher order differentiability could be proved similarly. For every real-valued function f and random variables F and G, we set $\triangle f(x)FG := \partial_i\partial_j f(x)F^iG^j$ and $D^{j,k}_{r,\tau}F = D^k_\tau D^j_r F$.

Lemma 5.2. *Assuming Hypothesis 3.1, for every $p > 1$ the unique strong solution of SDE* (3.1) *is in $\mathbb{D}^{2,p}$ and*

$$
\begin{aligned}
D^{j,k}_{r,\tau}X^i_t &= A^{ij}_{\tau,r}\\
&\quad + \int_{\tau\vee r}^t \Big[\langle \nabla\sigma^i_l(X_s), D^{j,k}_{r,\tau}X_s\rangle + \triangle\sigma^i_l(X_s)D^k_\tau X_s D^j_r X_s\Big]dW^l_s\\
&\quad + \int_{\tau\vee r}^t \langle \nabla b^i(X_s) + \nabla f^i(X_s), D^{j,k}_{r,\tau}X_s\rangle ds\\
&\quad + \int_{\tau\vee r}^t \Big[\triangle b^i(X_s) + \triangle f^i(X_s)\Big] D^k_\tau X_s D^j_r X_s ds,
\end{aligned}
$$

where

$$
A^{ij}_{\tau,r} = \langle \nabla\sigma^i_j(X_r), D^k_\tau X_r\rangle + \sum_{l=1}^d \langle \nabla\sigma^i_l(X_\tau), D^j_r X_\tau\rangle,
$$

and $D_\tau X_r = 0$ *for* $\tau > r$, *and* $D_r X_\tau = 0$ *for* $\tau < r$.

Proof. Since $X^n_t \in \mathbb{D}^\infty$, by Proposition 2.1 for $\tau_0 := \tau \vee r$ we have

$$
\begin{aligned}
D^{j,k}_{r,\tau}(X^n_t)^i &= A^{ij}_{n,\tau,r}\\
&\quad + \int_{\tau_0}^t \Big[\langle \nabla\sigma^i_l(X^n_s), D^{j,k}_{r,\tau}X^n_s\rangle + \triangle\sigma^i_l(X^n_s)D^k_\tau X^n_s D^j_r X^n_s\Big]dW^l_s\\
&\quad + \int_{\tau_0}^t \langle \nabla b^i_n(X^n_s) + \nabla f^i(X_s), D^{j,k}_{r,\tau}X^n_s\rangle ds\\
&\quad + \int_{\tau_0}^t \Big[\triangle b^i_n(X^n_s) + \triangle f^i(X^n_s)\Big] D^k_\tau X^n_s D^j_r X^n_s ds,
\end{aligned}
$$

where

$$
A^{ij}_{n,\tau,r} = \langle \nabla\sigma^i_j(X^n_r), D^k_\tau X^n_r\rangle + \sum_{l=1}^d \langle \nabla\sigma^i_l(X^n_\tau), D^j_r X^n_\tau\rangle,
$$

and $D_\tau X^n_r = 0$ for $\tau > r$. Similarly we have $D_r X^n_\tau = 0$ for $\tau < r$. By Lemma 1.2.3 in [15], now it is sufficient to find some $c_2 > 0$ such that

$$
\sup_n \mathbb{E}\Big[\|D^{j,k}X^n_t\|^p_{H\otimes H}\Big] < c_2. \tag{5.7}
$$

By Itô's formula, for every $1 \le i \le d$ we have

$$
\mathbb{E}\Big[|D^{j,k}_{r,\tau}(X^n_t)^i|^p\Big] = \mathbb{E}\Big[|A^{ij}_{n,\tau,r}|^p\Big] + \mathbb{E}\Big[\int_\tau^t \mathcal{G}^{ij}_n\Big(|D^{j,k}_{r,\tau}(X^n_s)^i|^p\Big)ds\Big] + \mathbb{E}\Big[M^{ij}_n(t))\Big], \tag{5.8}
$$

where

$$
\begin{aligned}
\mathcal{G}^{ij}_n\Big(|D^{j,k}_{r,\tau}(X^n_s)^i|^p\Big) &= p|D^{j,k}_{r,\tau}(X^n_s)^i|^{p-2}I_1 + \frac{p}{2}|D^{j,k}_{r,\tau}(X^n_s)^i|^{p-2}\sum_{l=1}^d I_2(l)\\
&\quad + \frac{p(p-2)}{2}|D^{j,k}_{r,\tau}(X^n_s)^i|^{p-4}I_3,
\end{aligned}
$$

in which

$$
\begin{aligned}
I_1 :=& D^{j,k}_{r,\tau}(X^n_s)^i\Big(\langle \nabla b^i_n(X^n_s) + \nabla f^i(X^n_s), D^{j,k}_{r,\tau}X^n_s\rangle\Big)\\
&\qquad\qquad + \Big[\triangle b^i_n(X^n_s) + \triangle f^i(X^n_s)\Big] D^k_\tau X^n_s D^j_r X^n_s,\\
I_2(l) :=& \Big[|\triangle\sigma^i_l(X^n_s) D^k_\tau X^n_s D^j_r X^n_s| + |\langle\nabla\sigma^i_l(X^n_s), D^{j,k}_{r,\tau}X^n_s\rangle|\Big]^2,\\
I_3 :=& |D^{j,k}_{r,\tau}(X^n_s)^i\Big(\triangle\sigma^i_l(X^n_s) D^k_\tau X^n_s D^j_r X^n_s + \langle\nabla\sigma^i_l(X^n_s), D^{j,k}_{r,\tau}X^n_s\rangle\Big)|^2,
\end{aligned}
$$

and

$$
M^{ij}_n(t) := \int_r^t p|D^{j,k}_{r,\tau}(X^n_s)^i|^{p-2}\langle D^{j,k}_{r,\tau}(X^n_s)^i, I_2(l)dW^l_s\rangle.
$$

Note that by Proposition 2.1, $M^{ij}_n(t)$ is a local martingale and thus $\mathbb{E}[M^{ij}_n(t)] = 0$.

Now, we are going to find appropriate upper bounds for $I_1, I_2(l)$ and I_3. As σ has bounded derivatives, we can find some $\gamma'_1 > 0$ such that

$$
\begin{aligned}
&\frac{p}{2}|D^{j,k}_{r,\tau}(X^n_s)^i|^{p-2}\sum_{l=1}^d I_2(l) + \frac{p(p-2)}{2}|D^{j,k}_{r,\tau}X^n_s|^{p-4} I_3 \leq\\
&\gamma'_1\frac{p(p-1)}{2}\Big(|D^{j,k}_{r,\tau}(X^n_s)^i|^{p-2}|D^{j,k}_{r,\tau}X^n_s|^2 + |D^{j,k}_{r,\tau}(X^n_s)^i|^{p-2}|D^j_r X^n_s|^2|D^k_\tau X^n_s|^2\Big).
\end{aligned}\tag{5.9}
$$

Also by the boundedness of f and the derivatives of σ, the polynomial growth of the derivatives of b and (4.3), there exist some $\gamma'_2 > 0$ and $q > 0$ such that

$$
\begin{aligned}
p|D^{j,k}_{r,\tau}(X^n_s)^i|^{p-2} I_1 &= p|D^{j,k}_{r,\tau}(X^n_s)^i|^{p-2}J_1 + p|D^{j,k}_{r,\tau}(X^n_s)^i|^{p-2}J_2\\
&\quad + p|D^{j,k}_{r,\tau}(X^n_s)^i|^{p-2}D^{j,k}_{r,\tau}(X^n_s)^i\langle\nabla f^i(X^n_s), D^{j,k}_{r,\tau}X^n_s\rangle\\
&\leq p|D^{j,k}_{r,\tau}(X^n_s)^i|^{p-2}J_1\\
&\quad + \gamma'_2 p|D^{j,k}_{r,\tau}(X^n_s)^i|^{p-2}|D^k_\tau X^n_s|^2|D^j_r X^n_s|^2(1+|X^n_s|^{p_2})^2\\
&\quad + p\gamma'_2|D^{j,k}_{r,\tau}(X^n_s)^i|^p + p\gamma'_2|D^{j,k}_{r,\tau}(X^n_s)^i|^{p-2}|D^{j,k}_{r,\tau}(X^n_s)^i|^2,
\end{aligned}\tag{5.10}
$$

where

$$
J_1 := D^{j,k}_{r,\tau}(X^n_s)^i\langle\nabla b^i_n(X^n_s), D^{j,k}_{r,\tau}X^n_s\rangle,
$$

and

$$
J_2 := D^{j,k}_{r,\tau}(X^n_s)^i\Big(\Big[\triangle b^i_n(X^n_s) + \triangle f^i(X^n_s)\Big]D^k_\tau X^n_s D^j_r X^n_s\Big).
$$

By using (3.7) and (4.1) for every $0 \leq t \leq T$, we have

$$\sum_{i=1}^{d} J_1 = \langle \nabla b_n(X_s^n) D_{r,\tau}^{j,k} X_s^n, D_{r,\tau}^{j,k} X_s^n \rangle = \phi_n(X_s^n) \langle \nabla b(X_s^n) D_{r,\tau}^{j,k} X_s^n, D_{r,\tau}^{j,k} X_s^n \rangle$$
$$+ \langle \langle b(X_s^n), \nabla \phi_n(X_s^n) \rangle D_{r,\tau}^{j,k} X_s^n, D_{r,\tau}^{j,k} X_s^n \rangle$$
$$\le (-K\phi_n(X_s^n) + M_1) |D_{r,\tau}^{j,k} X_s^n|^2 \le M_1 |D_{r,\tau}^{j,k} X_s^n|^2. \tag{5.11}$$

Now, substitute (5.10) and (5.9) in (5.8), sum up on i and then use (5.11) and part A of (5.6) to derive

$$\sum_{i=1}^{d} \mathbb{E}\Big[|D_{r,\tau}^{j,k}(X_t^n)^i|^p\Big] =$$
$$\sum_{i=1}^{d} \mathbb{E}\Big[|A_{n,\tau,r}^{ij}|^p\Big] + p(M_1 + 2d\gamma_2' + d\gamma_1' \frac{p(p-1)}{2}) \int_{\tau_0}^{t} \mathbb{E}\Big[|D_{r,\tau}^{j,k} X_s^n|^p\Big] ds$$
$$+ \sum_{i=1}^{d} \gamma_2' p \int_{\tau_0}^{t} \mathbb{E}\Big[|D_{r,\tau}^{j,k}(X_s^n)^i|^{p-2} |D_\tau^k X_s^n|^2 |D_r^j X_s^n|^2 (1 + |X_s^n|^{p_2})^2\Big] ds$$
$$+ \sum_{i=1}^{d} \gamma_1' \frac{p(p-1)}{2} \int_{\tau_0}^{t} \mathbb{E}\Big[|D_{r,\tau}^{j,k}(X_s^n)^i|^{p-2} |D_r^j X_s^n|^2 |D_\tau^k X_s^n|^2\Big] ds. \tag{5.12}$$

To bound the terms in the right hand side of the above inequality, we need the following version of the Young's inequality. For $p \ge 2$ and for all a, c and $\triangle_1 > 0$ we have:

$$a^{p-2} c^2 \le \triangle_1^2 \frac{p-2}{p} a^p + \frac{2}{p \triangle_1^{p-2}} c^p. \tag{5.13}$$

Using (5.13) with $\triangle_1 = 1$ we find some bounds for the last four terms in (5.12) which depend only on $\int_{\tau_0}^{t} \mathbb{E}\Big[|D_{r,\tau}^{j,k} X_s^n|^p\Big] ds$ and some terms which could be bounded by a constant. For the last term in (5.12) we have

$$\sum_{i=1}^{d} \gamma_1' \frac{p(p-1)}{2} \int_{\tau_0}^{t} \mathbb{E}\Big[|D_{r,\tau}^{j,k}(X_s^n)^i|^{p-2} |D_r^j X_s^n|^2 |D_\tau^k X_s^n|^2\Big] ds \le$$
$$d\gamma_1' \int_{\tau_0}^{t} \Big(\frac{(p-1)(p-2)}{2} \mathbb{E}\Big[|D_{r,\tau}^{j,k} X_s^n|^p\Big] + (p-1) \mathbb{E}\Big[|D_r^j X_s^n|^p |D_\tau^k X_s^n|^p\Big] \Big) ds,$$

and for the third term in (5.12) we have

$$\sum_{i=1}^{d} \gamma_2' p \int_{\tau_0}^{t} \mathbb{E}\Big[|D_{r,\tau}^{j,k}(X_s^n)^i|^{p-2} |D_\tau^k X_s^n|^2 |D_r^j X_s^n|^2 (1 + |X_s^n|^{p_2})^2\Big] ds \le$$
$$d\gamma_2' \int_{\tau_0}^{t} \Big((p-2) \mathbb{E}\Big[|D_{r,\tau}^{j,k} X_s^n|^p\Big] + 2\mathbb{E}\Big[|D_\tau^k X_s^n|^p |D_r^j X_s^n|^p (1 + |X_s^n|^{p_2})^p\Big] \Big) ds.$$

Substituting these bounds in the right hand side of (5.8) and using (4.5), (5.1) and part B of (5.6), we can find some positive constants $c_1(p)$ and $c_2(p)$ such that

$$\mathbb{E}\Big[|D^{j,k}_{r,\tau}X^n_s|^p\Big] \leq d^{\frac{p}{2}-1}\sum_{i=1}^{d}\mathbb{E}\Big[|A^{ij}_{n,\tau,r}|^p\Big] + c_2(p) + c_1(p)\int_{\tau_0}^{t}\mathbb{E}\Big[|D^{j,k}_{r,\tau}X^n_s|^p\Big]ds.$$

Now, from (5.1), (4.5) and the definition of $A^{ij}_{n,\tau,r}$ (in which we have used the boundedness of the derivatives of σ) and Gronwall's inequality, (5.7) will be derived. □

In the same way, one can easily show that for every multi-index α

$$\sup_n \mathbb{E}(\|D^\alpha X^n_t\|^p_{H^{\otimes\alpha}}) < \infty \tag{5.14}$$

and then by Lemma 1.2.3 in [15] deduce the following theorem.

Theorem 5.3. *The SDE (3.1) has a unique strong solution in $\mathbb{D}^\infty$.*

Here we give an example that will help us figure all this out.

Example 5.4. *Consider the following scalar SDE;*

$$dX_t = (-X_t^3 + X_t)dt + (sin^2(X_t)+1)dW_t, \qquad X_0 = 0. \tag{5.15}$$

Let $b(x) := -x^3$, $f(x) := x$ and $\sigma(x) := sin^2(x)+1$, then Hypothesis 3.1 is satisfied. Define $b_n(x) = b(x)\phi_n(x)$ as Lemma 4.1 and consider the following SDEs;

$$dX^n_t = -(X^n_t)^3\phi_n(X^n_t)dt + (sin^2(X^n_t)+1)dW_t, \qquad X_0 = 0.$$

By Theorem 5.3, SDE (5.15) has a unique strong solution in $\mathbb{D}^\infty$ and for every $r \leq t$

$$dDX_t = \Big(-3(X^n_t)^2 + 1\Big)DX_t dt + 2sin(X_t)cos(X_t)DX_t dW_t.$$

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(Mahdieh Tahmasebi) Department of Applied Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, P.O. Box 14115-134, Tehran, Iran
E-mail address: `tahmasebi@modares.ac.ir`

(Shiva Zamani) Graduate School of Management and Economics, Sharif University of Technology, P.O. Box 11155-9415, Tehran, Iran
E-mail address: `zamani@sharif.ir`