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ON SPAP-RINGS

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Abstract. In this paper we focus on a special class of commutative local rings called SPAP-rings and study the relationship between this class and other classes of rings. We characterize the structure of modules and especially, the prime submodules of free modules over an SPAP-ring and derive some basic properties. Then we answer the question of Lam and Reyes about strongly Oka ideals family. Finally, we characterize the structure of SPAP-ring in special cases.
Keywords: SPAP-Rings, perfect rings, quasi-Frobenius rings, Gorenstein rings, prime submodules.
MSC(2010): Primary: 13H10; Secondary: 16L30, 16L60.

1. Introduction

Throughout, R will be a commutative ring with identity. Prime ideals and factorization of ideals into prime ideals are two major topics in the history of ring theory. Recently the concept of prime ideals have been generalized and the factorization of ideals into generalized prime ideals have been studied. Bhatwadekar and Sharma in [9], defined a proper ideal I of R to be almost prime if for $a, b \in R$ with $ab \in I - I^2$, either $a \in I$ or $b \in I$.

D. D. Anderson and M. Bataineh in [4], have extended the concept of almost prime ideals to φ-prime ideals and then considered factorization into such ideals. They characterized rings for which all ideals can be factored into almost prime ideals and went on to discover and define a subclass of local rings of this type. They also defined a ring (R, m) which is a *special product of almost prime ideals ring* (abbreviated, SPAP-ring), as a local ring such that for each $x \in m - m^2$, $(x^2) = m^2$ and $m^3 = 0$. They [4], Theorem 22, verified that if R is a Noetherian ring, then every proper ideal of R is a product of almost prime ideals if and only if R is a finite direct product of Dedekind domains, Special Principal Ideal Rings (abbreviated, SPIRs), and (Noetherian) SPAP-rings.

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Note that the SPAP-ring is not Noetherian in general. For example D. D. Anderson and M. Bataineh in [4], Example 20, have constructed an SPAP-ring that is not Noetherian in general.

T.Y. Lam and M. L. Reyes [26] and [27], introduced an elementary " Prime Ideal Principle " which states that for suitable ideal families F in a ring, every ideal maximal with respect to not being in F is prime. This principle not only subsumes and unifies the some familiar results in commutative algebra, but also applies readily to retrieve all other results of the same kind in the literature that the authors are aware of. The key notion making this work possible is that of an Oka family and strongly Oka family of ideals in a ring, defined below. T.Y. Lam and M. L. Reyes gave some examples of rings such that the family of finitely generated ideals are strongly Oka, i.e. for ideals I and A of R, if $I + A$ and $(I : A)$ are finitely generated then, so is I. These include von Neumann regular rings, Bezout domains (in particular, all valuation domains) and Noetherian rings. However, they asked the following question:

Are there other rings R for which the family of finitely generated ideals is strongly Oka?

We answer this question affirmatively for SPAP-rings.

In this paper we consider SPAP-rings and state some basic properties of this class of rings. In section 1, we introduce the concept of strongly irreducible elements and characterize SPAP-rings with element factorization. In section 2, we consider quasi-Frobenius rings (abbreviated QF-rings) and Gorenstein rings, and state some of their properties. We then discuss the relation between these classes of rings and SPAP-rings and give a characterization of SPAP-rings in terms of QF-rings and Gorenstein rings. In section 3, we characterize the structure of R-modules and especially, the prime submodules of a free R-module over SPAP-rings. In section 4, we show that in SPAP-rings the family of finitely generated ideals is strongly Oka and consider some related results that are useful for answering the question of Lam and Reyes. Finally, in section 5, we consider the structure of SPAP-rings in special cases and we prove a version of the Cohen structure theorem for SPAP-rings.

2. SPAP-rings and unique factorization

For a commutative integral domain the terminology concerning divisibility and factorization is more or less standard. Much of the theory of factorization in an integral domain can be generalized to commutative rings with zero divisor, see [1], [2] and [3]. In this section we consider the factorization of elements in SPAP-rings.

Definition 2.1. *Let R be a ring. An element $a \in R$ is called strongly irreducible if $a = bc$ for $b, c \in R$ implies that b or c is unit. A ring R is called strongly atomic if each non-zero and non-unit element of R is a product of a unit and strongly irreducible elements. Also a ring R is called a bounded factorization* *ring (abbreviated BFR) if for each non-zero and non-unit element $a \in R$, there exists a natural number $N(a)$ such that for non-units a_i's, if $a = a_1 ... a_n$ then, $n \leq N(a)$.*

Lemma 2.2. *Let (R, m) be an SPAP-ring with $m \neq m^2$. Then R is strongly atomic.*

Proof. Let $0 \neq x \in R$ be a non-unit. Let $x \in m - m^2$ and assume that $x = rs$ for $r, s \in R$. If r and s are non-units then, $r, s \in m$ and hence $x \in m^2$, a contradiction. Thus r or s must be unit. So x is strongly irreducible. Now let $x \in m^2$. Since $m \neq m^2$, there exists $y \in m - m^2$ and we have $m^2 = (y^2)$. Therefore there exists $r \in R$ such that $x = ry^2$. If $r \in m$ then $x \in m^3 = 0$, a contradiction. Thus r is unit and since $y \in m - m^2$, y is strongly irreducible. So R is strongly atomic.

□

An R-module M is called multiplication, if for each submodule N of M there exists an ideal I of R such that $N = IM$. Clearly cyclic R-modules are multiplication.

Theorem 2.3. *Let R be a ring. Then R is an SPAP-ring with $m^2 \neq 0$ if and only if R is a BFR and contains a maximal ideal m such that m^2 is a minimal ideal of R and for all $x \in m - m^2$, $x \notin ann(x)$.*

Proof. Let (R, m) be an SPAP-ring with $m^2 \neq 0$. By Lemma 1.2, R is strongly atomic and since $m^3 = 0$, hence $N(a) \leq 3$, for all $a \in R$ and therefore R is a BFR. Now we show that m^2 is a minimal ideal. If $m = m^2$ then $m^2 = m^3 = 0$, a contradiction. Therefore $m \neq m^2$ and hence we can select $y \in m - m^2$. By definition of SPAP-ring, $m^2 = (y^2)$. Thus m^2 is a cyclic R-module and hence is a multiplication R-module. Now if J is a submodule(ideal) of m^2, there exists an ideal K of R such that $J = Km^2$. If $K = R$ then $J = m^2$ and if $K \neq R$ then $J = Km^2 \subseteq m^3 = 0$, hence $J = 0$. Therefore m^2 is a minimal ideal of R. Now let $x \in m - m^2$ and $x \in ann(x)$. Since $m^2 = (x^2)$, we have $x^2 = 0$. So $m^2 = 0$, a contradiction.

Conversely, let m be a maximal ideal such that m^2 is minimal and $x \notin ann(x)$, for all $x \in m - m^2$. If $m^3 = m^2$, since m^2 is minimal, then it is cyclic and [13], Proposition 2, Page 38, there exists $0 \neq a \in m$ such that $(1 - a)m^2 = 0$. Now if $x \in m$ then $x^2 = ax^2$ and hence for all natural numbers n, $x^2 = a^n x^2$. Since m is a maximal ideal, a is not unit and this is a contradiction. So $m^3 \neq m^2$ and since m^2 is a minimal ideal we deduce that $m^3 = 0$. Now if $P \in Spec(R)$ then $m^3 = 0 \subseteq P$ and hence $m \subseteq P$. Since m is maximal, hence $m = P$ and so $Spec(R) = \{m\}$. Thus R is a local ring with maximal ideal m and $m^3 = 0$. Now if $x \in m - m^2$, since $x \notin ann(x)$, then $0 \neq x^2 \in m^2$ and so by minimality of m^2, we have $m^2 = (x^2)$. Thus R is an SPAP-ring. □

Remark 2.4. *By proof of Theorem 1.3, for an SPAP-ring (R, m) with $m^2 \neq 0$, the ideal m^2 is a (unique) minimal ideal. Let I be a proper ideal in an SPAP-ring (R, m). If $I \subseteq m^2$, by part one, $I = 0$ or $I = m^2$. If $I \nsubseteq m^2$, then there exists $y \in I - m^2$. Now $m^2 = (y^2)$, hence $m^2 = (y^2) \subseteq I^2$. Thus $I^2 = m^2$. Therefore for every ideal $I \neq R$, we have $I = 0$ or $I = m^2$ or $I^2 = m^2$. Also in the SPAP-ring (R, m), $Spec(R) = \{m\}$ and hence $dim(R) = 0$. Furthermore, by the Cohen Theorem, R is Noetherian if and only if it is Artinian if and only if m is finitely generated.*

3. SPAP-rings, QF-rings and Gorenstein rings

In this section we introduce some classes of rings and consider the relationship between them and SPAP-rings. We begin by the concept of perfect rings. A subset A of a ring R is called T-nilpotent, if for any sequence of elements $\{a_1, a_2, a_3, ...\} \subseteq A$, there exists a natural number $n > 1$ such that $a_1...a_n = 0$. A ring R is called semisimple (or completely reducible) if every ideal of R is a direct summand of R. For a ring R with Jacobson radical $J(R)$, we say that R is perfect if $R/J(R)$ is semisimple and $J(R)$ is T-nilpotent. The pioneering work on perfect rings was done by H. Bass(see [5]). Now we have the following Lemma.

Lemma 3.1. *Let (R, m) be an SPAP-ring. Then R is a perfect ring.*

Proof. Since for an SPAP-ring (R, m), $J(R)^3 = 0$, hence $J(R)$ is T-nilpotent. By [23], Page 345, R is a semiprimary ring and so is perfect. $\square$

Note that, in general the converse of Lemma 2.1 is not true. For example, let F be a field and $R = F \times F$. By [23], Theorem 23.24, R is a perfect ring and since it is not local, it can not be an SPAP-ring.

Lemma 3.2. *Let (R, m) be an SPAP-ring. For an R-module M, the concepts flat, projective, free and faithfully flat are equivalent.*

Proof. Flat means projective, by [5], Theorem 28.4 and Lemma 2.1. Projective R-modules are free, by [5], Corollary 26.7. Clearly free implies faithfully flat and faithfully flat implies flat(also see [14]).

$\square$

Now we assume that m is a finitely generated ideal of R. This leads us to consider a class of rings that are self-injective (a ring R which is an injective R-module). The class of rings that are self-injective has been under close scrutiny by ring theorists and there is a vast literature on the structure of self-injective rings satisfying various conditions. We focus our attention on QF-ring, which are self-injective and Noetherian ring (see [24], for more information). The socle of a an R-module M (abbreviated $Soc(M)$) is defined to be the sum of all simple submodules of M (see [23], [25]). According to Remark 1.4, for an

SPAP-ring (R, m) with $m^2 \neq 0$, we have $Soc(R) = m^2$ and hence $Soc(R)$ is the unique minimal (and so simple R-module) ideal of R.

Lemma 3.3. *Let (R, m) be an SPAP-ring with $m^2 \neq 0$. Then the following statements are equivalent:*
a) m is finitely generated;
b) R is a Noetherian ring;
c) R is an Artinian ring;
d) For every R-module, the concepts projective, injective, free and flat are equivalent;
e) R is a QF-ring.

Proof. By Remark 1.4, (a), (b) and (c) are equivalent. Parts (d) and (e) are equivalent by [24], Theorem 15.1, Theorem 15.9 and Lemma 2.2. Clearly, (e) implies (b). Since for an SPAP-ring (R, m), $Soc(R)$ is a minimal ideal and hence is a simple R-module in R, thus (b) implies (e) by [24], Theorem 15.27. □

Note that if (R, m) is an SPAP-ring with finitely generated m and $m^2 \neq 0$, then R is a QF-ring and so is self-injective.
Now we introduce another important class of local rings and derive some relations between this class of rings and SPAP-rings. A Noetherian local ring R is called a Gorenstein ring if $injdimR < \infty$, where $injdim(R)$ is the injective dimension of R.

Remark 3.4. *By [20], Section 3.4, Theorem 17 and [10], Exercise 3.2.15, a zero dimensional Noetherian ring R is a Gorenstein ring if and only if R has a unique minimal ideal. Also since a QF-ring is Noetherian and $ann(ann(I)) = I$, for every ideal I of R, [24], Theorem 15.1 and Theorem 15.9, hence R is a Gorenstein ring. Thus by Lemma 2.3, every SPAP-ring (R, m) with finitely generated m is a Gorenstein ring.*

The embedding dimension of a local ring (R, m) denoted by $\nu(m)$ is defined as the minimum number of elements of m that generate m as an ideal. Also the codimension of a ring R (denoted by $codim(R)$) is defined to be $\nu(m) - dimR$.

Let (R, m) be a d-dimensional Noetherian local ring and M be a finitely generated R-module. We know that the *Samuel* function $\chi_M^m(n) = length_R(\frac{M}{m^{n+1}M})$, can be expressed for $n >> 0$ as a polynomial in n with rational coefficients and degree equal to $dimM$ and therefore at most d.
It is well-known that, this polynomial is of the form, $\chi_M^m(n) = \frac{e}{d!}n^d +$(terms of lower order), where e is an integer. If $M = R$, the integer number e is called the multiplicity of R and denoted by $e = e(R)$.
By [21], page 3462, If R is a Gorenstein ring with $codimR \geq 2$ then the multiplicity of R is at least $codimR + 2$ and when $codimR \leq 1$, the multiplicity is at least $codimR + 1$. In either case, when equality holds we say that R is Gorenstein of minimal multiplicity. Furthermore, if R is a Gorenstein Artinian

ring then R is of minimal multiplicity if and only if $m^3 = 0$ (see [30] and [35] for more details).

Theorem 3.5. *Let (R, m) be a local ring. Then the following statements are equivalent:*

1) R is an SPAP-ring with finitely generated m and $m^2 \neq 0$;

2) R is a QF-ring with $m^3 = 0$ and for all $x \in m - m^2$, $x \notin ann(x)$;

3) R is a Gorenstein ring of minimal multiplicity and for all $x \in m - m^2$, $x \notin ann(x)$.

Proof. By Lemma 2.3 and Theorem 1.3, (1) implies (2). Suppose that (2) holds. Since $m^3 = 0$, we have $dim(R) = 0$. So by Remark 2.4, R is a Gorenstein ring and is of minimal multiplicity. Now let (3) hold. By Remark 2.4, $m^3 = 0$ and since R is a Gorenstein ring, hence R is Noetherian. Therefore m is finitely generated. Now suppose that $x \in m - m^2$ and $r, s \in m$. If $rs = 0$ then $rs \in (x^2)$. Now assume that $rs \neq 0$. Since $m^3 = 0$, thus $mrs = 0$. Therefore $m \subseteq ann(rs) \neq R$ and hence $m = ann(rs)$. By assumption $x \notin ann(x)$, thus $x^2 \neq 0$ and so $ann(x^2) \neq R$. By a similar argument, $ann(x^2) = m$. Therefore $ann(rs) \subseteq ann(x^2)$. Now by [20], Section 3.4, Theorem 17 and [10], Exercise 3.2.15, R is injective and hence is divisible. Since $ann(rs) \subseteq ann(x^2)$, there exists $t \in R$ such that $x^2 = trs$. Now if $t \in m$ then $x^2 = trs \in m^3 = 0$, a contradiction. Thus $t \notin m$ and therefore t is unit. So $rs \in (x^2)$, thus $m^2 \subseteq (x^2)$ and hence $m^2 = (x^2)$. This shows that R is an SPAP-ring. □

Remark 3.6. *A ring R is called Kasch if every simple R-module is isomorphic to a (minimal) ideal of R. Clearly for an SPAP-ring (R, m), $M = \frac{R}{m}$ is the unique simple R-module (up to isomorphism). If $m \neq m^2 \neq 0$ and $x \in m - m^2$, hence $(x^2) = m^2$ and so the map $1 + m \to x^2$ is an isomorphism between M and m^2. Now m^2 is simple and thus R is a Kasch ring (for more details see [24], Section 8 and [32]). A ring R is said to be secular, if every nonzero R-module has nonzero socle. By [5], Theorem 28.4 and Lemma 2.1, every nonzero module has a nonzero simple submodule and hence its socle is nonzero. Therefore SPAP-rings are secular(for more information see [12], Proposition 22, 10 A).*

4. **Prime submodules of free module over Quasi-Frobenius and SPAP-rings**

A proper submodule N of an R-module M is called prime, if for $r \in R$ and $m \in M$ such that $rm \in N$ we have $m \in N$ or $rM \subseteq N$. One of the most important aims in module theory is to find and distinguish the structure of modules and more recently, prime submodules of a module. For this purpose, authors limit their attentions to a special class of rings and modules. For example, in [15], [16] and [17], the authors determined the structure of prime submodules of a free module over PIDs and UFDs. In this section we do this for QF-rings and specially for SPAP-rings. A submodule N of M is called small (or superfluous) in M, if for every submodule L of M, $N + L = M$ implies $L = M$. Also an R-module M is called a small module, if M is small (or superfluous) in $E(M)$(where $E(M)$ is the injective hull of M). Now let R be a perfect ring. We say that the family of all injective R-modules is closed under small covers, if for any epimorphism $\varphi : M \to E$ such that E is an injective R-module and $Ker\varphi$ is small in M, M is also injective. In the early 1980s, Harada found a new class of Artinian rings which contain QF-rings. A ring R is a Harada ring (abbreviated H-ring), if it is a perfect ring and every non-small R-module contains a non-zero injective submodule(see [7] for more details of Harada rings). Now we consider the relation between SPAP-rings and H-rings.

Let (R, m) be an SPAP-ring with finitely generated m and $m^2 \neq 0$. By Lemma 2.3, R is a QF-ring. Since every QF-ring is H-ring, hence R is an H-ring. In Lemma 3.1, we give another proof of this statement.

Lemma 4.1. *Let (R, m) be an SPAP-ring with finitely generated m and $m^2 \neq 0$. Then R is an H-ring.*

Proof. By Lemma 2.1, R is a perfect ring. Now let $\varphi : M \to E$ be an epimorphism such that E is an injective R-module and $Ker\varphi$ is small in M. So we have the following exact sequence:

$$0 \to Ker\varphi \hookrightarrow M \to E \to 0.$$

Since E is injective, by Lemma 2.3 it is projective and hence there exists a submodule N of M such that $N \cong E$ and $M = Ker\varphi \oplus N$. Since $Ker\varphi$ is small in M, thus $M = N$. Therefore $M \cong E$ and so M is injective. It follows that the family of all injective R-modules is closed under small covers. So by [7], Theorem 3.1.12, R is an H-ring. $\square$

Corollary 4.2. *Let R be a QF-ring. Then every R-module can be expressed as a direct sum of a free module and a small module. In particular, if (R, m) is an SPAP-ring with finitely generated m and $m^2 \neq 0$, then every R-module can be expressed as a direct sum of a free module and a small module.*

Proof. By [24], Theorem 15.9 and [5], Corollary 26.7, an injective R-module over an QF-ring is projective and so is free. Since every QF-ring is H-ring, by [7], Theorem 3.1.12, every R-module can be expressed as a direct sum of a free module and a small module. Now if (R, m) is an SPAP-ring with finitely generated m and $m^2 \neq 0$, by Theorem 2.5, R is QF-ring. $\square$

Let K be a prime submodule of an R-module M. Clearly if $(K : M) = \{r \in R : rM = K\}$ is a maximal ideal of R, then K is a prime submodule of M (for more information see [31]).

Theorem 4.3. *Let (R, m) be a local QF-ring with $Spec(R) = \{m\}$. Let $\phi : F \to \oplus_K R$ be an $R-$isomorphism (i.e. F is a free R-module) and P*

a submodule of F. Then P is a prime submodule of F if and only if there exist two sets I and J with $J \neq \emptyset$ and $K = I \cup J$such that P is isomorphic with $(\oplus_{\alpha\in I} R) \oplus (\oplus_{\beta\in J} m)$ under the isomorphism ϕ.

Proof. Let P be a submodule of F and isomorphic with $(\oplus_{\alpha\in I} R) \oplus (\oplus_{\beta\in J} m)$ which is a submodule of $\oplus_K R$. Since $J \neq \emptyset$, we have $(P : F) \neq R$. But $m(\oplus_K R) \subseteq (\oplus_K R) \oplus (\oplus_{\beta\in J} m)$. Hence $mF \subseteq P$ and we have $(P : F) = m \in Max(R)$. This implies that P is a prime submodule of F.

Conversely, let P be a prime submodule of free R-module F. Since every QF-ring is H-ring, so by [7], Theorem 3.1.12, P can be expressed as a direct sum of a free module and a small module. Let $P = (\oplus_{\alpha\in I} R) \oplus M$, where M is a small module. Since R is an injective R-module and R is Noetherian, by [24], Theorem 3.46 , $\oplus_{\alpha\in I} R$ is an injective R-module and hence it is a direct summand of F. Therefore there exists a submodule N of F, such that $F = (\oplus_{\alpha\in I} R) \oplus N$. Thus N is a projective R-module and since R is local, by [5], Corollary 26.7, is free (note that $M \subseteq N$). Let $N \cong \oplus_{\beta\in J} R$. Then $F \cong (\oplus_{\alpha\in I} R) \oplus (\oplus_{\beta\in J} R)$. Now if M' is the image of M in $(\oplus_{\beta\in J} R)$, hence $P \cong (\oplus_{\alpha\in I} R) \oplus M' \leq (\oplus_{\alpha\in I} R) \oplus (\oplus_{\beta\in J} R)$. Since P is a prime submodule of F and $Spec(R) = \{m\}$, we have $(P : F) = m$. Thus $m((\oplus_{\alpha\in I} R) \oplus (\oplus_{\beta\in J} R)) \subseteq (\oplus_{\alpha\in I} R) \oplus M'$. So we have $m(\oplus_{\beta\in J} R) \subseteq M'$. Then $\oplus_{\beta\in J} m \subseteq M'$. By the injectivity of $\oplus_{\beta\in J} R$ and the definition of injective hull $E(M')$ of M', we have $M' \subseteq E(M') \subseteq \oplus_{\beta\in J} R$. Now since M' is a small module, hence it is a small submodule of $E(M')$ and thus is small in $\oplus_{\beta\in J} R$ (see [29]). Now if $x \in M'$, then $x \in \oplus_{\beta\in J} R$. Therefore $x = (x_\beta)_{\beta\in J}$, where a finite number of x_β's are non-zero.

We now show that $x_\beta \in m$ for $\beta \in J$. We verify this for the first component. The proof for other components is similar. Assume that $x_1 \notin m$. So x_1 is unit and we have $x = (x_1, x_2, \dots, x_n, 0, 0, \dots)$ and hence $x_1^{-1}x = (1, x_1^{-1}x_2, \dots, x_1^{-1}x_n, 0, 0, \dots) \in M'$. We claim that $M' + (0 \oplus (\oplus_{\beta\in J-\{1\}} R)) = \oplus_{\beta\in J} R$. Clearly, $M' + (0 \oplus (\oplus_{\beta\in J-\{1\}} R)) \subseteq \oplus_{\beta\in J} R$. If $y \in \oplus_{\beta\in J} R$, we can write $y = (y_\beta)_{\beta\in J}$, where finite number of y_β's are non- zero. So we have

$$\begin{aligned} y &= (y_1, y_2, \dots, y_n, 0, 0, \dots) = y_1 x_1^{-1}(x_1, x_2, \dots, x_n, 0, 0, \dots) \\ &\quad -(0, y_1 x_1^{-1}x_2, \dots, y_1 x_1^{-1}x_n, 0, 0, \dots) \\ &\quad +(o, y_2, \dots, y_n, 0, 0, \dots) \in M' + (0 \oplus (\oplus_{\beta\in J-\{1\}} R)). \end{aligned}$$

Therefore, $y \in M' + (0 \oplus (\oplus_{\beta\in J-\{1\}} R))$ and thus $M' + (0 \oplus (\oplus_{\beta\in J-\{1\}} R)) = \oplus_{\beta\in J} R$. Now since M' is small in $\oplus_{\beta\in J} R$, hence $(0 \oplus (\oplus_{\beta\in J-\{1\}} R)) = \oplus_{\beta\in J} R$, which is a contradiction. So $x_1 \in m$. Similarly, for all $\beta \in J$, $x_\beta \in m$. Therefore $M' \subseteq \oplus_{\beta\in J} m$ and since $\oplus_{\beta\in J} m \subseteq M'$, we have $M' = \oplus_{\beta\in J} m$. Thus $P \cong (\oplus_{\alpha\in I} R) \oplus M' = (\oplus_{\alpha\in I} R) \oplus (\oplus_{\beta\in J} m)$ and $F = (\oplus_{\alpha\in I} R) \oplus (\oplus_{\beta\in J} R)$. Now if $J = \emptyset$, clearly P is not prime. So $J \neq \emptyset$ and the proof is complete. □

Corollary 4.4. *Let (R, m) be an SPAP-ring with finitely generated m and $m^2 \neq 0$. Let $\phi : F \to \oplus_K R$ be an $R-$isomorphism (i.e. F is a free R-module) and P be a submodule of F. Then P is a prime submodule of F if and only if there exist two sets I and J with $J \neq \emptyset$ and $K = I \cup J$ such that P is isomorphic with $(\oplus_{\alpha \in I} R) \oplus (\oplus_{\beta \in J} m)$ under the isomorphism ϕ.*

Proof. By Theorem 2.5 and Theorem 3.3. $\square$

Corollary 4.5. *Let M be a small submodule of a free R-module $\oplus_{\beta \in J} R$, over a local ring (R, m). Then $M \subseteq \oplus_{\beta \in J} m$.*

Proof. Similar to the proof of Theorem 3.3. $\square$

5. Strongly Oka family and SPAP-rings

In [26] and [27], T. Y. Lam and M. L. Reyes gave some examples of rings such that the family of finitely generated ideals are strongly Oka, i.e. if $I + A$ and $(I : A)$ are finitely generated then so is I. These include von Neumann regular rings, Bezout domains (in particular, all valuation domains) and Noetherian rings. However, they asked the following question:

Are there other rings R for which the family of finitely generated ideals is strongly Oka?

We begin this section by a basic Lemma that shows the relation between SPAP-rings and strongly Oka families.

Lemma 5.1. *Let (R, m) be an SPAP-ring, R is not Noetherian in general, and $I, A \trianglelefteq R$ such that $I + A$ be a proper finitely generated ideal. Then I and A are finitely generated.*

Proof. Since $I + A$ is finitely generated, hence for some $n \in \mathbf{N}$, there exist $a_i \in I$ and $b_i \in A$ such that $I + A = (a_i + b_i)_{i=1}^n$. Suppose that $\{a_i + b_i\}_{i=1}^n$ is a minimal generating set for $I + A$ with minimal number of nonzero b_i. Let k be the number of nonzero b_i. Now let $I + A = (a_1 + b_1, a_2 + b_2, \ldots, a_k + b_k, a_{k+1}, \ldots, a_n)$. Let $x \in I - m^2$, there exist $r_i \in R$ such that $x = r_1(a_1 + b_1) + \cdots + r_k(a_k + b_k) + r_{k+1}a_{k+1} + \cdots + r_n a_n$. If for $i = 1, \ldots, k$, $r_i \in m$ then we have $I \subseteq m^2 + (a_i)_{k+1}^n$ and since m^2 is the unique minimal ideal, hence $I \supseteq m^2 + (a_i)_{i=k+1}^n$. So $I = m^2 + (a_i)_{k+1}^n$ and I is finitely generated. Suppose there exists $x \in I - m^2$ and say, $r_1 \notin m$, then r_1 is unit and we have $a_1 + b_1 = r_1^{-1}(x - (r_2(a_2 + b_2) + \cdots + r_k(a_k + b_k) + r_{k+1}a_{k+1} + \cdots + r_n a_n))$. So we have $I + A = (r_1^{-1}(x - (r_2(a_2 + b_2) + \cdots + r_k(a_k + b_k) + r_{k+1}a_{k+1} + \cdots + r_n a_n)), a_2 + b_2, \ldots, a_k + b_k, a_{k+1}, \ldots, a_n) = (x, a_2 + b_2, \ldots, a_k + b_k, a_{k+1}, \ldots, a_n)$.

Clearly $\{x, a_2 + b_2, \ldots, a_k + b_k, a_{k+1}, \ldots, a_n\}$ is a minimal generating set for $I + A$ with $\{x, a_2, \ldots, a_k, a_{k+1}, \ldots, a_n\} \subseteq I$ and $\{b_2, \ldots, b_k\} \subseteq A$. By minimality of k, this is a contradiction and the proof is complete. So I is finitely generated. Similarly this statement holds for A. $\square$

Let (R, m) be an SPAP-ring, I be a finitely generated proper ideal of R and $J \subseteq I$ be another ideal of R. By Lemma 4.1, J is finitely generated and $\mu(J) \leq \mu(I)$, where $\mu(I)$ denotes the least cardinal μ such that I can be generated by μ elements.

Corollary 5.2. *Let (R, m) be an SPAP-ring. Then the family of finitely generated ideals is strongly Oka.*

Proof. Let $I, A \trianglelefteq R$ and $I + A$ and $(I : A)$ be finitely generated ideals. If $I + A = R$ then $I = (I : R) = (I : I + A) = (I : A)$. So $I = (I : A)$ and hence I is finitely generated. Suppose that $I + A$ is a proper ideal. By Lemma 4.1, I is finitely generated. So the family of finitely generated ideals is an strongly Oka. □

Theorem 5.3. *If every proper (principal) ideal of the ring R is almost prime then the family of finitely generated ideals is strongly Oka.*

Proof. By [4], Theorem 17, R is von Neumann regular or (R, m) is local with $m^2 = 0$. If R is Von Neumann regular then by [26] and [27], the family of finitely generated ideals is strongly Oka. But if (R, m) is a local ring with $m^2 = 0$, then R is an SPAP-ring. Thus by Corollary 4.2, the theorem holds. □

There is also another kind of rings such that the family of finitely generated ideals is strongly Oka. For example, if for all proper ideals I and J of a ring R we have $IJ = J$ or $IJ = I$ or $IJ = 0$, then by [18], Proposition 3, every ideal of R is weakly prime. Since any weakly prime ideal is almost prime, by Theorem 4.3 the family of finitely generated ideals is strongly Oka.

In the proof of Lemma 4.1, we proved that for ideals $J \subseteq I$ of R, if we have a special minimal generating set for I then we can produce a generating set for J by this minimal generating set of I . Now we consider the inverse: if we have a generating set for J, can we extend this set to a generating set for I? To answer this question we need the concept of Steinitz rings. We recall that a Steinitz ring is such that for any free R-module F and any free submodule U of F, F/U is again free (and U a direct summand of F). Now let (R, m) be an SPAP-ring. By Lemma 2.1, R is perfect. Since R is local hence by [28], Theorem, part(e), R is an Steinitz ring. This means that any linearly independent subset of a free R-module F can be extended to a basis of F. So we have a partial answer for above question.

6. **Structure of SPAP-rings**

D. D. Anderson and M. Bataineh in [4], Example 20, construct an SPAP-ring that is not Noetherian as follows:
Let k be an ordered field and $\{x_\alpha\}_{\alpha\in\Lambda}$ a nonempty set of indeterminates. Put $R = k[[\{x_\alpha\}_{\alpha\in\Lambda}]], m = (\{x_\alpha\}_{\alpha\in\Lambda})$ and $I = (\{x_\alpha x_\beta, x_\alpha^2 - x_\beta^2\}_{\alpha\neq\beta}, \{x_\alpha^3\}_\alpha)$. Let $\overline{R} = R/I$. Then $\overline{R}$ is an SPAP-ring. If the index set Λ is infinite then $\overline{m} = \frac{m}{I}$,

the maximal ideal of $\overline{R}$ is not finitely generated. So $\overline{R}$ is not Noetherian. Note that not all SPAP-rings are of this form. For example, let S be a ring and n a maximal ideal of S. Put $R = \frac{S}{n^2}$. Clearly R is a local ring with maximal ideal $\overline{n} = \frac{n}{n^2}$. Since $\overline{n}^2 = 0$, R is an SPAP-ring. Suppose that there exists a field k and $\{x_\alpha\}_{\alpha\in\Lambda}$ is a nonempty set of indeterminates such that $R \cong k[[\{x_\alpha\}_{\alpha\in\Lambda}]]/(\{x_\alpha x_\beta, x_\alpha^2 - x_\beta^2\}_{\alpha\neq\beta}, \{x_\alpha^3\}_\alpha)$. Clearly the maximal ideal of $k[[\{x_\alpha\}_{\alpha\in\Lambda}]]/(\{x_\alpha x_\beta, x_\alpha^2 - x_\beta^2\}_{\alpha\neq\beta}, \{x_\alpha^3\}_\alpha)$ is $\frac{(\{x_\alpha\}_{\alpha\in\Lambda})}{(\{x_\alpha x_\beta, x_\alpha^2 - x_\beta^2\}_{\alpha\neq\beta}, \{x_\alpha^3\}_\alpha)}$ and $(\frac{(\{x_\alpha\}_{\alpha\in\Lambda})}{(\{x_\alpha x_\beta, x_\alpha^2 - x_\beta^2\}_{\alpha\neq\beta}, \{x_\alpha^3\}_\alpha)})^2 \neq 0$, which is contradiction with $\overline{n}^2 = 0$. So R is not of the above form.

We now state a conjecture and verify it in a special case.

Conjecture: For any SPAP-ring (R, m) with finitely generated m and $m^2 \neq 0$. There exists a regular local ring (S, n) and a nonempty subset $\{x_\alpha\}_{\alpha\in\Lambda}$ of n such that $R \cong S/I$, where $I =< \{x_\alpha x_\beta\}_{\alpha\neq\beta}, \{x_\alpha^2\}_{\alpha\neq 1}, \{x_\alpha^2 u_\alpha x_1^2\} >$.

The embedding dimension of a local ring (R, m) denoted by $v(m)$, is the number of elements of a minimal generating set for m. Suppose that I is an ideal of the regular local ring (R, n) such that $I \subseteq n^2$. Put $A := R/I$, $m := n/I$, $k := R/n \cong A/m$. Let $d = dim(A)$, e be its multiplicity and $h = v(m)$ the embedding dimension of A. In this notation we say that an Artinian local ring (A, m), not necessarily Gorenstein, is stretched if $\mu(m^2) = 1$(where μ is the minimum number of generator of an ideal). We denote the Hilbert function of A by $H_A(n) := dim_k \frac{m^n}{m^{n+1}}$, $n \geq 0$. The socle degree of an Artinian ring A is the least integer s such that $H_A(s) \neq 0$, denoted by $s = s(R)$, and the Cohen-Macaulay type of A, denoted by $\tau(A)$, is defined as $\tau(A) := dim_k(0 : m)$.

Sally in [34], studied several properties of stretched local rings and proved a structure theorem for stretched Artinian local rings in the Gorenstein case. J. Elias and G. Valla in [11] proved another structure theorem for stretched Artinian local rings(also see [19]). The following theorem verifies our conjecture in a special case.

Theorem 6.1. *Let (R, m) be an SPAP-ring with finitely generated m and $m^2 \neq 0$. Let $h = v(m)$. If there exists a regular local ring (S, n) with $dim(S) \leq h$ and $char(\frac{S}{n}) \neq 2$ such that $R \cong \frac{S}{I}$, then there exists $\{x_\alpha\}_{1\leq\alpha\leq h} \subseteq n$ such that $(\{x_\alpha\}_{1\leq\alpha\leq h}) = n$ and $R \cong \frac{S}{(\{x_i x_j\}_{i\neq j}, \{x_i^2 - u_i x_1^2\}_{2\leq i\leq h})}$, where $u_i \in U(R)$.*

Proof. Let $\{a_1, \ldots, a_h\}$ be a minimal generating set for m. Since R is an SPAP-ring and $m^2 \neq 0$, there exists $g \in R$ such that $m^2 = (g)$. We know that $\frac{m}{m^2}$ is an $\frac{R}{m}$-vector space, with an $\frac{R}{m}-$ basis $\{a_1 + m^2, \ldots, a_h + m^2\}$. Let $i, j \in \{1, 2, ..., h\}$. Since $a_i a_j \in m^2 = (g)$, there exists $u_{ij} \in R$ such that $a_i a_j = u_{ij} g$. We now define an $\frac{R}{m}$-bilinear form $F : \frac{m}{m^2} \times \frac{m}{m^2} \to \frac{R}{m}$ as follows:

$$F(a_i + m^2, a_j + m^2) = u_{ij} + m.$$

Clearly F is symmetric. Now if $v = b + m^2 = \sum_{j=1}^{n} r_j a_j + m^2$ is an arbitrary element of $\frac{m}{m^2}$ such that $F(v, w) = 0$ for all $w \in \frac{m}{m^2}$, then $F(v, a_i + m^2) = 0$ for all $i = 1, \ldots, h$. Therefore, by the definition of F, $\sum_{j=1}^{n} r_j u_{ji} \in m$. So $ba_i = \sum_{j=1}^{n} r_j u_{ji} g \in m^3 = 0$, for all $i = 1, \ldots, h$ and hence $b \in (0 : m)$. Now, since $(0 : m) = m^2$, we have $b \in m^2$. Thus $v = 0$ and hence F is a non-degenerate form. By [33], Theorem 9.8, since $char(\frac{S}{n} = \frac{R}{m}) \neq 2$, F can be diagonalized. This means that there exists a basis $\{z_1 + m^2, \ldots, z_h + m^2\}$ of $\frac{m}{m^2}$ such that $F(z_i + m^2, z_j + m^2) = 0$ for all $i \neq j$ and $F(z_i + m^2, z_i + m^2)$ is unit for all i(non zero). Now let $F(z_i + m^2, z_i + m^2) = u_i + m$, where u_i is unit in R. From $F(z_i + m^2, z_j + m^2) = 0$ and the definition of F, we have $z_i z_j = 0$ for all $i \neq j$. Since $z_i \in m$, so $z_i = \sum_{k=1}^{n} r_k a_k$. Then $F(z_i + m^2, z_i + m^2) = F(\sum_{k=1}^{n} r_k a_k, \sum_{k=1}^{n} r_k a_k) = \sum_{k=1}^{n} \sum_{k'=1}^{n} r_k r_{k'} F(a_k + m^2, a_{k'} + m^2) = \sum_{k=1}^{n} \sum_{k'=1}^{n} r_k r_{k'} u_{kk'} + m$. Since $F(z_i + m^2, z_i + m^2)$ is unit, so $\sum_{k=1}^{n} \sum_{k'=1}^{n} r_k r_{k'} u_{kk'} \notin m$. Thus $\sum_{k=1}^{n} \sum_{k'=1}^{n} r_k r_{k'} u_{kk'}$ is unit in R, and $z_i^2 = z_i z_i = \sum_{k=1}^{n} \sum_{k'=1}^{n} r_k r_{k'} a_k a_{k'} = \sum_{k=1}^{n} \sum_{k'=1}^{n} r_k r_{k'} u_{kk'} g$. Now, putting $u_i = \sum_{k=1}^{n} \sum_{k'=1}^{n} r_k r_{k'} u_{kk'}$, we have $z_i^2 = u_i g$, and thus for all i, $u_i^{-1} z_i^2 = u_1^{-1} z_1 = g$. So for all i, $z_i^2 = v_i z_1$, where $v_i = u_i u_1^{-1} \in U(R)$ (we can also write $z_i^2 - v_i z_1 = 0$). Now let $R \cong \frac{S}{I}$ with $dim(S) = h$. Then there exist $x_i, w_i \in S$ such that $z_i = x_i + I$ and $v_i = w_i + I$ (clearly $w_i \in U(S)$). Let $J = (\{x_i x_j\}_{i \neq j}, \{x_i^2 - w_i x_1^2\})$ in S. Since $\frac{R}{m} = \frac{S}{n}$, we have $h = dim_{\frac{R}{m}}(\frac{m}{m^2}) = dim_{\frac{S}{n}}(\frac{\frac{n}{I}}{\frac{n^2+I}{I}}) = dim_{\frac{S}{n}}(\frac{n}{n^2+I})$, so $dim_{\frac{S}{n}}(\frac{n}{n^2+I}) = h$ and since (S, n) is a regular local ring of dimension h, we have $dim_{\frac{S}{n}}(\frac{n}{n^2}) = h$. This shows that $dim_{\frac{S}{n}}(\frac{n}{n^2}) = dim_{\frac{S}{n}}(\frac{n}{n^2+I})$, which implies that the surjection $\frac{n}{n^2} \longrightarrow \frac{n}{n^2+I}$, as an $\frac{S}{n}$−space is an isomorphism, and so $n^2 + I = n^2$. Hence $I \subseteq n^2$ and $\frac{m}{m^2} \cong \frac{n}{n^2}$. Now, since $z_i + m^2$ forms a basis for $\frac{R}{m}$−space $\frac{m}{m^2}$, hence $x_i + n^2$ forms a basis for $\frac{n}{n^2}$ as $\frac{S}{n}$−space. Therefore $n = (x_1, x_2, ..., x_h)$. Now we have the following surjection of S-modules

$$\frac{S}{(\{x_i x_j\}_{i \neq j}, \{x_i^2 - w_i x_1^2\}_{2 \leq i \leq h})} \to \frac{S}{I} \to 0.$$

If we show that both of these S-modules have the same finite length $h + 1$, we reach the desired conclusion.

Since $\{z_i + m^2\}$ is a basis for the $\frac{R}{m}$−space $\frac{m}{m^2}$, so clearly the chain $0 \subseteq (z_1) \subseteq (z_1, z_2) \subseteq ... \subseteq (z_1, z_2, ..., z_h) = m \subseteq R$, is a composition series for R as S-module. Hence $length_S(\frac{S}{I}) = length_S(R) = h + 1$. By the definition of $\{x_i\}$, $\{x_i + n^2\}$ is a basis for $\frac{S}{n}$−space $\frac{n}{n^2}$, so $dim_{\frac{S}{n}}(\frac{(x_1, x_2, ..., x_{r+1})}{(x_1, x_2, ..., x_r)}) \leq 1$, and $dim_{\frac{S}{n}}(\frac{J+(x_1, x_2, ..., x_{r+1})}{J+(x_1, x_2, ..., x_r)}) \leq 1$. Thus, after omitting the same factor, the chain $0 \subseteq \frac{J+(x_1)}{J} \subseteq \frac{J+(x_1, x_2)}{J} \subseteq ... \subseteq \frac{J+(x_1, x_2, ..., x_h)}{J} \subseteq \frac{S}{J}$, is a composition series as S-module. So $length_S(\frac{S}{J}) \leq h + 1$, and by the above surjection, we have $length_S(\frac{S}{J}) \geq h + 1$, and hence the equality holds. Now if $dim(S) < h$, put $S' = S[[X]]$ and $I' = (I, X)$. Clearly S' is a regular local ring with $dim(S') = dim(S) + 1$ and $R \cong \frac{S}{I} \cong \frac{S'}{I'}$. By this argument over $dim(S)$ we can assume that $dim(S) = h$. □

A subring A of R is called a coefficient ring of R if A is a complete local ring such that the inclusion $A \hookrightarrow R$ is local and induces an isomorphism on the residue fields and the maximal ideal of A is pA, where $p = char(\frac{R}{m})$(see [8]).

Proposition 6.2. *(Cohen structure theorem for SPAP-rings) Let (R, m) be an SPAP-ring with finitely generated m and $m^2 \neq 0$ such that $char(R) = p^3$ and $h = v(m)$, where p is a prime number. Then there exists a complete discrete valuation ring W such that R is a quotient of $S = W[[x_1, x_2, \ldots, x_{h-1}]]$ as W-algebra.*

Proof. Since R is an Artinian ring and $m^3 = 0$, R is complete with respect to m-adic topology. Now by $char(R) = p^3$, R is inequicharacteristic and hence by [8], Theorem 10.2, there exists a subring A of R and complete discrete valuation ring W with regular parameter p such that A is a coefficient ring of R and $A \cong W$ or $A \cong \frac{W}{p^tW}$, for some $t \geq 2$. Suppose that $p \in m^2$. Since for an SPAP-ring, $m^3 = 0$ hence $p^2 = 0$, is contradiction with $char(R) = p^3$. Therefore $p \notin m^2$, hence $p + m^2$ is a nonzero element of the $\frac{R}{m}$-vector space $\frac{m}{m^2}$. So we can extend it to a basis for $\frac{m}{m^2}$ (since m is finitely generated, this basis is finite). Thus there exists $a_1, a_2, \ldots, a_{h-1} \in m$ such that $\{p + m^2, a_1 + m^2, \ldots, a_{h-1} + m^2\}$ is a basis. Recall that since $char(\frac{R}{m}) = p$, hence by definition of coefficient ring of R, pA is the unique maximal ideal of A and since (A, pA) is a coefficient ring of (R, m) and $(p, a_1, a_2, \ldots, a_{h-1}) = m$, by [8], Lemma 8.3, R is a quotient of $A[[x_1, x_2, \ldots, x_{h-1}]]$ as A-algebras. Since $A \cong W$ or $A \cong \frac{W}{P^tW}$, hence R is a quotient of $W[[x_1, x_2, ..., x_{h-1}]]$. The proof is complete. □

Proposition 6.3. *(Structure of SPAP-rings) Let (R, m) be an SPAP-ring with finitely generated m and $h := v(m)$ where $m^2 \neq 0$. If $char(R) \neq p^2$ and $char(R/m) \neq 2$, where p is a prime number then there exists a regular local ring (S, n) and $\{x_\alpha\}_{1 \leq \alpha \leq h}$ a subset of n such that $R \cong S/I$ and I is minimally generated by the elements $\{x_i x_j\}_{1 \leq i < j \leq h}$, $\{x_j^2\}_{2 \leq j \leq \tau}$ and $\{x_i^2 u_i x_1^2\}_{\tau+1 \leq i \leq h}$, where the u_i are units in R and τ is the Cohen-Macaulay type of A.*

Proof. Since R is an SPAP-ring, we have one of the following cases.

Case 1. $char(R) = 0$. Since R is an Artinian ring and $m^3 = 0$, R is complete with respect to m-adic topology. So by Cohen Structure Theorem [8], Theorem 8.1, R is a quotient of a regular local ring (S, n). Let $R \cong S/I$. If $I \nsubseteq n^2$ then there exists $x \in I - n^2$. Since x is a nonzero divisor, $S' = S/(x)$ is again a regular ring. Put $A \cong S'/I'$ where $I' = I + (x)/(x)$ in S'. Now we can write R as a quotient of a ring of dimension smaller than $dim(S)$. Since $dim(S)$ is finite then there exists an expression $R \cong S/I$ of R as a quotient of a regular local ring (S, n) with $I \subseteq n^2$. Furthermore, in an SPAP-ring since m^2 is cyclic,

$\mu(m^2) = 1$ and hence the ring is stretched. Now since $m^2 \neq 0$, $s = s(R) = 2$. Therefore in all cases $\tau \leq h$, and so by [11], Theorem 3.1, the statement holds.

Case 2. $char(R) = p$. Then char $(R/m) = p$ and hence R is an equicharacteristic complete Noetherian local ring. So by [8], Theorem 8.1, there exists a subfield K of R such that $R \cong K[[x_1, x_2, \ldots, x_n]]/I$ as K-algebras, for some ideal I. By Remark 2.4, $n = v(m) = h$. So R is a quotient of a regular local ring $S = K[[x_1, x_2, \ldots, x_n]]$ such that $dim(S) = h$. Therefore, the Proposition is true by Theorem 5.1.

Case 3. $char(R) = p^3$. If R is an SPAP-ring with $char(R) = p^3$ then by Proposition 5.2, there exists a complete discrete valuation ring W such that R is a quotient of $S = W[[x_1, x_2, \ldots, x_{n-1}]]$. Clearly $dim(S) = n$ and hence by Theorem 5.1, the statement is true. □

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