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SOME NUMERICAL RADIUS INEQUALITIES WITH POSITIVE DEFINITE FUNCTIONS

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Dedicated to Professor Abbas Salemi

Abstract. Using several examples of positive definite functions, some inequalities for the numerical radius of matrices are investigated. Also, some open problems are stated.
Keywords: Inequalities, Numerical radius, Positive definite function.
MSC(2010): Primary: 15A60; Secondary: 15A16, 47A63, 47A12.

1. Introduction and preliminaries

Let $\mathbb{M}_n$ be the algebra of all $n \times n$ complex matrices. The numerical range of $A \in \mathbb{M}_n$, denoted by $W(A)$, is defined as the subset of complex plane given by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}.$$

It is known that $W(A)$ is a compact and convex subset of $\mathbb{C}$. It can be viewed as a picture of A containing useful information of A. Even if the matrix A is not known explicitly, $W(A)$ would allow one to see many properties of the matrix. For example, the numerical range can be used to locate eigenvalues, deduced algebraic and analytic properties, help find dilations with simple structure, etc. For more information see [10] and [15, Chapter 1].
The numerical radius of $A \in \mathbb{M}_n$, denoted by $\omega(A)$, is defined as

$$\omega(A) = \max\{|\lambda| : \lambda \in W(A)\}.$$

The quantity $\omega(A)$ is useful in studying perturbations, convergence, stability, approximation problems, iterative methods, etc. For more information see [3, 8, 20]. In this paper, the symbol $|||.|||$ stands for unitarily invariant norm on the space of matrices. The spectral matrix norm $\|.\|$ (i.e., the matrix norm

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subordinate to the Euclidean vector norm) is an example of unitarily invariant norms; see [4] and [14] for more information.

It is well known that $\omega(.)$ is a vector norm on $\mathbb{M}_n$ but is not unitarily invariant norm. We recall the following results about the numerical radius of matrices which can be found in [10] (see also [15, Chapter 1]).

Lemma 1.1. *Let $A \in \mathbb{M}_n$ and k be a positive integer. Then the following assertions are true:*
(i) $\omega(U^*AU) = \omega(A)$, where $U \in \mathbb{M}_n$ is unitary;
(ii) If $A = A_1 \oplus \cdots \oplus A_k$, then $\omega(A) = \max\{\omega(A_1), \ldots, \omega(A_k)\}$;
(iii) $\frac{1}{2}\|A\| \leqslant \omega(A) \leqslant \|A\|$;
(iv) $\omega(A) = \|A\|$ if (but not only if) A is normal;
(v) $\omega(A^k) \leqslant \omega^k(A)$.

In $\mathbb{M}_n$, beside the usual matrix product, the entrywise product is quite important and interesting. The entrywise product of two matrices A and B is called the Schur (or Hadamard) product of A and B, and denoted by $A \circ B$. With this multiplication, $\mathbb{M}_n$ becomes a commutative algebra for which the matrix with all entries equal to one is the unit.
If $A = (a_{ij}) \in M_n$ is positive semidefinite, then for any matrix $X \in \mathbb{M}_n$ (see [2]), we have

$$\omega(A \circ X) \leqslant (\max_{1 \leqslant i \leqslant n} a_{ii})\, \omega(X). \tag{1.1}$$

Throughout the paper, we use the notation $A \geq 0$ to mean that A is positive semidefinite and $A > 0$ to mean it is positive definite. In [17–19], the matrix Young inequality for numerical radius and operator norm, even for the special case arithmetic geometric mean inequality, are investigated. In section 2 of this paper, using several examples of positive definite functions, we will consider some important inequalities for numerical radius; in particular for the Heinz inequality. In section 3, by using some results in [6, 7], we obtain some inequalities about the numerical radius of matrices, and we also give some open problems.

2. Main results

A complex valued function φ on $\mathbb{R}$ is said to be positive definite if for each positive integer n, the matrix $[\varphi(x_i - x_j)]$ is positive semidefinite for every choice of real numbers $x_1, \ldots, x_n$. Let I be any interval and $K(x, y)$ be a bounded continuous complex valued function on $I \times I$. We say that K is a positive definite kernel if

$$\int_I \int_I K(x, y) f(x) \overline{f(y)} dx dy \geqslant 0,$$

for every continuous function f on the interval I. We repeatedly use the connection between positive definite matrices, positive definite functions on $\mathbb{R}$ and positive definite kernels. Now, we state the following proposition.

Proposition 2.1. *[4, Exercise 5.1.6] A bounded continuous function $K(x,y)$ on $I \times I$ is a positive definite kernel if and only if for all choices of points $x_1, \dots, x_n$ in I, the $n \times n$ matrix $[K(x_i, x_j)]$ is positive semidefinite. Also, a bounded continuous function φ on $\mathbb{R}$ is positive definite if and only if the kernel $K(x,y) = \varphi(x-y)$ is positive definite.*

Let f be a function in $\mathrm{L}^1(\mathbb{R})$. The Fourier transform of f is the function $\hat{f}$ defined as

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-itx} dx.$$

By a known theorem of Bochner, if $f \in \mathrm{L}^1(\mathbb{R})$, then $\hat{f}$ is positive definite if and only if $f(t) \geqslant 0$ for almost all t; see, e.g. [12, p. 70]. So, we have the following lemma.

Lemma 2.2. *[6, 7] The following functions are positive definite:*

- *(i)* $\dfrac{\cosh(\alpha x)}{\cosh(x)}$, *for* $-1 < \alpha < 1$;
- *(ii)* $\dfrac{\sinh(\alpha x)}{\sinh(x)}$, *for* $0 < \alpha < 1$;
- *(iii)* $\dfrac{\sin(x)}{x}$;
- *(iv)* $\dfrac{x}{\sinh(x)}$;
- *(v)* $\dfrac{x\cosh(\alpha x)}{\sinh(x)}$, *for* $-1/2 < \alpha < 1/2$.

Bhatia and Parthasarathy in [7, p. 216 and 217], by using Lemma 2.2(i), Proposition 2.1 and congruence relation, proved the following result.

Lemma 2.3. *Let $\lambda_1, \dots, \lambda_n$ be positive real numbers (not necessarily distinct), and $0 \leqslant \nu \leqslant 1$. If $Y = [y_{ij}] \in \mathbb{M}_n$, where $y_{ij} = \dfrac{\lambda_i^{\nu}\lambda_j^{1-\nu} + \lambda_i^{1-\nu}\lambda_j^{\nu}}{\lambda_i + \lambda_j}$, then $Y \geq 0$.*

In [11], Heinz proved the following inequality:

$$\left|\left|\left| A^{\nu} X B^{1-\nu} + A^{1-\nu} X B^{\nu} \right|\right|\right| \leqslant |||AX + XB|||\,, \tag{2.1}$$

where $A, B \in \mathbb{M}_n$ are positive semidefinite matrices, $0 \leqslant \nu \leqslant 1$ and $X \in \mathbb{M}_n$. For another proof of this inequality, see [7]. Now, we state a similar inequality for the numerical radius of matrices.

Theorem 2.4. *Let $A, X \in \mathbb{M}_n$ be such that $A \geq 0$. Suppose that $0 \leqslant \nu \leqslant 1$. Then*

$$\omega(A^{\nu} X A^{1-\nu} + A^{1-\nu} X A^{\nu}) \leqslant \omega(AX + XA).$$

Proof. In view of Lemma 1.1(i), we assume that $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, where $\lambda_i \geqslant 0$. At first, let $\lambda_i > 0$ for all $i = 1, \ldots, n$. A simple calculation shows that:

$$A^\nu X A^{1-\nu} + A^{1-\nu} X A^\nu = Y \circ (AX + XA),$$

where $Y \in \mathbb{M}_n$ is the matrix with entries given by $y_{ij} = \dfrac{\lambda_i^\nu \lambda_j^{1-\nu} + \lambda_i^{1-\nu}\lambda_j^\nu}{\lambda_i + \lambda_j}$. Then by Lemma 2.3, $Y \geq 0$, and hence the result in this case follows from (1.1).

In the general case, assume $A = A_1 \oplus 0_{n-k}$, where $A_1 \in \mathbb{M}_k (k < n)$ is a positive definite matrix. Let $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \in \mathbb{M}_n$, where $X_1 \in \mathbb{M}_k$ and $X_4 \in \mathbb{M}_{n-k}$. Then we have

$$A^\nu X A^{1-\nu} + A^{1-\nu} X A^\nu = (A_1^\nu X_1 A_1^{1-\nu} + A_1^{1-\nu} X_1 A_1^\nu) \oplus 0_{n-k},$$

$$AX + XA = \begin{bmatrix} A_1X_1 + X_1A_1 & A_1X_2 \\ X_3A_1 & 0_{n-k} \end{bmatrix}.$$

Now, by Lemma 1.1(ii), the argument in the first case and [9, Lemma 2.1], we have:

$$\begin{aligned} \omega(A^\nu X A^{1-\nu} + A^{1-\nu} X A^\nu) &= \omega(A_1^\nu X_1 A_1^{1-\nu} + A_1^{1-\nu} X_1 A_1^\nu) \\ &\leqslant \omega(A_1X_1 + X_1A_1) \\ &\leqslant \omega(AX + XA). \end{aligned}$$

So, the proof is complete. □

The following example shows that the inequality (2.1) need not be true for the numerical radius norm.

Example 2.5. Let $\nu = \frac{1}{2}$, $A = I,\ B^{1/2} = \operatorname{diag}(2 \pm \sqrt{3}, 1)$ and $X = \begin{bmatrix} 1/(4 \pm 2\sqrt{3}) & 3 \\ 0 & -2 \end{bmatrix}$. Then a simple calculation shows that:

$$2.7025 = \omega(A^\nu X B^{1-\nu} + A^{1-\nu} X B^\nu) > \omega(AX + XB) = 2.6213.$$

By setting $X = I$ in (2.1), we have the following inequality:

$$\left|\left|\left| A^\nu B^{1-\nu} + A^{1-\nu} B^\nu \right|\right|\right| \leqslant |||A + B|||\,.$$

In the following theorem, we prove this inequality for the numerical radius of matrices.

Theorem 2.6. *Let $A, B \in \mathbb{M}_n$ be positive semidefinite matrices and $0 \leqslant \nu \leqslant 1$. Then*

$$\omega(A^\nu B^{1-\nu} + A^{1-\nu} B^\nu) \leqslant \omega(A + B). \tag{2.2}$$

Proof. By using Lemma 1.1((iii) and (iv)), the result is easy to verify. □

In [7, p. 218 and 219], by using Lemma 2.2(ii), Proposition 2.1 and congruence relation, we have the following result:

Lemma 2.7. *Let $\lambda_1, \dots, \lambda_n$ be positive real numbers (not necessarily distinct), and $0 \leqslant \nu \leqslant 1$. If $Y = [y_{ij}] \in \mathbb{M}_n$, where*

$$y_{ij} = \begin{cases} \dfrac{\lambda_i^{\nu}\lambda_j^{1-\nu} - \lambda_i^{1-\nu}\lambda_j^{\nu}}{\lambda_i - \lambda_j} & if\ i \neq j\ and\ \lambda_i \neq \lambda_j, \\ 1 & if\ i \neq j\ and\ \lambda_i = \lambda_j, \\ 2\nu - 1 & if\ i = j, \end{cases} \tag{2.3}$$

then $Y \geq 0$.

Theorem 2.8. *Let $A, X \in \mathbb{M}_n$, where $A \geq 0$. Suppose that $0 \leqslant \nu \leqslant 1$. Then*

$$\omega(A^{\nu}XA^{1-\nu} - A^{1-\nu}XA^{\nu}) \leqslant |2\nu - 1|\omega(AX - XA).$$

Proof. In view of Lemma 1.1(i), we assume that $A = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, where $\lambda_i \geqslant 0$. At first, let $\lambda_i > 0$ for all $i = 1, \dots, n$. A simple calculation shows that:

$$A^{\nu}XA^{1-\nu} - A^{1-\nu}XA^{\nu} = Y \circ (AX - XA),$$

where $Y \in \mathbb{M}_n$ is the matrix as in (2.3). By Lemma 2.7, $Y \geq 0$, and hence the result in this case follows from (1.1).

By the same manner in the proof of Theorem 2.4, the result in the general case also holds. So, the proof is complete. □

Lemma 2.9. *[7, p. 222] Let $\lambda_1, \dots, \lambda_n$ be positive real numbers (not necessarily distinct). If $Y = [y_{ij}] \in \mathbb{M}_n$, where*

$$y_{ij} = \begin{cases} \dfrac{\log\lambda_i - \log\lambda_j}{\lambda_i^{1/2}\lambda_j^{-1/2} - \lambda_i^{-1/2}\lambda_j^{1/2}} & if\ i \neq j\ and\ \lambda_i \neq \lambda_j, \\ 1 & if\ i = j\ or\ \lambda_i = \lambda_j, \end{cases} \tag{2.4}$$

then $Y \geq 0$.

Theorem 2.10. *Let $A, X \in \mathbb{M}_n$ be such that $A > 0$. Then*

$$\omega((\log A)X - X(\log A)) \leqslant \omega(A^{1/2}XA^{-1/2} - A^{-1/2}XA^{1/2}).$$

In particular, if H is Hermitian, then

$$\omega(HX - XH) \leqslant \omega(e^{H/2}Xe^{-H/2} - e^{-H/2}Xe^{H/2}).$$

Proof. In view of Lemma 1.1(i), we assume that $A = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, where $\lambda_i > 0$. A simple calculation shows that:

$$(\log A)X - X(\log A) = Y \circ (A^{1/2}XA^{-1/2} - A^{-1/2}XA^{1/2}),$$

where $Y \in \mathbb{M}_n$ is the matrix as in (2.4). By Lemma 2.9, $Y \geq 0$, and hence the result follows from (1.1).

The second inequality follows from this fact that every Hermitian matrix can be written as the logarithm of a positive definite matrix. □

Theorem 2.11. *Let $H, X \in \mathbb{M}_n$ be such that $H = H^*$. Then*

$$\omega((\sin H)X(\cos H) - (\cos H)X(\sin H)) \leqslant \omega(HX - XH).$$

Proof. In view of Lemma 1.1(i), we assume that $H = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, where $\lambda_i \in \mathbb{R}$. A simple calculation shows that:

$$(\sin H)X(\cos H) - (\cos H)X(\sin H) = Y \circ (HX - XH),$$

where $Y = (y_{ij}) \in \mathbb{M}_n$ is the following matrix:

$$y_{ij} = \begin{cases} \dfrac{\sin(\lambda_i - \lambda_j)}{\lambda_i - \lambda_j} & if\ i \neq j\ and\ \lambda_i \neq \lambda_j, \\ 1 & if\ i = j\ or\ \lambda_i = \lambda_j. \end{cases}$$

By Lemma 2.2(iii), $Y \geq 0$, and hence the result follows from (1.1). □

Theorem 2.12. *Let $H, X \in \mathbb{M}_n$ be such that $H = H^*$. Then*

$$\omega(HX - XH) \leqslant \omega((\sinh H)X(\cosh H) - (\cosh H)X(\sinh H)).$$

Proof. In view of Lemma 1.1(i), we assume that $H = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, where $\lambda_i \in \mathbb{R}$. A simple calculation shows that:

$$HX - XH = Y \circ ((\sinh H)X(\cosh H) - (\cosh H)X(\sinh H)),$$

where $Y = (y_{ij}) \in \mathbb{M}_n$ is the following matrix:

$$y_{ij} = \begin{cases} \dfrac{\lambda_i - \lambda_j}{\sinh(\lambda_i - \lambda_j)} & if\ i \neq j\ and\ \lambda_i \neq \lambda_j, \\ 1 & if\ i = j\ or\ \lambda_i = \lambda_j. \end{cases}$$

By Lemma 2.2(iv), $Y \geq 0$, and hence the result follows from (1.1). □

Bhatia and Kosaki in [6, p. 46], by using Lemma 2.2(v), Proposition 2.1 and congruence relation, proved the following lemma.

Lemma 2.13. *Let $\lambda_1, \ldots, \lambda_n$ be positive real numbers (not necessarily distinct), and $1/4 \leqslant \nu \leqslant 3/4$. If $Y = [y_{ij}] \in \mathbb{M}_n$, where*

$$\text{(2.5)} \quad y_{ij} = \begin{cases} \dfrac{(\lambda_i^{\nu}\lambda_j^{1-\nu} + \lambda_i^{1-\nu}\lambda_j^{\nu})(\log \lambda_i - \log \lambda_i)}{2(\lambda_i - \lambda_j)} & if\ i \neq j\ and\ \lambda_i \neq \lambda_j, \\ 1 & if\ i = j\ or\ \lambda_i = \lambda_j, \end{cases}$$

then $Y \geq 0$.

Theorem 2.14. *Let $A, X \in \mathbb{M}_n$, $A > 0$ and $1/4 \leqslant \nu \leqslant 3/4$. Then*

$$\frac{1}{2}\omega(A^{\nu}XA^{1-\nu} + A^{1-\nu}XA^{\nu}) \leqslant \omega(\int_0^1 A^t X A^{1-t} dt).$$

Proof. In view of Lemma 1.1(i), we assume that $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, where $\lambda_i > 0$ for all $i = 1, \ldots, n$. A simple calculation shows that:

$$A^\nu X A^{1-\nu} + A^{1-\nu} X A^\nu = Y \circ (\int_0^1 A^t X A^{1-t} dt),$$

where $Y \in \mathbb{M}_n$ is the matrix as in (2.5). By Lemma 2.13, $Y \geq 0$, and hence the result follows from (1.1). □

The following lemma is a consequence of the spectral theorem for positive operators and Jensen's inequality (see, e.g., [16]).

Lemma 2.15. *Let $A \in \mathbb{M}_n$ be such that $A \geq 0$, and $x \in \mathbb{C}^n$ be a unit vector. Then for all $0 \leqslant r \leqslant 1$,*

$$x^* A^r x \leqslant (x^* A x)^r, \tag{2.6}$$

and for all $r \geqslant 1$,

$$(x^* A x)^r \leqslant x^* A^r x. \tag{2.7}$$

Theorem 2.16. *Let $A, B \in \mathbb{M}_n$ be positive semidefinite matrices, and $0 \leqslant \nu \leqslant 1$. Then*

$$\omega(A^\nu B^{1-\nu} + B^\nu A^{1-\nu}) \leqslant \sqrt{2}\ \omega^{1/2}(A^2 + B^2).$$

Proof. By the Schwarz inequality, Young inequality for positive numbers and convexity of the function $f(t) = t^2$, for every unit vector $x \in \mathbb{C}^n$, we have

$$\begin{aligned}
\left|x^*(A^\nu B^{1-\nu} + B^\nu A^{1-\nu})x\right| &= \left|x^* A^\nu B^{1-\nu} x + x^* B^\nu A^{1-\nu} x\right| \\
&\leqslant \|B^{1-\nu}x\|\ \|A^\nu x\| + \|A^{1-\nu}x\|\ \|B^\nu x\| \\
&= (x^* B^{2(1-\nu)} x)^{1/2}\ (x^* A^{2\nu} x)^{1/2} \\
&+ (x^* A^{2(1-\nu)} x)^{1/2}\ (x^* B^{2\nu} x)^{1/2} \\
&\leqslant \sqrt{2}\ [(x^* B^{2(1-\nu)} x)(x^* A^{2\nu} x) \\
&+ (x^* A^{2(1-\nu)} x)(x^* B^{2\nu} x)]^{1/2} \\
&\underset{(2.6)}{\leqslant} \sqrt{2}\ [(x^* B^2 x)^{1-\nu}(x^* A^2 x)^\nu \\
&+ (x^* A^2 x)^{1-\nu}(x^* B^2 x)^\nu]^{1/2} \\
&\leqslant \sqrt{2}\ [x^*((1-\nu)B^2 + \nu A^2)x \\
&+ x^*((1-\nu)A^2 + \nu B^2)x]^{1/2} \\
&= \sqrt{2}\left(x^*(A^2 + B^2)x\right)^{1/2}.
\end{aligned}$$

Now, by taking the supremum over all unit vectors in $\mathbb{C}^n$, the result holds. □

It is known, see [5, Theorem 3], that if $A, B \in \mathbb{M}_n$ are positive semidefinite matrices, and $r \geqslant 1$, then

$$\|A^r + B^r\| \leqslant \|(A+B)^r\|. \tag{2.8}$$

Theorem 2.17. *Let $A, B \in \mathbb{M}_n$ be positive semidefinite matrices. Then*

$$\omega(A+B) \leqslant \sqrt{2}\ \omega^{1/2}(A^2+B^2) \leqslant \sqrt{2}\ \omega(A+B).$$

Proof. By convexity of the function $f(t) = t^2$, for every unit vector $x \in \mathbb{C}^n$, we have

$$\begin{aligned} x^*(A^2+B^2)x &= x^*A^2x + x^*B^2x \\ &\underset{(2.7)}{\geqslant} (x^*Ax)^2 + (x^*Bx)^2 \\ &\geqslant \frac{1}{2}\left((x^*Ax) + (x^*Bx)\right)^2 \\ &= \frac{1}{2}(x^*(A+B)x)^2. \end{aligned}$$

Now, by taking the supremum over all unit vectors in $\mathbb{C}^n$, the left inequality holds.

Using the inequality in relation (2.8) and Lemma 1.1((iv) and (v)), we have

$$\omega^{1/2}(A^2+B^2) \leqslant \omega^{1/2}((A+B)^2) \leqslant \omega(A+B),$$

and hence, the right inequality also holds. $\square$

At the end of this section, by Theorems 2.6, 2.16 and 2.17, and this fact, see [19], that $\omega(A^{1/2}B^{1/2}) \leqslant \omega(A+B)/2$ for any two positive semidefinite matrices $A, B \in \mathbb{M}_n$, we get the following corollary:

Corollary 2.18. *Let $A, B \in \mathbb{M}_n$ be two positive semidefinite matrices, and $0 \leqslant \upsilon \leqslant 1$. Then*

$$\max\{\omega(A^\nu B^{1-\nu} + A^{1-\nu}B^\nu), \omega(A^\nu B^{1-\nu} + B^\nu A^{1-\nu})\} \leqslant \sqrt{2}\ \omega^{1/2}(A^2+B^2),$$

and

$$\max\{2\omega(A^{1/2}B^{1/2}), \omega(A^{1/2}B^{1/2} + B^{1/2}A^{1/2})\} \leqslant \omega(A+B).$$

3. **Additional results and some questions**

Let $A \in \mathbb{M}_n$. The linear operator S_A on $\mathbb{M}_n$, called the Schur multiplier operator, is defined as $S_A(X) := A \circ X$. The induced norm of S_A with respect to a unitarily invariant norm $|||.|||$ will be denoted by

$$|||S_A||| := \sup_{X \neq 0} \frac{|||S_A(X)|||}{|||X|||} = \sup_{X \neq 0} \frac{|||A \circ X|||}{|||X|||}.$$

Also, the induced norm of S_A with respect to numerical radius norm will be denoted by

$$\|S_A\|_\omega := \sup_{X\neq 0} \frac{\omega(S_A(X))}{\omega(X)} = \sup_{X\neq 0} \frac{\omega(A\circ X)}{\omega(X)}.$$

Ando and Okubo in 1991 proved the following two lemmas:

Lemma 3.1. *[2, Corollary 1] For all $A \in \mathbb{M}_n$,*

$$\|S_A\| \leqslant \|S_A\|_\omega \leqslant 2\|S_A\|.$$

Zhan in [21] claimed that the following useful result can be deduced from [1].

Lemma 3.2. *If $A \in \mathbb{M}_n$, then*

$$|||S_A||| \leqslant \|S_A\|.$$

Remark 3.3. By using Lemmas 3.1 and 3.2, in the proof of Theorems 2.4, 2.8, 2.10, 2.11, 2.12 and 2.14, we obtain the inequalities in this theorems for any unitarily invariant norm. Now, if in this inequalities, we replace the matrices A and X by $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ and $\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$, respectively, then we get the following corollaries, which were obtained in [6, relation (20)] and [7, relations (3.1), (3.2), (4.5), (4.6), (4.7), (4.8)].

Corollary 3.4. *Let $A, B \in \mathbb{M}_n$ be positive semidefinite matrices and $0 \leqslant \nu \leqslant 1$. Then for all $X \in \mathbb{M}_n$,*

$$\left|\left|\left| A^\nu X B^{1-\nu} + A^{1-\nu} X B^\nu \right|\right|\right| \leqslant |||AX + XB|||\,.$$

Corollary 3.5. *Let $A, B \in \mathbb{M}_n$ be positive semidefinite matrices and $0 \leqslant \nu \leqslant 1$. Then for all $X \in \mathbb{M}_n$,*

$$\left|\left|\left| A^\nu X B^{1-\nu} - A^{1-\nu} X B^\nu \right|\right|\right| \leqslant |2\nu - 1|\, |||AX - XB|||\,.$$

By setting $X = I \in \mathbb{M}_n$ in Corollary 3.5 and using Lemma 1.1((iii) and (iv)), and this fact that the spectral matrix norm $\|.\|$ is unitarily invariant, we have the following result:

Proposition 3.6. *Let $A, B \in \mathbb{M}_n$ be positive semidefinite matrices and $0 \leqslant \nu \leqslant 1$. Then*

$$\omega(A^\nu B^{1-\nu} - A^{1-\nu} B^\nu) \leqslant |2\nu - 1|\, \omega(A - B).$$

Question 3.7. Let $A, B \in \mathbb{M}_n$ be positive semidefinite matrices and $0 \leqslant \nu \leqslant 1$. Is it true that for all $X \in \mathbb{M}_n$,

$$\omega(A^\nu X B^{1-\nu} - A^{1-\nu} X B^\nu) \leqslant |2\nu - 1|\, \omega(AX - XB)?$$

Corollary 3.8. *Let $H, K, X \in \mathbb{M}_n$ be such that $H = H^*$ and $K = K^*$. Then*

$$|||(\sin H)X(\cos K) - (\cos H)X(\sin K)||| \leqslant |||HX - XK|||\,,$$

and

$$|||HX - XK||| \leqslant |||(\sinh H)X(\cosh K) - (\cosh H)X(\sinh K)|||\,.$$

By setting $X = I \in \mathbb{M}_n$ in Corollary 3.8 and using Lemma 1.1((iii) and (iv)), we have the following result:

Proposition 3.9. *Let $H, K, \in \mathbb{M}_n$ be Hermitian matrices. Then*

$$\omega(\sin H \cos K - \cos H \sin K) \leqslant \omega(H - K).$$

In particular, if $HK = KH$, then

$$\omega(H - K) \leqslant \omega(\sinh H \cosh K - \cosh H \sinh K).$$

Question 3.10. Let $H, K, X \in \mathbb{M}_n$ be such that $H = H^*$ and $K = K^*$. Is it true that

$$\omega((\sin H)X(\cos K) - (\cos H)X(\sin K)) \leqslant \omega(HX - XK)?$$

and

$$\omega(HX - XK) \leqslant \omega((\sinh H)X(\cosh K) - (\cosh H)X(\sinh K))?$$

Corollary 3.11. *Let $A, B, X \in \mathbb{M}_n$ be such that $A, B > 0$. Then*

$$|||(\log A)X - X(\log B)||| \leqslant \left|\left|\left| A^{1/2}XB^{-1/2} - A^{-1/2}XB^{1/2} \right|\right|\right|.$$

In particular, if $H, K \in \mathbb{M}_n$ are Hermitian, then

$$|||HX - XK||| \leqslant \left|\left|\left| e^{H/2}Xe^{-K/2} - e^{-H/2}Xe^{K/2} \right|\right|\right|.$$

By setting $X = I \in \mathbb{M}_n$ in Corollary 3.11 and using Lemma 1.1((iii) and (iv)), we have the following result:

Proposition 3.12. *Let $A, B \in \mathbb{M}_n$ be such that $A, B > 0$ and $AB = BA$. Then*

$$\omega(\log A - \log B) \leqslant \omega(A^{1/2}B^{-1/2} - A^{-1/2}B^{1/2}).$$

In particular, if $H, K \in \mathbb{M}_n$ are Hermitian and $HK = KH$, then

$$\omega(H - K) \leqslant \omega(e^{H/2}e^{-K/2} - e^{-H/2}e^{K/2}).$$

Question 3.13. Let $A, B, X \in \mathbb{M}_n$ be such that $A, B > 0$. Is it true that

$$\omega((\log A)X - X(\log B)) \leqslant \omega(A^{1/2}XB^{-1/2} - A^{-1/2}XB^{1/2})?$$

Corollary 3.14. *Let $A, B, X \in \mathbb{M}_n$ be such that A, B are positive semidefinite matrices. If $1/4 \leqslant \nu \leqslant 3/4$, then*

$$\frac{1}{2}\,|||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}||| \leqslant \left|\left|\left| \int_0^1 A^t XB^{1-t}dt \right|\right|\right|.$$

By setting $X = I \in \mathbb{M}_n$ in Corollary 3.14 and using Lemma 1.1((iii) and (iv)), we have the following result:

Proposition 3.15. *Let $A, B \in \mathbb{M}_n$ be positive semidefinite matrices such that $AB = BA$. If $1/4 \leqslant \nu \leqslant 3/4$, then*

$$\omega(A^\nu B^{1-\nu} + A^{1-\nu} B^\nu) \leqslant 2\omega(\int_0^1 A^t B^{1-t} dt). \tag{3.1}$$

Remark 3.16. Hiai and Kosaki [13, Corollary 2.3] proved that for all unitarily invariant norms and $0 \leqslant t \leqslant 1$,

$$2 \left|\left|\left| \int_0^1 A^t X B^{1-t} dt \right|\right|\right| \leqslant |||AX + XB||| .$$

That is in this case, (3.1) is a refinement of (2.2).

Question 3.17. Let $A, B, X \in \mathbb{M}_n$ be such that A, B are positive semidefinite matrices. If $1/4 \leqslant \nu \leqslant 3/4$, then is it true that

$$\frac{1}{2}\omega(A^\nu X B^{1-\nu} + A^{1-\nu} X B^\nu) \leqslant \omega(\int_0^1 A^t X B^{1-t} dt)?$$

and

$$\omega(\int_0^1 A^t X B^{1-t} dt) \leqslant \frac{1}{2}\omega(AX + XB)?$$

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