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## $\sigma$ -NORMALITY OF TOPOLOGICAL SPACES

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ABSTRACT. We generalize the notion of normality on topological spaces to  $\sigma$ -normality.  $\sigma$ -version of the Urysohn's lemma is established and  $\sigma$ -normality is characterized in terms of semi-continuity and  $\sigma$ -continuity.

### 1. Introduction

As a known notion, a topological space  $(X, \tau)$  is called normal if for each pair of disjoint closed subsets E, F of X there are two disjoint open subsets U, V of X such that  $E \subseteq U$  and  $F \subseteq V$ . The Urysohn's lemma states that the normality is equivalent to the existence of a continuous function  $f: (X, \tau) \to [0, 1]$  for each pair of disjoint closed subsets E, Fof X such that f is 0 on E and 1 on F (see, for example, [2], [4] and [5]).

In many situations we are faced with a space X with a pair of topologies on it such that one is contained in the other. This happens for example for weak topologies for a normed space and its continuous dual (see [3]) or for several topologies on the space of all norm bounded operators on a Hilbert space (see [1]). In these cases, we are interested in continuity of a certain family of mappings with respect to different topologies. It is also happened to consider two families of subsets of the space and separate two members of one family using members of the

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other. For instance, in the Banach-Mazur topological game, one deals with such problems (for a history and recent developments, see [6]).

Consider a pair of topologies  $\tau_1 \subseteq \tau_2$  on a set X. We say that X is weakly normal if for each pair of  $\tau_1$ -closed disjoint subsets E, F of X there are two  $\tau_2$ -open disjoint subsets U, V of X such that  $E \subseteq U$  and  $F \subseteq V$ . Here, we find some interesting facts concerning weakly normal spaces and we characterize  $\sigma$ -normal spaces in terms of semi-continuity and  $\sigma$ -continuity. In the next section we introduce the notion of weak normality and give some elementary facts concerning the concept. All topological spaces are assumed to be Hausdorff.

#### 2. Preliminaries

**Definition 2.1.** Let  $\tau_1 \subseteq \tau_2$  be two topologies on a set X. We say that  $\tau_2$  is  $\sigma$ -obtainable from  $\tau_1$  and write  $\tau_1 \leq_{\sigma} \tau_2$  if each member of  $\tau_2$  is a  $G_{\delta}$  set with respect to  $\tau_1$ . The triple  $(X, \tau_1, \tau_2)$  is then called a couple topological space. We say that X is  $(\tau_1, \tau_2)$ -normal, or weakly normal with respect to  $(\tau_1, \tau_2)$ , if for each pair of disjoint  $\tau_1$ -closed subsets E, F of X there are a pair of disjoint  $\tau_2$ -open subsets U, V of X such that  $E \subseteq U$  and  $F \subseteq V$ . If the couple topological space  $(X, \tau_1, \tau_2)$  is weakly normal then it is called  $\sigma$ -weakly normal, or simply  $\sigma$ -normal. We say that the topological space  $(X, \tau)$  is upper (resp. lower)  $\sigma$ -normal if there is a topology  $\tau_1$  (resp.  $\tau_2$ ) on X with  $\tau_1 \leq_{\sigma} \tau$  (resp.  $\tau \leq_{\sigma} \tau_2$ ) such that the couple topological space  $(X, \tau_1, \tau)$  is  $\sigma$ -normal.  $(X, \tau)$  is called  $\sigma$ -normal if it is both upper and lower  $\sigma$ -normal.

Note that  $\sigma$ -obtainability does not imply that each  $G_{\delta}$  set with respect to  $\tau_1$  belong to  $\tau_2$ . In fact,  $\tau_1 \leq_{\sigma} \tau_2$  if and only if  $\tau_1 \subseteq \tau_2 \subseteq G_{\delta}(\tau_1)$ , where  $G_{\delta}(\tau_1)$  is the family of all  $G_{\delta}$  sets with respect to  $\tau_1$ . As a nontrivial example, let  $X = \mathbb{R}, \tau_1 = \tau_{\varepsilon}$ , the Euclidean topology, and  $\tau_2$  be the lower limit topology on  $\mathbb{R}$ . Then,  $\tau_1 \leq_{\sigma} \tau_2$ .

Clearly, if  $\sigma$ -Nor is the category of all  $\sigma$ -normal spaces then Nor  $\subseteq \sigma$ -Nor. Prior to anything, an example of a  $\sigma$ -normal space which is not normal is therefore noticeable.

**Example 2.2.** (Nor  $\subset \sigma$ -Nor) Let  $X = [0,1] \cap \mathbb{Q}$  and  $K = \{\frac{1}{n} : n \in \mathbb{N}\}$ . For each  $x \in X$  and r > 0 let  $N_r(x) = \{y \in X : |x - y| < r\}$ . Then,  $\{N_r(x)\}_{r>0,x\in X} \cup \{N_r(0) \setminus K\}_{r>0}$  is a topological basis for a topology  $\tau$  on X. As a famous result, we know that  $(X, \tau)$  is a Hausdorff topological space which is not normal (see, for example [4]). Let  $\tau_1$  be the Euclidean topology on X. Then,  $(X, \tau_1, \tau)$  is a couple topological space, since for an arbitrary element  $N_r(0) \setminus K$  we can write  $N_r(0) \setminus K = \bigcap_{m=1}^{\infty} U_{n,m}$ , where  $U_{n,m} = N_r(0) \setminus [\frac{1}{n} - \frac{1}{m}, \frac{1}{n} + \frac{1}{m}] \in \tau_1$ . The couple topological space  $(X, \tau_1, \tau)$  is  $\sigma$ -normal, since  $(X, \tau_1)$  is itself normal. Thus,  $(X, \tau)$ is upper  $\sigma$ -normal. Moreover, if  $\tau_2 = \tau_{\delta}$ , the discrete topology, then  $\tau \leq_{\sigma} \tau_2$  and the couple topological space  $(X, \tau, \tau_{\delta})$  is obviously  $\sigma$ -normal. Hence  $(X, \tau)$  is lower  $\sigma$ -normal.

We also show that the lower  $\sigma$ -normality does not imply the upper one. This is done by the Niemytzki's tangent disc topology (see [7]).

**Example 2.3.** Suppose  $\mathcal{P}$  is the upper half plane  $\{(x, y) \in \mathbb{R}^2 : y > 0\}$ and  $\mathcal{L}$  is the real line. Let  $\tau_{\varepsilon}$  be the Euclidean topology on  $\mathbb{R}^2$ ,  $\tau'_{\varepsilon}$  be the topology induced by  $\tau_{\varepsilon}$  on  $\mathcal{P}$  and  $\tau_{\delta}$  be the discrete topology on  $\mathcal{L}$ . For  $x \in \mathcal{L}$  and r > 0, let  $D_r(x)$  be the  $\tau_{\varepsilon}$ -interior of the circle with radius r centered in  $\mathcal{P}$  tangent to  $\mathcal{L}$  at x and  $D_r[x] = D_r(x) \cup \{x\}$ . Then  $\tau'_{\varepsilon} \cup \{D_r[x]\}_{r>0, x \in \mathcal{L}}$  is a topological basis and forms a topology  $\tau$  on X. We show that  $(X, \tau)$  is lower  $\sigma$ -normal but it is not upper.

Let  $\tau_2$  be the topology generated by the basis  $\tau_{\varepsilon} \cup \tau_{\delta}$  on X. Then it is not hard to show that the couple topological space  $(X, \tau, \tau_2)$  is  $\sigma$ -normal which implies that  $(X, \tau)$  is lower  $\sigma$ -normal.

Now, let  $\tau_1$  be an arbitrary topology on X with  $\tau_1 \leq_{\sigma} \tau$ . We prove that the couple topological space  $(X, \tau_1, \tau)$  is not  $\sigma$ -normal.  $\mathbb{Q}^c$  is a  $\tau$ -closed subset of X. As  $\tau_1 \leq_{\sigma} \tau$ , there is a sequence  $\{F_n\}$  of  $\tau_1$ -closed subsets of X such that  $\mathbb{Q}^c = \bigcup_{n=1}^{\infty} F_n$ . At least one of the  $F_n$ 's, say  $F_{n_0}$ , must be uncountable. Let U be a  $\tau$ -open neighborhood of  $F = F_{n_0}$ . Then for each  $x \in U$ , there is an  $r_x > 0$  such that  $D_{r_x}[x] \subseteq U$ . Let  $S_k = \{x \in F : r_x > \frac{1}{k}\}$ . Then,  $F \subseteq \bigcup_{k=1}^{\infty} S_k$  and so there is a  $k_0 \in \mathbb{N}$  such that  $S_{k_0}$  is not  $\tau_{\varepsilon}$ -nowhere dense, since F is of second category equipped with the Euclidean topology on  $\mathcal{L}$ . This shows that the  $\tau_{\varepsilon}$ -closure of  $S_{k_0}$  has a  $\tau_{\varepsilon}$ -interior point and so there is an interval [a, b] such that the  $\tau_{\varepsilon}$ -closure of  $S_{k_0}$  contains [a, b]. This implies that for every rational number  $r \in [a, b]$  in  $\tau_{\varepsilon}$ -closure of F, each  $\tau$ -open neighborhood V of rmust intersect U. Now, F and  $\{r\}$  are two disjoint  $\tau_1$ -closed subsets of X which can not be separated by any  $\tau$ -open disjoint sets.

Recall that if  $(X, \tau)$  is a topological space then a function  $f : (X, \tau) \to \mathbb{R}$  is called upper (resp. lower) semi-continuous if  $f^{-1}((-\infty, t))$  (resp.

 $f^{-1}((t,\infty))$  is open for each  $t \in \mathbb{R}$ . If f is upper (resp. lower) semicontinuous then for each  $t \in \mathbb{R}$  the set  $\{x \in X : f(x) \leq t\}$  (resp.  $\{x \in X : f(x) \geq t\}$ ) is  $G_{\delta}$  (see [2] and [5]).

**Definition 2.4.** Let  $(X, \tau)$  and  $(Y, \tau')$  be two topological spaces. A function  $f : (X, \tau) \to (Y, \tau')$  is called  $\sigma$ -continuous if  $f^{-1}(U)$  is a  $G_{\delta}$  subset of X for each open set U in Y.

Let A be an  $F_{\sigma}$  subset of X which is neither open nor closed. Then, the characteristic function  $\chi_A$  is a  $\sigma$ -continuous function which is neither upper nor lower semi-continuous. Thus,  $\sigma$ -continuity does not imply upper or lower semi-continuity.

Obviously, a function  $f: (X, \tau) \to (Y, \tau')$  is  $\sigma$ -continuous if and only if  $f: (X, G_{\delta}(\tau)) \to (Y, \tau')$  is continuous. Moreover, if  $\tau_1$  and  $\tau_2$  are two topologies on a set X with  $\tau_1 \leq_{\sigma} \tau_2$  and  $f: (X, \tau_2) \to (Y, \tau')$  is continuous then  $f: (X, \tau_1) \to (Y, \tau')$  is  $\sigma$ -continuous.

#### 3. The $\sigma$ -Urysohn's lemma and its applications

We prove the  $\sigma$ -version of Urysohn's lemma, but prior to that, we state the following lemma which can be proved easily. We denote the  $\tau$ -closure of a set A by  $\tau$ -cl(A).

**Lemma 3.1.** Let  $\tau_1 \subseteq \tau_2$  be two toplogies on a set X. Then, X is  $(\tau_1, \tau_2)$ -normal if and only if for each  $\tau_1$ -closed subset F and  $\tau_1$ -open subset G of X with  $F \subseteq G$  there is a  $\tau_2$ -open subset V of X such that  $F \subseteq V \subseteq \tau_2$ -cl $(V) \subseteq G$ .

 $\sigma$ -Urysohn's Lemma 3.2. Let  $(X, \tau_1, \tau_2)$  be a  $\sigma$ -normal couple topological space. Then, for each pair of disjoint  $\tau_1$ -closed subsets E, F of X there is a continuous function  $f : (X, \tau_2) \to [0, 1]$  such that f is 0 on E and 1 on F.

**Proof.** Let  $V_1 = F^c$ . Then, by Lemma 3.1, there is a  $\tau_2$ -open set  $V_{\frac{1}{2}}$  such that  $E \subseteq V_{\frac{1}{2}} \subseteq \tau_2$ -cl $(V_{\frac{1}{2}}) \subseteq V_1$ . Now,  $V_{\frac{1}{2}}$  is  $\tau_2$ -open and thus there is a sequence  $\{V_{\frac{1}{2},n}\}$  of  $\tau_1$ -open sets such that  $V_{\frac{1}{2}} = \bigcap_{n=1}^{\infty} V_{\frac{1}{2},n}$  and we may assume that  $V_{\frac{1}{2},n} \subseteq V_1$ , since  $V_{\frac{1}{2}} = V_{\frac{1}{2}} \cap V_1 = \bigcap_{n=1}^{\infty} (V_{\frac{1}{2},n} \cap V_1)$ . For  $E \subseteq V_{\frac{1}{2},n}$  by Lemma 3.1, there is a  $\tau_2$ -open set  $V_{\frac{1}{4},n}$  such that

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 $E \subseteq V_{\frac{1}{4},n} \subseteq \tau_2 \operatorname{-cl}(V_{\frac{1}{4},n}) \subseteq V_{\frac{1}{2},n}$ . Now,  $\tau_2 \operatorname{-cl}(V_{\frac{1}{2},n})$  is  $\tau_2$ -closed and so there is a sequence  $\{E_{n,m}\}$  of  $\tau_1$ -closed sets such that  $\tau_2 \operatorname{-cl}(V_{\frac{1}{2},n}) = \bigcup_{m=1}^{\infty} E_{n,m}$ and we may assume that  $E \subseteq E_{n,m}$ , since  $\tau_2 \operatorname{-cl}(V_{\frac{1}{2},n}) = \tau_2 \operatorname{-cl}(V_{\frac{1}{2},n}) \cup E = \bigcup_{m=1}^{\infty} (E_{n,m} \cup E)$ . For  $E_{n,m} \subseteq V_1$ , we can find a  $\tau_2$ -open set  $V_{\frac{3}{4},n,m}$  such that

$$E_{n,m} \subseteq V_{\frac{3}{4},n,m} \subseteq \tau_2 \operatorname{-cl}(V_{\frac{3}{4},n,m}) \subseteq V_1.$$

Continuing this process by induction, we can obtain for each binary fraction  $r = k2^{-\ell}, 1 \leq k \leq 2^{\ell}, k \text{ odd}, a \tau_2$ -open set  $V_{r,n_1,\ldots,n_{\varphi(r)}}$  such that

$$E \subseteq V_{r,n_1,\ldots,n_{\varphi(r)}} \subseteq \tau_2 \text{-cl}(V_{r,n_1,\ldots,n_{\varphi(r)}}) \subseteq V_{s,m_1,\ldots,m_{\varphi(s)}}$$

for all r < s and  $n_i, m_i \in \mathbb{N}$  with  $n_i = m_i$  for each  $i = 1, \ldots, \varphi(s) \leq \varphi(r)$ , where  $\varphi : \mathbb{N} \to \mathbb{N}$  is defined by

$$\varphi(r) = \varphi(\frac{k}{2^{\ell}}) = \begin{cases} \varphi(\frac{k}{2^{\ell-1}}) & \text{if } k < 2^{\ell-1} \\ \varphi(\frac{k-2^{\ell-1}}{2^{\ell}}) + 1 & \text{if } k > 2^{\ell-1} \end{cases}$$

We now define  $f : (X, \tau_2) \to [0, 1]$  by letting f(x) = 1 if  $x \notin \bigcup V_{r,n_1,\ldots,n_{\varphi(r)}}(=V_1)$  and  $f(x) = \inf\{r : x \in V_{r,n_1,\ldots,n_{\varphi(r)}}\}$  otherwise. Clearly, f is 1 on F and 0 on E. We show that f is continuous. If  $0 < t \leq 1$ , then f(x) < t if and only if  $x \in V_{r,n_1,\ldots,n_{\varphi(r)}}$  for some r < t. Thus,

$$f^{-1}([0,t)) = \bigcup_{r < t} V_{r,n_1,\dots,n_{\varphi(r)}},$$

which is a  $\tau_2$ -open set in X. If  $0 \leq s < 1$ , then  $f(x) \leq s$  if and only if for each r > s there is a binary fraction p < r such that  $x \in V_{p,n_1,\ldots,n_{\varphi(p)}}$ . Thus,

$$f^{-1}([0,s]) = \bigcap_{r>s} \bigcup_{p < r} V_{p,n_1,\dots,n_{\varphi(p)}} = \bigcap_{q>s} \tau_2 \operatorname{-cl} (V_{q,n_1,\dots,n_{\varphi(q)}}),$$

since if  $x \in \tau_2$ -cl  $(V_{q,n_1,\ldots,n_{\varphi(q)}})$  for each q > s, then for given r > s we can find p and q such that r > p > q > s and so  $x \in \tau_2$ -cl  $(V_{q,n_1,\ldots,n_{\varphi(q)}}) \subseteq V_{p,n_1,\ldots,n_{\varphi(p)}}$ . Hence,  $f^{-1}((s,1]) = (f^{-1}([0,s]))^c$  is a  $\tau_2$ -open set in X. This shows that f is continuous.  $\Box$ 

The  $\sigma$ -Urysohn's lemma has some beautiful consequences.

**Corollary 3.3.** Let  $(X, \tau_1, \tau_2)$  be a  $\sigma$ -normal couple topological space. Then, for each pair of disjoint  $\tau_1$ -closed subsets E, F of X there is a  $\sigma$ -continuous function  $f : (X, \tau_1) \to [0, 1]$  such that f is 0 on E and 1 on F.

Noting the fact that the pointwise limit of an increasing sequence of continuous functions is lower semi-continuous, we have the following result.

**Corollary 3.4.** Let  $(X, \tau_1, \tau_2)$  be a  $\sigma$ -normal couple topological space. Then, for each pair of disjoint  $\tau_2$ -closed subsets E, F of X there is a lower semi-continuous function  $f : (X, \tau_2) \to [0, 1]$  such that f is 0 on E and 1 on F.

**Theorem 3.5.** A topological space  $(X, \tau)$  is lower  $\sigma$ -normal if and only if for each pair of disjoint closed subsets E, F of X there is a  $\sigma$ continuous function  $f : (X, \tau) \to [0, 1]$  such that f is 0 on E and 1 on F.

**Proof.** Let  $(X, \tau)$  be a lower  $\sigma$ -normal topological space. Then, there is a topology  $\tau_2$  on X such that  $\tau \leq_{\sigma} \tau_2$  and the couple topological space  $(X, \tau, \tau_2)$  is  $\sigma$ -normal. Thus, by Corollary 3.3, for each pair of disjoint  $\tau$ -closed subsets E, F of X there is a  $\sigma$ -continuous function  $f: (X, \tau) \to [0, 1]$  such that f is 0 on E and 1 on F.

Conversely, suppose that for each pair of disjoint closed subsets E, Fof X there is a  $\sigma$ -continuous function  $f_{E,F} : (X, \tau) \to [0, 1]$  such that  $f_{E,F}$  is 0 on E and 1 on F. Let  $\mathcal{F}$  be the family of all  $f_{E,F}$ 's where E, F ranges over all disjoint closed subsets of X. Suppose that  $\tau_2$  is the weakest topology, containing  $\tau$ , such that  $f : (X, \tau_2) \to [0, 1]$  is continuous for each  $f \in \mathcal{F}$ . Thus, each  $f \in \mathcal{F}$  is continuous on  $(X, G_{\delta}(\tau))$ and hence  $\tau_2 \subseteq G_{\delta}(\tau)$ . This shows that  $\tau \trianglelefteq_{\sigma} \tau_2$  and so  $(X, \tau, \tau_2)$  is a couple topological space. We claim that  $(X, \tau, \tau_2)$  is  $\sigma$ -normal. Let E, Fbe a pair of  $\tau$ -closed subsets of X. Regarding to the construction of  $\tau_2$ , we can deduce that  $f_{E,F} : (X, \tau_2) \to [0, 1]$  is continuous. Thus,  $U = f^{-1}([0, \frac{1}{2}))$  and  $V = f^{-1}((\frac{1}{2}, 1])$  are disjoint  $\tau_2$ -open subsets of Xand  $E \subseteq U, F \subseteq V$ .

**Theorem 3.6.** A topological space  $(X, \tau)$  is upper  $\sigma$ -normal if and only if for each pair of disjoint closed subsets E, F of X there is a lower semi-continuous function  $f : (X, \tau) \to [0, 1]$  such that f is 0 on E and 1 on F.  $\sigma\text{-Normality}$  Of Topological Spaces

**Proof.** Let  $(X, \tau)$  be an upper  $\sigma$ -normal topological space. Then, there is a topology  $\tau_1$  on X such that  $\tau_1 \leq_{\sigma} \tau$  and the couple topological space  $(X, \tau_1, \tau)$  is  $\sigma$ -normal. Thus, by Corollary 3.4, for each pair of disjoint  $\tau$ -closed subsets E, F of X there is a lower semi-continuous function  $f: (X, \tau) \to [0, 1]$  such that f is 0 on E and 1 on F.

Conversely, suppose that for each pair of disjoint closed subsets E, Fof X there is a lower semi-continuous function  $f_{E,F}: (X, \tau) \to [0, 1]$  such that  $f_{E,F}$  is 0 on E and 1 on F. Let  $\mathcal{F}$  be the family of all  $f_{E,F}$ 's where E, F ranges over all disjoint closed subsets of X. Suppose that  $\tau_1$  is the weakest topology such that  $f: (X, \tau_2) \to [0, 1]$  is lower semi-continuous for each  $f \in \mathcal{F}$ . Each  $f \in \mathcal{F}$  is lower semi-continuous on  $(X, \tau)$  and hence  $\tau_1 \subseteq \tau$ . We shows that  $\tau_1 \leq_{\sigma} \tau$ . Let F be an arbitrary  $\tau$ -closed subset of X. For each  $x \in F^c$ , let  $E_x = \{x\}$ . Then,  $f_{E_x,F}: (X, \tau_1) \to [0, 1]$  is lower semi-continuous and so  $f_{E_x,F}^{-1}(\{1\})$  is  $F_{\sigma}$  with respect to  $\tau_1$ . Thus  $F = \bigcap_{x \in F^c} f_{E_x,F}^{-1}(\{1\})$  is also  $F_{\sigma}$  with respect to  $\tau_1$ .

Now we claim that the couple topological space  $(X, \tau_1, \tau)$  is  $\sigma$ -normal. Let E, F be a pair of disjoint  $\tau_1$ -closed subsets of X. We know that  $f_{E,F}: (X, \tau_1) \to [0, 1]$  is lower semi-continuous. Thus,  $W = f_{E,F}^{-1}((\frac{1}{2}, 1])$  is  $\tau_1$ -open. Let  $U = \tau$ -cl  $(W)^c \cap F^c$  and  $V = W \cap E^c$ . Since  $E^c, F^c$  are  $\tau_1$ -open and hence  $\tau$ -open, U, V is a pair of disjoint  $\tau$ -open subsets of X such that  $E \subseteq U$  and  $F \subseteq V$ . To see this, note that  $f_{E,F}(F) = \{1\}$  and so  $F \subseteq W$ , whence  $F \subseteq V$ . Moreover,  $\tau$ -cl  $(W) \cap E$  is the empty set, since  $f_{E,F}$  is 0 on E. Thus  $E \subseteq \tau$ -cl  $(W)^c$  and so  $E \subseteq U$ .

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