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ON RESIDUATED LATTICES WITH UNIVERSAL QUANTIFIERS

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Abstract. We consider properties of residuated lattices with universal quantifier and show that, for a residuated lattice X, $(X, \forall)$ is a residuated lattice with a quantifier if and only if there is an m-relatively complete substructure of X. We also show that, for a strong residuated lattice X, $\bigcap\{P_\lambda \mid P_\lambda \text{ is an } m\text{-filter}\} = \{1\}$ and hence that any strong residuated lattice is a subdirect product of a strong residuated lattice with a universal quantifier $\{X/P_\lambda\}$, where P_λ is a prime m-filter. As a corollary of this result, we prove that every strong monadic MTL-algebra (BL- and MV-algebra) is a subdirect product of linearly ordered strong monadic MTL-algebras (BL- and MV-algebras, respectively).
Keywords: residuated lattice, universal quantifier, m-filter.
MSC(2010): Primary: 06B10 ; Secondary: 03G10.

1. Introduction

The notion of monadic MV-algebras (MMV-algebras) was firstly introduced and investigated their properties in [4] as an algebraic semantics for the Łukasiewicz infinite valued logic. Since then, many papers about similar algebraic structures deeply considered their properties for various logics, such as monadic intuitionistic logic ([1]), monadic many valued logic ([2]), monadic basic logic ([3]) and so on. On the other hand these algebraic semantics have common algebras - residuated lattices - as support algebras. In [5], several important results were proved in general forms, in particular a characterization theorem of monadic Rℓ-monoids (Theorem 4 and 5): For an Rℓ-monoids M, there exists a universal quantifier $\forall$ such that $(M, \forall)$ is a monadic Rℓ-monoid if and only if there exists an m-relatively complete substructure M_0 and M satisfies the condition

$$(*) \quad \bigwedge_{i\in I} x_i \to y = \bigvee_{i\in I}(x_i \to y).$$

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In this paper we further generalize the result above to the case of residuated lattices and give a simpler characterization theorem of monadic residuated lattices, which says the condition (*) above is redundant even in the case of residuated lattices: For a residuated lattice X, $(X, \forall)$ is a residuated lattice with a universal quantifier if and only if there is an m-relatively complete substructure of X.

We also show that, for a strong residuated lattice X,

$$\bigcap\{P_\lambda \mid P_\lambda \text{ is an } m\text{−filter}\} = \{1\}$$

and hence that any strong residuated lattice is a subdirect product of strong residuated lattices $\{X/P_\lambda\}$, where P_λ is a prime m-filter. As a corollary of this result, we prove that every strong monadic MTL-algebra (BL- and MV-algebra) is a subdirect product of linearly ordered strong monadic MTL-algebras (BL- and MV-algebras, respectively).

2. **Residuated lattices with universal quantifiers**

We recall a definition of bounded integral commutative residuated lattices. An algebraic structure $(X; \wedge, \vee, \odot, \to, 0, 1)$ is called a bounded integral commutative residuated lattice (simply called *residuated lattice*) if

(1) $(X; \wedge, \vee, 0, 1)$ is a bounded lattice;
(2) $(X; \odot, 1)$ is a commutative monoid;
(3) For all $x, y, z \in L$, $x \odot y \leq z$ if and only if $x \leq y \to z$.

For all $x \in X$, we define x' by $x \to 0$. The following result is easy to prove.

Proposition 2.1. *For all $x, y, z \in X$, we have*

(1) $0' = 1, 1 = 0$,
(2) $(x \vee y)' = x' \wedge y'$,
(3) $x \leq y \iff x \to y = 1$,
(4) $x \leq y \implies x \odot z \leq y \odot z,\ z \odot x \leq z \odot y$,
(5) $x \leq y \implies z \to x \leq z \to y,\ y \to z \leq x \to z$,
(6) $x \to y \leq (y \to z) \to (x \to z)$,
(7) $x \to y \leq (z \to x) \to (z \to y)$.

Some well-known algebras, MTL-algebras, BL-algebras, MV-algebras, Heyting algebras and so on, are considered as algebraic semantics for so-called fuzzy logics, monoidal t-norm logic, Basic logic, many valued logic, intuitionistic logic and so on, respectively.

Any residuated lattice satisfying the divisibility condition

$$\text{(div)}\quad x \odot (x \to y) = x \wedge y$$

is called an Rℓ-monoid ([5]). For example, an MV-algebra is a residuated lattice with satisfying the conditions

(div) : $x \odot (x \to y) = x \wedge y$,
(dn) : $x'' = x$,
(p-lin) : $(x \to y) \vee (y \to x) = 1$.

Moreover, those algebras are axiomatic extensions of residuated lattices as follows:

$$\begin{aligned} \mathrm{MTL} &= \mathrm{RL} + \{\mathrm{p-lin}\} \\ \mathrm{BL} &= \mathrm{RL} + \{\mathrm{div}\} + \{\mathrm{p-lin}\} \\ &= \mathrm{MTL} + \{\mathrm{div}\} \\ \mathrm{MV} &= \ \mathrm{BL} + \{\mathrm{dn}\} \end{aligned}$$

A map $\forall : X \to X$ is called a *universal quantifier* if it satisfies the conditions:

(m1) $\forall x \leq x$,
(m2) $\forall(x \wedge y) = \forall x \wedge \forall y$,
(m3) $\forall((\forall x)') = (\forall x)'$,
(m4) $\forall(\forall x \odot \forall y) = \forall x \odot \forall y$,
(m5) $\forall(x \odot x) = \forall x \odot \forall x$,
(m6) $\forall(x \oplus x) = \forall x \oplus \forall x$, where $x \oplus y = (x' \odot y')'$.

Let $\forall$ be a universal quantifier on a residuated lattice X. Then the algebra $(X; \wedge, \vee, \odot, \to, 0, 1, \forall)$, or briefly $(X, \forall)$, is called a *residuated lattice with a universal quantifier*. We note that $(X, \forall)$ is called a *monadic Rℓ-monoid* in [5] in the case of X being a Rℓ-monoid. Therefore, our notion of residuated lattices with universal quantifiers is a generalization of monadic residuated lattices. We also note that the following result can be proved without using (div).

Proposition 2.2. ([5]) *For any residuated lattice with a universal quantifier* $(X, \forall)$*, we have,*

[(1)]

(1) $\forall 0 = 0, \forall 1 = 1$,
(2) $\forall\forall x = \forall x$,
(3) $x \leq y \Longrightarrow \forall x \leq \forall y$,
(4) $\forall(\forall x \oplus \forall y) = \forall x \oplus \forall y$,
(5) $\forall x \odot \forall y \leq \forall(x \odot y),\ \forall x \oplus \forall y \leq \forall(x \oplus y)$,
(6) $\forall(x \to y) \leq \forall x \to \forall y$,
(7) $\forall x' \leq (\forall x)'$,
(8) $x \odot y \leq z \Longrightarrow \forall x \odot \forall y \leq \forall z$,
(9) $\forall(x' \oplus \forall y) \leq (\forall x)' \oplus \forall y$.

Proof. We only show the last case which was proved in [5] under the assumption of normality:$(x \odot y)'' = x'' \odot y''$ and Rℓ-monoid. We here provide a proof without using such assumptions. Since $\forall(x'' \odot (\forall y)')' \odot (\forall x)'' \odot (\forall y)' \leq (x'' \odot (\forall y)')' \odot x'' \odot (\forall y)' = (x'' \to (\forall y)'') \odot x'' \odot (\forall y)' \leq (\forall y)'' \odot (\forall y)' = 0$, we have $\forall(x'' \odot (\forall y)')' \odot (\forall x)'' \odot (\forall y)' = 0$ and this means that $\forall(x' \oplus \forall y) = \forall(x'' \odot (\forall y)')' \leq ((\forall x)'' \odot (\forall y)')' = (\forall x)' \oplus \forall y$. □

Let $(X, \forall)$ be a residuated lattice with a universal quantifier. We put

$$\forall X = \{\forall x \in X \mid x \in X\}.$$

It follows from the result $\forall\forall x = \forall x$ above that $\forall X$ is identical with the set $\{x \in X \mid \forall x = x\}$.

Lemma 2.3. *The set $\forall X$ is closed under the operations $\wedge, \vee, \odot, \forall, ', 0$ and 1. Thus, $\forall X$ is a subalgebra of the reduct $(X; \wedge, \vee, \odot, \forall, ', 0, 1)$ of the residuated lattice with a universal quantifier $(X, \forall)$.*

Proof. We only show the case of $\forall(x \vee y) = x \vee y$ for $x, y \in \forall X$. Suppose that $x, y \in \forall X$. Since $\forall x = x$ and $\forall y = y$, we have $\forall(x \vee y) \leq x \vee y = \forall x \vee \forall y \leq \forall(x \vee y)$ and thus $\forall(x \vee y) = x \vee y$. □

Proposition 2.4. *For all elements $x, y \in \forall X$, $\forall(x \to y)$ is a greatest element w in $\forall X$ such that $x \odot w \leq y$.*

Proof. Suppose that $x, y \in \forall X$. It is obvious that $x \odot \forall(x \to y) = \forall x \odot \forall(x \to y) \leq \forall y = y$. For all $w \in \forall X$ such that $x \odot w \leq y$, since $w \leq x \to y$, we have $w = \forall w \leq \forall(x \to y)$. This means that $\forall(x \to y)$ is the greatest element $w \in \forall X$ such that $x \odot w \leq y$. □

We note that $(\forall X; \wedge, \vee, \odot, \to_\forall, 0, 1)$ is a residuated lattice, where $\to_\forall$ is defined by $x \to_\forall y = \forall(x \to y)$ for all $x, y \in \forall X$.

According to [5], we define an m-relatively complete substructure. Let $(X, \forall)$ be a residuated lattice and further X_0 be a subalgebra of the reduct $(X; \wedge, \vee, \odot, ', 0, 1)$, which is moreover a residuated lattice. We call X_0 a *relatively complete substructure* of X, if for any $a \in X$ there exists a greatest element of the set $\{x \in X_0 \mid x \leq a\}$. Further, X_0 is called an *m-relatively complete substructure* of X if X_0 is a relatively complete substructure and the following conditions are satisfied:

(MRL1) For any $a \in X$ and $x \in X_0$ such that $x \leq a \odot a$ there is $v \in X_0$ such that $v \leq a$ and $x \leq v \odot v$.
(MRL2) For any $a \in X$ and $x \in X_0$ such that $x \leq a \oplus a$ there is $v \in X_0$ such that $v \leq a$ and $x \leq v \oplus v$.

It is easy to prove the next result.

Theorem 2.5. *If $(X, \forall)$ is a residuated lattice with a universal quantifier, then $\forall X$ is an m-relatively complete substructure of X.*

Conversely, we have the following result.

Theorem 2.6. *Let X be a residuated lattice. If there exists an m-relatively complete substructure X_0, then the algebra $(X, \forall)$ is a residuated lattice with a universal quantifier, where $\forall$ operator is defined by $\forall a = \max\{x \in X_0 \mid x \leq a\}$.*

Proof. Suppose that X_0 is an m-relatively complete substructure of a residuated lattice X. Since X_0 is also a relatively complete substructure, we note that there is a greatest element of the set $\{x \in X_0 \,|\, x \leq a\}$ for all $a \in X$. We denote such an element by $\forall a$, that is,

$$\forall a = \max\{x \in X_0 \,|\, x \leq a\}.$$

We show that $\forall$-operator is a monadic operator. We only show the cases of (m2), (m5) and (m6).

Case of (m2): $\forall(a \wedge b) = \forall a \wedge \forall b$ for $a, b \in X$. For all $u \in X_0$, if $u \leq a \wedge b$, since $u \leq a$ and $u \leq b$, then $u \leq \forall a$ and $u \leq \forall b$ and thus $u \leq \forall a \wedge \forall b$. This means that $\forall(a \wedge b) \leq \forall a \wedge \forall b$. Conversely, if $u \leq \forall a \wedge \forall b$, since $u \leq \forall a$ and $u \leq \forall b$, then we have $u \leq a \wedge b$ and hence $u \leq \forall(a \wedge b)$. Since X_0 is closed under $\wedge$ and $\forall a \wedge \forall b \in X_0$, it follows that $\forall(a \wedge b) \leq \forall a \wedge \forall b$.

For the case of (m5):$\forall(a \odot a) = \forall a \odot \forall a$, if we take any $u \in X_0$ such that $u \leq \forall a \odot \forall a$, since $u \leq a \odot a$ and $\forall a \odot \forall a \in X_0$, we have $\forall a \odot \forall a \leq \forall(a \odot a)$. Conversely, suppose that $x \leq a \odot a$ for $x \in X_0$. It follows from (MRL1) that there exists $v \in X_0$ such that $v \leq a$ and $x \leq v \odot v$. Since $v = \forall v \leq \forall a$ and $x \leq \forall a \odot \forall a$. This means that $\forall(a \odot a) \leq \forall a \odot \forall a$. Therefore $\forall(a \odot a) = \forall a \odot \forall a$.

For the last case (m6): $\forall(a \oplus a) = \forall a \oplus \forall a$. For any $u \in X_0$ such that $u \leq \forall a \oplus \forall a$, since $\forall a \leq a$, we have $u \leq a \oplus a$ and hence $u \leq \forall(a \oplus a)$ by definition of $\forall$. We note that $\forall a \in X_0$ and X_0 is closed under $\odot$ and $'$. This implies $\forall a \oplus \forall a = ((\forall a)' \odot (\forall a)')' \in X_0$ and hence that $\forall a \oplus \forall a \leq \forall a \oplus a)$. Conversely, if $u \leq a \oplus a$ for $u \in X_0$ then there exists $v \in X_0$ such that $v \leq a$ and $u \leq v \oplus v$ by (MRL2). Since $v = \forall v \leq \forall a$, we have $u \leq v \oplus v \leq \forall a \oplus \forall a$ and hence $\forall(a \oplus a) \leq \forall a \oplus \forall a$ by definition of $\forall$. □

We note that the result above is stronger than that of Theorem 4 and 5 in [5], where the same result was proved under the conditions (div) of Rℓ-monoid and

$$\bigwedge_{i \in I} x_i \to y = \bigvee_{i \in I} (x_i \to y).$$

Our proof does not require such assumptions to get the result. Thus, we have a characterization theorem of residuated lattices with universal quantifiers.

Theorem 2.7. *For a residuated lattice X, there exists a universal quantifier $\forall$ satisfying (m1)-(m6) if and only if there is an m-relatively complete substructure of X.*

3. Filter and m-filter

Let $(X, \forall)$ be a residuated lattice with a universal quantifier. A non-empty subset F of X is called a *filter* of X if it satisfies the conditions

[(F1)]

If $x, y \in F$ then $x \odot y \in F$;

(2) If $x \in F$ and $x \leq y$ then $y \in F$.

Moreover a filter F is called an *m-filter* of X if it satisfies the condition that $x \in F$ implies $\forall x \in F$.

For a non-empty subset $S \subseteq X$, By $[S)$ we mean the smallest filter containing S. Similarly we denote $[S)_m$ the smallest m-filter containing S. We also denote the class of all filters of X by $Fil(X)$ and the class of all m-filters by $Fil_m(X)$. It is easy to show that $Fil(X)$ and $Fil_m(X)$ are both distributive lattices with respect to the set-inclusion order.

It is easy to show that

Proposition 3.1. *For a non-empty subset S of a monadic residuated lattice $(X, \forall)$,*

(1) $[S) = \{x \in X \,|\, \text{there are elements } s_1, \cdots, s_n \in S \text{ such that } s_1 \odot \cdots \odot s_n \leq x\}$

(2) $[S)_m = [\forall S)$

Proof. We only show that $[\forall S)$ is an m-filter in the case (2). Let $x \in [\forall S)$. There are elements $s_i \in S$ such that $\forall s_1 \odot \cdots \odot \forall s_n \leq x$. Since $\forall(\forall s_1 \odot \cdots \odot \forall s_n) \leq \forall x$ and $\forall(\forall s_1 \odot \cdots \odot \forall s_n) = \forall s_1 \odot \cdots \odot \forall s_n$, we have $\forall s_1 \odot \cdots \odot \forall s_n \leq \forall x$. This means that $\forall x \in [\forall S)$ and thus $[\forall S)$ is the m-filter. □

Let F be an m-filter of X. For all $x, y \in X$, we define $x \equiv_F y$ by $x \to y, y \to x \in F$. It is clear that the relation $\equiv_F$ is a congruence. Since the class of all monadic residuated lattices forms a variety, a quotient algebra X/F by the congruence $\equiv_F$ induced from the m-filter is also a monadic residuated lattice. Moreover, it is easy to prove that $Fil_m(X)$ is isomorphic to the set $Con(X)$ of all congruences, that is,,

$$Fil_m(X) \cong Con(X).$$

Lemma 3.2. *Let F be an m-filter of X and $a \in X$. Then the smallest m-filter $[F \cup \{a\})_m$ containing $F \cup \{a\}$ is*

$$\{x \in X \,|\, \text{there exist } n \geq 1 \text{ and } u \in F \text{ such that } u \odot (\forall a)^n \leq x\} = F \vee [\forall a).$$

Proof. The result can be proved easily from the fact $[F \cup \{a\})_m = [F \cup \{\forall a\}) = F \vee [\forall a)$. □

A residuated lattice with a universal quantifier $(X, \forall)$ is called *strong* ([5]) if $\forall(x \vee y) = \forall x \vee \forall y$ for $x, y \in X$. An m-filter P is called *prime* if $x \vee y \in P$ implies $x \in P$ or $y \in P$ for all $x, y \in X$. It is easy to prove that a filter P is prime if and only if $P = F \cap G$ implies $P = F$ or $P = G$ for any filter F and G of X.

Lemma 3.3. *Let $(X, \forall)$ be a strong residuated lattice with a universal quantifier and $a \in X$ such that $a \neq 1$. Then there is a prime m-filter P such that $a \notin P$.*

Proof. Let $\Gamma = \{F \,|\, a \notin F, F \text{ is an } m-\text{filter}\}$. It follows from Zorn's lemma that there exists a maximal element P in Γ. We only show that P is prime. Suppose that P is not prime. There are $x, y \in X$ such that $x \vee y \in P$ but $x, y \notin P$. Since P is maximal, we have $a \in [P \cup \{x\})_m = P \vee [\forall x)$ and $a \in [P \cup \{y\})_m = P \vee [\forall y)$. It follows from strong property that

$$\begin{aligned} a &\in (P \vee [\forall x)) \wedge (P \vee [\forall y)) \\ &= P \vee ([\forall x) \wedge [\forall y)) \\ &= P \vee [\forall x \vee \forall y) \\ &= P \vee [\forall (x \vee y)) = P. \end{aligned}$$

But this is a contradiction. Hence P is prime. □

If we take the class $\{P_\lambda\}_{\lambda \in \Lambda}$ of all prime m-filters of X, then it follows from the result above that $\bigcap_{\lambda \in \Lambda} P_\lambda = \{1\}$. It follows from the above

Theorem 3.4. *Any strong residuated lattice with a universal quantifier is a subdirect product of a strong residuated lattice with a universal quantifier X/P_λ, where $\{P_\lambda\}_{\lambda \in \Lambda}$ is the set of all prime m-filters of X.*

If X satisfies the condition (p-lin): $(x \to y) \vee (y \to x) = 1$, then it is easy to show that the quotient algebra X/P by a prime m-filter P is linearly ordered monadic residuated lattice. It follows that

Corollary 3.5.

(1) *Every strong monadic MTL-algebra is a subdirect product of linearly ordered strong monadic MTL-algebras.*
(2) *Every strong monadic BL-algebra is a subdirect product of linearly ordered strong monadic BL-algebras.*
(3) *Every strong monadic MV-algebra is a subdirect product of linearly ordered strong monadic MV-algebras.*

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