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CHARACTERIZATION OF LIE HIGHER DERIVATIONS ON C^*-ALGEBRAS

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(Communicated by Ali Abkar)

Abstract. Let $\mathcal{A}$ be a C^*-algebra and $Z(\mathcal{A})$ the center of $\mathcal{A}$. A sequence $\{L_n\}_{n=0}^{\infty}$ of linear mappings on $\mathcal{A}$ with $L_0 = I$, where I is the identity mapping on $\mathcal{A}$, is called a Lie higher derivation if $L_n[x, y] = \sum_{i+j=n}[L_i x, L_j y]$ for all $x, y \in \mathcal{A}$ and all $n \geqslant 0$. We show that $\{L_n\}_{n=0}^{\infty}$ is a Lie higher derivation if and only if there exist a higher derivation $\{D_n : \mathcal{A} \to \mathcal{A}\}_{n=0}^{\infty}$ and a sequence of linear mappings $\{\Delta_n : \mathcal{A} \to Z(\mathcal{A})\}_{n=0}^{\infty}$ such that $\Delta_0 = 0$, $\Delta_n([x, y]) = 0$ and $L_n = D_n + \Delta_n$ for every $x, y \in \mathcal{A}$ and all $n \geq 0$.
Keywords: Lie Derivations, Lie Higher derivations.
MSC(2010): Primary: 16W25; Secondary: 46L05.

1. Introduction

Let $\mathcal{A}$ be an algebra and $[x, y] = xy - yx$ the *commutator* (the Lie product) of the elements $x, y \in \mathcal{A}$. A linear mapping $d : \mathcal{A} \to \mathcal{A}$ is called a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{A}$. A linear mapping $l : \mathcal{A} \to \mathcal{A}$ is called a *Lie derivation* if $l([x, y]) = [l(x), y] + [x, l(y)]$ for all $x, y \in \mathcal{A}$. Clearly, every derivation is a Lie derivation. Johnson [4] proved that every continuous Lie derivation from a C^*-algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-module M is standard, that is, can be decomposed as the form $d + \delta$, where $d : \mathcal{A} \to M$ is a derivation and δ is a linear map from $\mathcal{A}$ into the center of M vanishing at each commutator. Mathieu and Villena [7] showed that every Lie derivation (without continuity) on a C^*-algebra is standard. In [11], Qi and Hou proved that the same is true for Lie derivations of nest algebras on Banach spaces. For other results, see [3]. A sequence $\{D_n\}_{n=0}^{\infty}$ of linear mappings from $\mathcal{A}$ into $\mathcal{A}$ with $D_0 = I$ is called a *higher derivation* if $D_n(xy) = \sum_{i+j=n} D_i(x)D_j(y)$ for all $x, y \in \mathcal{A}$ and all $n \geqslant 0$. Let d be a derivation on $\mathcal{A}$ and define the sequence $\{D_n\}_{n=0}^{\infty}$ of linear

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mappings on $\mathcal{A}$ by $D_0 = I$ and $D_n = \frac{d^n}{n!}$. Then the Leibnitz rule ensures that $\{D_n\}_{n=0}^{\infty}$ is a higher derivation. Higher derivations were introduced by Hasse and Schmidt [1], and algebraists sometimes call them Hasse-Schmidt derivations. In [8], higher derivations are applied to study generic solving of higher differential equations. For more information about higher derivations and its applications see [2], [5], and [14]. The last author [9] characterized higher derivations in terms of derivations. A sequence $\{L_n\}_{n=0}^{\infty}$ of linear mappings from $\mathcal{A}$ into $\mathcal{A}$ with $L_0 = I$ is called a *Lie higher derivation* if $L_n[x, y] = \sum_{i+j=n}[L_i x, L_j y]$ for all $x, y \in \mathcal{A}$ and all $n \geqslant 0$. Clearly, every higher derivation is a Lie higher derivation but the converse is not true in general. Let $Z(\mathcal{A})$ be the center of $\mathcal{A}$ and $\{D_n\}_{n=0}^{\infty}$ be a higher derivation on $\mathcal{A}$. For any $n \geqslant 0$, let $L_n = D_n + \Delta_n$, where $\{\Delta_n : \mathcal{A} \to Z(\mathcal{A})\}_{n=0}^{\infty}$ is a sequence of linear mappings such that $\Delta_0 = 0$ and $\Delta_n([x, y]) = 0$ for each $x, y \in \mathcal{A}$ and all $n \geqslant 0$. It is easily checked that $\{L_n\}_{n=0}^{\infty}$ is a Lie higher derivation and not a higher derivation if $\Delta_n \neq 0$ for some n. Lie higher derivations of the above form are called proper. The natural problem that one considers in this context is whether or not every Lie higher derivation is proper. In [10], the author discussed the properties of Lie higher derivations. In [12], Qi and Hou showed that each Lie higher derivation is proper on nest algebras. In [6], Li and Shen proved that the same is true for triangular algebras. In this paper we are going to show that every Lie higher derivation on C^*-algebras is standard. This is an extension of Johnson's result in [4].

2. Main results

In the following theorem we give a representation of Lie higher derivations in terms of Lie derivations.

Theorem 2.1. *Let $\{L_n\}_{n=0}^{\infty}$ be a Lie higher derivation on $\mathcal{A}$. Then there exists a sequence $\{l_n\}_{n=1}^{\infty}$ of Lie derivations on $\mathcal{A}$ such that for every $n \geq 1$, we have*

$$L_n = \sum_{i=1}^{n}\left(\sum_{\sum_{j=1}^{i} r_j = n}\left(\Pi_{j=1}^{i}\frac{1}{r_1 + r_2 + \cdots + r_i}\right) l_{r_1} l_{r_2} \ldots l_{r_i}\right), \tag{2.1}$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = n$.

Proof. Let $\{L_n\}_{n=0}^{\infty}$ be a Lie higher derivation on $\mathcal{A}$. First we show that there exists a sequence $\{l_n\}_{n=1}^{\infty}$ of Lie derivations on $\mathcal{A}$ such that $(n+1)L_{n+1} = \sum_{k=0}^{n} l_{k+1} L_{n-k}$ for every $n \geq 0$. We use induction on n. For $n = 0$, We have

$$L_1([x, y]) = \sum_{i+j=1}[L_i x, L_j y] = [L_1 x, y] + [x, L_1 y].$$

Thus if $l_1 := L_1$, then l_1 is a Lie derivation on $\mathcal{A}$. Suppose that l_k is defined and is a Lie derivation for $k \leqslant n$ and $(r+1)L_{r+1} = \sum_{k=0}^{r} l_{k+1}L_{r-k}$, for $r \leqslant n$. Define $l_{n+1} := (n+1)L_{n+1} - \sum_{k=0}^{n-1} l_{k+1}L_{n-k}$. We are going to show that l_{n+1} is a Lie derivation on $\mathcal{A}$. For $x, y \in \mathcal{A}$, we have

$$\begin{aligned}
l_{n+1}([x,y]) &= (n+1)L_{n+1}([x,y]) - \sum_{k=0}^{n-1} l_{k+1}L_{n-k}([x,y]) \\
&= \sum_{i+j=n+1}(n+1)[L_i x, L_j y] - \sum_{k=0}^{n-1} l_{k+1}\Big(\sum_{r+s=n-k}[L_r x, L_s y]\Big) \\
&= \sum_{i+j=n+1}(i+j)[L_i x, L_j y] \\
&- \sum_{k=0}^{n-1}\Big(\sum_{r+s=n-k}[l_{k+1}L_r x, L_s y] + [L_r x, l_{k+1}L_s y]\Big) \\
&= \sum_{i+j=n+1} i[L_i x, L_j y] - \sum_{k=0}^{n-1}\sum_{r+s=n-k}[l_{k+1}L_r x, L_s y] \\
&+ \sum_{i+j=n+1} j[L_i x, L_j y] - \sum_{k=0}^{n-1}\sum_{r+s=n-k}[L_r x, l_{k+1}L_s y] \\
&= K_1 + K_2.
\end{aligned}$$

Note that for $0 \leqslant k \leqslant n-1$ and $r+s = n-k$ if we put $u := r+k$, then $0 \leqslant u \leqslant n$, $u+s = n$, $0 \leqslant k \leqslant u$ and $k \neq n$. Thus by $L_0 = I$, we have

$$\begin{aligned}
K_1 &= \sum_{i+j=n+1} i[L_i x, L_j y] - \sum_{k=0}^{n-1}\sum_{r+s=n-k}[l_{k+1}L_r x, L_s y] \\
&= \sum_{i+j=n+1} i[L_i x, L_j y] - \sum_{u=0}^{n}\sum_{k=0, u+s=n, k\neq n}^{u}[l_{k+1}L_{u-k} x, L_s y] \\
&= \sum_{i+j=n+1} i[L_i x, L_j y] - \sum_{u=0}^{n-1}\sum_{k=0, u+s=n, k\neq n}^{u}[l_{k+1}L_{u-k} x, L_s y] \\
&- \sum_{k=0}^{n-1}[l_{k+1}L_{n-k} x, L_0 y].
\end{aligned}$$

Thus

$$\begin{aligned} & K_1 + \sum_{k=0}^{n-1} [l_{k+1} L_{n-k} x, y] \\ = & \sum_{i+j=n+1} i[L_i x, L_j y] - \sum_{u=0}^{n-1} \sum_{k=0, u+s=n, k\neq n}^{u} [l_{k+1} L_{u-k} x, L_s y] \\ = & \sum_{u=0, u+s=n}^{n} (u+1)[L_{u+1} x, L_s y] - \sum_{u=0}^{n-1} \sum_{k=0, u+s=n, k\neq n}^{u} [l_{k+1} L_{u-k} x, L_s y] \\ = & [(n+1)L_{n+1} x, y] + \sum_{u=0, u+s=n}^{n-1} [((u+1)L_{u+1} x - \sum_{k=0}^{u} l_{k+1} L_{u-k} x), L_s y]. \end{aligned}$$

The second equality above is obtained by replacing i, j by $u+1$ and r, respectively, in the first summation. By our assumption we have $(u+1)L_{u+1}x - \sum_{k=0}^{u} l_{k+1} L_{u-k} x = 0$ for $0 \leqslant u \leqslant n-1$ and $x \in M$. Therefore,

$$\begin{aligned} K_1 &= [(n+1)L_{n+1} x, y] - \sum_{k=0}^{n-1} [l_{k+1} L_{n-k} x, y] \\ &= [((n+1)L_{n+1} x - \sum_{k=0}^{n-1} l_{k+1} L_{n-k} x), y] \\ &= [l_{n+1} x, y]. \end{aligned}$$

By a similar argument we have

$$\begin{aligned} K_2 &= \sum_{i+j=n+1} j[L_i x, L_j y] - \sum_{k=0}^{n-1} \sum_{r+s=n-k} [L_r x, l_{k+1} L_s y] \\ &= [x, l_{n+1} y]. \end{aligned}$$

Thus

$$l_{n+1}([x, y]) = K_1 + K_2 = [l_{n+1} x, y] + [x, l_{n+1} y].$$

Whence l_{n+1} is a derivation and clearly, $(n+1)L_{n+1} = \sum_{k=0}^{n} l_{k+1} L_{n-k}$. Now, Theorem 2.3 of [9] implies that for $n \geq 1$ we have

$$L_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\Pi_{j=1}^{i} \frac{1}{r_1 + r_2 + ... + r_i} \right) l_{r_1} l_{r_2} ... l_{r_i} \right),$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = n$. □

The following lemma is our key to prove our main result.

Lemma 2.2. *([13]) Every derivation on a C^*-algebra annihilates its center.*

Theorem 2.3. *Let $\mathcal{A}$ be a C^*- algebra and $\{L_n\}_{n=0}^{\infty}$ a sequence of linear mappings from $\mathcal{A}$ into $\mathcal{A}$ with $L_0 = I$. Then $\{L_n\}_{n=0}^{\infty}$ is a Lie higher derivation if and only if there exist a higher derivation $\{D_n : \mathcal{A} \to \mathcal{A}\}_{n=0}^{\infty}$ and a sequence of linear mappings $\{\Delta_n : \mathcal{A} \to Z(\mathcal{A})\}_{n=0}^{\infty}$ such that $\Delta_0 = 0$, $\Delta_n([x, y]) = 0$ and $L_n = D_n + \Delta_n$ for every $x, y \in \mathcal{A}$ and all $n \geq 0$.*

Proof. Let $\{L_n\}_{n=0}^{\infty}$ be a Lie higher derivation on $\mathcal{A}$. Define $\Delta_0, d_0 : \mathcal{A} \to \mathcal{A}$ by $\Delta_0 = 0$ and $d_0 = I$. Lie derivations l_{r_i} satisfying (2.1) can be decomposed as $d_{r_i} + \delta_{r_i}$ where $d_{r_i} : \mathcal{A} \to \mathcal{A}$ is a derivation and $\delta_{r_i} : \mathcal{A} \to Z(\mathcal{A})$ vanishes at each commutator, see [7]. Therefore, for $n \geq 1$ we have

$$\begin{aligned} L_n &= \sum_{i=1}^{n}\Bigg(\sum_{\sum_{j=1}^{i} r_j = n}\bigg(\Pi_{j=1}^{i}\frac{1}{r_1 + r_2 + \cdots + r_i}\bigg)(d_{r_1} + \delta_{r_1})(d_{r_2} + \delta_{r_2})...(d_{r_i} + \delta_{r_i})\Bigg) \\ &= \sum_{i=1}^{n}\Bigg(\sum_{\sum_{j=1}^{i} r_j = n}\bigg(\Pi_{j=1}^{i}\frac{1}{r_1 + r_2 + \cdots + r_i}\bigg)d_{r_1}d_{r_2}...d_{r_i}\Bigg) + \Delta_n. \end{aligned}$$

If we define

$$D_n = \sum_{i=1}^{n}\Bigg(\sum_{\sum_{j=1}^{i} r_j = n}\bigg(\Pi_{j=1}^{i}\frac{1}{r_1 + r_2 + \cdots + r_i}\bigg)d_{r_1}d_{r_2}...d_{r_i}\Bigg),$$

then Theorem 2.5 of [9] implies that $\{D_n\}_{n=0}^{\infty}$ is a higher derivation. Clearly, $\Delta_1 = \delta_1$ and by Lemma 2.2 for $n \geq 2$ we have

$$\begin{aligned} \Delta_n &= \sum_{i=1}^{n}\Bigg(\sum_{\sum_{j=1}^{i} r_j = n}\bigg(\Pi_{j=1}^{i}\frac{1}{r_1 + r_2 + \cdots + r_i}\bigg)\delta_{r_1}\delta_{r_2}...\delta_{r_{i-1}}\delta_{r_i}\Bigg) \\ &+ \sum_{i=2}^{n}\Bigg(\sum_{\sum_{j=1}^{i} r_j = n}\bigg(\Pi_{j=1}^{i}\frac{1}{r_1 + r_2 + \cdots + r_i}\bigg)\delta_{r_1}\delta_{r_2}...\delta_{r_{i-1}}d_{r_i}\Bigg). \end{aligned}$$

Therefore, $\Delta_n : \mathcal{A} \to Z(\mathcal{A})$ is defined for every $n \geq 0$, $\Delta_n([x, y]) = 0$ and $L_n = D_n + \Delta_n$ for every $x, y \in \mathcal{A}$ and all $n \geq 0$. The converse is easy to verify. □

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