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Author(s):
M. Kondo

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ON RESIDUATED LATTICES WITH UNIVERSAL QUANTIFIERS

M. KONDO

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Abstract. We consider properties of residuated lattices with universal quantifier and show that, for a residuated lattice $X$, $(X, \forall)$ is a residuated lattice with a quantifier if and only if there is an $m$-relatively complete substructure of $X$. We also show that, for a strong residuated lattice $X$, $\bigcap\{P_\lambda | P_\lambda$ is an $m$-filter} = $\{1\}$ and hence that any strong residuated lattice is a subdirect product of a strong residuated lattice with a universal quantifier $\langle X/P_\lambda \rangle$, where $P_\lambda$ is a prime $m$-filter. As a corollary of this result, we prove that every strong monadic MTL-algebra (BL- and MV-algebra) is a subdirect product of linearly ordered strong monadic MTL-algebras (BL- and MV-algebras, respectively).

Keywords: residuated lattice, universal quantifier, $m$-filter.


1. Introduction

The notion of monadic MV-algebras (MMV-algebras) was firstly introduced and investigated their properties in [4] as an algebraic semantics for the Lukasiewicz infinite valued logic. Since then, many papers about similar algebraic structures deeply considered their properties for various logics, such as monadic intuitionistic logic ([1]), monadic many valued logic ([2]), monadic basic logic ([3]) and so on. On the other hand these algebraic semantics have common algebras - residuated lattices - as support algebras. In [5], several important results were proved in general forms, in particular a characterization theorem of monadic $R\ell$-monoids (Theorem 4 and 5): For an $R\ell$-monoids $M$, there exists a universal quantifier $\forall$ such that $(M, \forall)$ is a monadic $R\ell$-monoid if and only if there exists an $m$-relatively complete substructure $M_0$ and $M$ satisfies the condition

\[
(*) \quad \bigwedge_{i \in I} x_i \rightarrow y = \bigvee_{i \in I} (x_i \rightarrow y).
\]
In this paper we further generalize the result above to the case of residuated lattices and give a simpler characterization theorem of monadic residuated lattices, which says the condition (*) above is redundant even in the case of residuated lattices: For a residuated lattice $X$, $(X, \forall)$ is a residuated lattice with a universal quantifier if and only if there is an $m$-relatively complete substructure of $X$.

We also show that, for a strong residuated lattice $X$, 

$$\bigcap \{ P_\lambda \mid P_\lambda \text{ is an } m\text{-filter} \} = \{1\}$$

and hence that any strong residuated lattice is a subdirect product of strong residuated lattices $\{X/P_\lambda\}$, where $P_\lambda$ is a prime $m$-filter. As a corollary of this result, we prove that every strong monadic MTL-algebra (BL- and MV-algebra) is a subdirect product of linearly ordered strong monadic MTL-algebras (BL- and MV-algebras, respectively).

2. Residuated lattices with universal quantifiers

We recall a definition of bounded integral commutative residuated lattices. An algebraic structure $(X; \wedge, \lor, \cdot, 0, 1)$ is called a bounded integral commutative residuated lattice (simply called *residuated lattice*) if

1. $(X; \wedge, \lor, 0, 1)$ is a bounded lattice;
2. $(X; \cdot, 1)$ is a commutative monoid;
3. For all $x, y, z \in L$, $x \cdot y \leq z$ if and only if $x \leq y \rightarrow z$.

For all $x \in X$, we define $x'$ by $x \rightarrow 0$. The following result is easy to prove.

**Proposition 2.1.** For all $x, y, z \in X$, we have

1. $0' = 1, 1 = 0$,
2. $(x \lor y)' = x' \land y'$,
3. $x \leq y \iff x \rightarrow y = 1$,
4. $x \leq y \implies x \circ z \leq y \circ z, z \circ x \leq z \circ y$,
5. $x \leq y \implies z \rightarrow x \leq z \rightarrow y, y \rightarrow z \leq x \rightarrow z$,
6. $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$,
7. $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$.

Some well-known algebras, MTL-algebras, BL-algebras, MV-algebras, Heyting algebras and so on, are considered as algebraic semantics for so-called fuzzy logics, monoidal t-norm logic, Basic logic, many valued logic, intuitionistic logic and so on, respectively.

Any residuated lattice satisfying the divisibility condition

$$(\text{div}) \ x \circ (x \rightarrow y) = x \land y$$

is called an Rℓ-monoid ([5]). For example, an MV-algebra is a residuated lattice with satisfying the conditions
We only show the last case which was proved in [8],

\[(\text{div}) : x \odot (x \rightarrow y) = x \land y,\]

\[(\text{dn}) : x^\prime\prime = x,\]

\[(\text{p-lin}) : (x \rightarrow y) \lor (y \rightarrow x) = 1.\]

Moreover, those algebras are axiomatic extensions of residuated lattices as follows:

\[\text{MTL} = \text{RL} + \{p - \text{lin}\}\]
\[\text{BL} = \text{RL} + \{\text{div}\} + \{p - \text{lin}\}\]
\[\text{MV} = \text{BL} + \{\text{dn}\}\]

A map \(\forall : X \rightarrow X\) is called a universal quantifier if it satisfies the conditions:

\[
\begin{align*}
(\text{m1}) & \forall x \leq x, \\
(\text{m2}) & \forall(x \land y) = \forall x \land \forall y, \\
(\text{m3}) & \forall(\forall x) = \forall x, \\
(\text{m4}) & \forall(\forall x \lor y) = \forall x \lor \forall y, \\
(\text{m5}) & \forall(x \lor x) = \forall x \lor \forall x, \\
(\text{m6}) & \forall(x \lor x) = \forall x \lor \forall x, \text{ where } x \lor y = (x' \lor y')'.
\end{align*}
\]

Let \(\forall\) be a universal quantifier on a residuated lattice \(X\). Then the algebra \((X; \land, \lor, \odot, \rightarrow, 0, 1, \forall)\), or briefly \((X, \forall)\), is called a residuated lattice with a universal quantifier. We note that \((X, \forall)\) is called a monadic \(\text{RL}\text{-monoid}\) in [5] in the case of \(X\) being a \(\text{RL}\text{-monoid}\). Therefore, our notion of residuated lattices with universal quantifiers is a generalization of monadic residuated lattices. We also note that the following result can be proved without using \(\text{(div)}\).

**Proposition 2.2.** ([5]) For any residuated lattice with a universal quantifier \((X, \forall)\), we have,

\[
\begin{align*}
\text{(1)} & \forall 0 = 0, \forall 1 = 1, \\
\text{(2)} & \forall x = \forall x, \\
\text{(3)} & x \leq y \implies \forall x \leq \forall y, \\
\text{(4)} & \forall(\forall x \land y) = \forall x \land \forall y, \\
\text{(5)} & \forall x \odot \forall y \leq \forall(x \odot y), \forall x \odot \forall y \leq \forall(x \odot y), \\
\text{(6)} & \forall(x \rightarrow y) \leq \forall x \rightarrow \forall y, \\
\text{(7)} & \forall x' \leq (\forall x)', \\
\text{(8)} & x \odot y \leq z \implies \forall x \odot \forall y \leq \forall z, \\
\text{(9)} & \forall(x' \land \forall y) \leq (\forall x)' \land \forall y.
\end{align*}
\]

**Proof.** We only show the last case which was proved in [5] under the assumption of normality:\((x \odot y)^\prime \prime = x'' \odot y''\) and \(\text{RL}\text{-monoid}\). We here provide a proof without using such assumptions. Since \(\forall(x'' \odot (\forall y)')' \odot (\forall y)' \leq (x'' \odot (\forall y)')' \odot x'' \odot (\forall y)' = (x'' \rightarrow (\forall y)')' \odot x'' \odot (\forall y)' \leq (\forall y)'' \odot (\forall y)' = 0\), we have \(\forall(x'' \odot (\forall y)')' \odot (\forall y)' \leq (\forall x)' \odot (\forall y)' = 0\) and this means that \(\forall(x' \lor \forall y) = \forall(x'' \odot (\forall y)')' \leq ((\forall x)' \odot (\forall y)'')' = (\forall x)' \odot \forall y\). \(\square\)
Let $(X, \forall)$ be a residuated lattice with a universal quantifier. We put
\[ \forall X = \{ \forall x \in X \mid x \in X \}. \]
It follows from the result $\forall \forall x = \forall x$ above that $\forall X$ is identical with the set
\[ \{ x \in X \mid \forall x = x \}. \]

**Lemma 2.3.** The set $\forall X$ is closed under the operations $\land, \lor, \odot, \forall', 0$ and $1$. Thus, $\forall X$ is a subalgebra of the reduct $(X; \land, \lor, \odot, \forall', 0, 1)$ of the residuated lattice with a universal quantifier $(X, \forall)$.

**Proof.** We only show the case of $\forall(x \lor y) = x \lor y$ for $x, y \in \forall X$. Suppose that $x, y \in \forall X$. Since $\forall x = x$ and $\forall y = y$, we have $\forall(x \lor y) = \forall x \lor \forall y \leq \forall(x \lor y)$ and thus $\forall(x \lor y) = x \lor y$. □

**Proposition 2.4.** For all elements $x, y \in \forall X$, $\forall(x \rightarrow y)$ is a greatest element $w$ in $\forall X$ such that $x \odot w \leq y$.

**Proof.** Suppose that $x, y \in \forall X$. It is obvious that $x \odot \forall(x \rightarrow y) = \forall x \odot \forall(x \rightarrow y) \leq \forall y = y$. For all $w \in \forall X$ such that $x \odot w \leq y$, since $w \leq x \rightarrow y$, we have $w = \forall w \leq \forall(x \rightarrow y)$. This means that $\forall(x \rightarrow y)$ is the greatest element $w \in \forall X$ such that $x \odot w \leq y$. □

We note that $(\forall X; \land, \lor, \odot, \rightarrow_\forall, 0, 1)$ is a residuated lattice, where $\rightarrow_\forall$ is defined by $x \rightarrow_\forall y = \forall(x \rightarrow y)$ for all $x, y \in \forall X$.

According to [5], we define an $m$-relatively complete substructure. Let $(X, \forall)$ be a residuated lattice and further $X_0$ be a subalgebra of the reduct $(X; \land, \lor, \odot, \forall', 0, 1)$, which is moreover a residuated lattice. We call $X_0$ a *relatively complete substructure* of $X$, if for any $a \in X$ there exists a greatest element of the set $\{ x \in X_0 \mid x \leq a \}$. Further, $X_0$ is called an *$m$-relatively complete substructure* of $X$ if $X_0$ is a relatively complete substructure and the following conditions are satisfied:

- (MRL1) For any $a \in X$ and $x \in X_0$ such that $x \leq a \circ a$ there is $v \in X_0$ such that $v \leq a$ and $x \leq v \circ v$.
- (MRL2) For any $a \in X$ and $x \in X_0$ such that $x \leq a \odot a$ there is $v \in X_0$ such that $v \leq a$ and $x \leq v \odot v$.

It is easy to prove the next result.

**Theorem 2.5.** If $(X, \forall)$ is a residuated lattice with a universal quantifier, then $\forall X$ is an $m$-relatively complete substructure of $X$.

Conversely, we have the following result.

**Theorem 2.6.** Let $X$ be a residuated lattice. If there exists an $m$-relatively complete substructure $X_0$, then the algebra $(X, \forall)$ is a residuated lattice with a universal quantifier, where $\forall$ operator is defined by $\forall a = \max\{ x \in X_0 \mid x \leq a \}$. 
Proof. Suppose that $X_0$ is an $m$-relatively complete substructure of a residuated lattice $X$. Since $X_0$ is also a relatively complete substructure, we note that there is a greatest element of the set $\{x \in X_0 \mid x \leq a\}$ for all $a \in X$. We denote such an element by $\forall a$, that is,

$$\forall a = \max\{x \in X_0 \mid x \leq a\}.$$ 

We show that $\forall$-operator is a monadic operator. We only show the cases of (m2), (m5) and (m6).

Case of (m2): $\forall(a \land b) = \forall a \land \forall b$ for $a, b \in X$. For all $u \in X_0$, if $u \leq a \land b$, since $u \leq a$ and $u \leq b$, then $u \leq \forall a$ and $u \leq \forall b$ and thus $u \leq \forall a \land \forall b$. This means that $\forall(a \land b) \leq \forall a \land \forall b$. Conversely, if $u \leq \forall a \land \forall b$, since $u \leq \forall a$ and $u \leq \forall b$, then we have $u \leq a \land b$ and hence $u \leq \forall(a \land b)$. Since $X_0$ is closed under $\land$ and $\forall a \land \forall b \in X_0$, it follows that $\forall(a \land b) \leq \forall a \land \forall b$.

For the case of (m5): $\forall(a \circ a) = \forall a \circ \forall a$, if we take any $u \in X_0$ such that $u \leq \forall a \circ \forall a$, since $u \leq a \circ a$ and $\forall a \circ \forall a \in X_0$, we have $\forall a \circ \forall a \leq \forall(a \circ a)$. Conversely, suppose that $x \leq a \circ a$ for $x \in X_0$. It follows from (MRL1) that there exists $v \in X_0$ such that $v \leq a$ and $x \leq v \circ v$. Since $v = \forall v \leq \forall a$ and $x \leq \forall a \circ \forall a$. This means that $\forall(a \circ a) \leq \forall a \circ \forall a$. Therefore $\forall(a \circ a) = \forall a \circ \forall a$.

For the last case (m6): $\forall(a \circ a) = \forall a \circ \forall a$. For any $u \in X_0$ such that $u \leq \forall a \circ \forall a$, since $\forall a \leq a$, we have $u \leq a \circ a$ and hence $u \leq \forall(a \circ a)$ by definition of $\forall$. We note that $\forall a \in X_0$ and $X_0$ is closed under $\circ$ and $'$. This implies $\forall a \circ \forall a = ((\forall a)' \circ (\forall a)')' \in X_0$ and hence $\forall a \circ \forall a \leq \forall a \circ a$.

Conversely, if $u \leq a \circ a$ for $u \in X_0$ then there exists $v \in X_0$ such that $v \leq a$ and $u \leq v \circ v$ by (MRL2). Since $v = \forall v \leq \forall a$, we have $u \leq v \circ v \leq \forall a \circ \forall a$ and hence $\forall(a \circ a) \leq \forall a \circ \forall a$ by definition of $\forall$. \hfill \Box

We note that the result above is stronger than that of Theorem 4 and 5 in [5], where the same result was proved under the conditions (div) of $\forall$-monoid and

$$\bigwedge_{i \in I} x_i \rightarrow y = \bigvee_{i \in I} (x_i \rightarrow y).$$

Our proof does not require such assumptions to get the result. Thus, we have a characterization theorem of residuated lattices with universal quantifiers.

**Theorem 2.7.** For a residuated lattice $X$, there exists a universal quantifier $\forall$ satisfying (m1)-(m6) if and only if there is an $m$-relatively complete substructure of $X$.

3. Filter and $m$-filter

Let $(X, \forall)$ be a residuated lattice with a universal quantifier. A non-empty subset $F$ of $X$ is called a filter of $X$ if it satisfies the conditions

\[ [(F1)] \]

If $x, y \in F$ then $x \circ y \in F$;

$$\forall a = \max\{x \in X_0 \mid x \leq a\}.$$
(2) If \( x \in F \) and \( x \leq y \) then \( y \in F \).

Moreover a filter \( F \) is called an \textit{m-filter} of \( X \) if it satisfies the condition that \( x \in F \) implies \( \forall x \in F \).

For a non-empty subset \( S \subseteq X \), By \( [S] \) we mean the smallest filter containing \( S \). Similarly we denote \( [S]_m \) the smallest \( m \)-filter containing \( S \). We also denote the class of all filters of \( X \) by \( Fil(X) \) and the class of all \( m \)-filters by \( Fil_m(X) \).

It is easy to show that \( Fil(X) \) and \( Fil_m(X) \) are both distributive lattices with respect to the set-inclusion order.

It is easy to show that

**Proposition 3.1.** For a non-empty subset \( S \) of a monadic residuated lattice \((X, \forall)\),

\[ 1. \quad [S] = \{ x \in X \mid \text{there are elements } s_1, \ldots, s_n \in S \text{ such that } s_1 \odot \cdots \odot s_n \leq x \}\]

\[ 2. \quad [S]_m = [\forall S] \]

**Proof.** We only show that \( [\forall S] \) is an \( m \)-filter in the case (2). Let \( x \in [\forall S] \).

There are elements \( s_1, \ldots, s_n \in S \) such that \( \forall s_1 \odot \cdots \odot s_n \leq x \). Since \( \forall (s_1 \odot \cdots \odot s_n) \leq \forall x \) and \( \forall(s_1 \odot \cdots \odot s_n) = \forall s_1 \odot \cdots \odot s_n \), we have \( \forall s_1 \odot \cdots \odot s_n \leq \forall x \).

This means that \( \forall x \in [\forall S] \) and thus \( [\forall S] \) is the \( m \)-filter.

Let \( F \) be an \( m \)-filter of \( X \). For all \( x, y \in X \), we define \( x \equiv_F y \) by \( x \to y, y \to x \in F \). It is clear that the relation \( \equiv_F \) is a congruence. Since the class of all monadic residuated lattices forms a variety, a quotient algebra \( X/F \) by the congruence \( \equiv_F \) induced from the \( m \)-filter is also a monadic residuated lattice.

Moreover, it is easy to prove that \( Fil_m(X) \) is isomorphic to the set \( \text{Con}(X) \) of all congruences, that is,

\[ Fil_m(X) \cong \text{Con}(X). \]

**Lemma 3.2.** Let \( F \) be an \( m \)-filter of \( X \) and \( a \in X \). Then the smallest \( m \)-filter \( [F \cup \{ a \}]_m \) containing \( F \cup \{ a \} \) is

\[ \{ x \in X \mid \text{there exist } n \geq 1 \text{ and } u \in F \text{ such that } u \odot (\forall a)^n \leq x \} = F \lor [\forall a]. \]

**Proof.** The result can be proved easily from the fact \( [F \cup \{ a \}]_m = [F \cup \{ \forall a \}] = F \lor [\forall a]. \)

A residuated lattice with a universal quantifier \((X, \forall)\) is called strong \([5]\) if \( \forall(x \lor y) = \forall x \lor \forall y \) for \( x, y \in X \). An \( m \)-filter \( P \) is called \textit{prime} if \( x \lor y \in P \) implies \( x \in P \) or \( y \in P \) for all \( x, y \in X \). It is easy to prove that a filter \( P \) is prime if and only if \( P = F \cap G \) implies \( P = F \) or \( P = G \) for any filter \( F \) and \( G \) of \( X \).

**Lemma 3.3.** Let \((X, \forall)\) be a strong residuated lattice with a universal quantifier and \( a \in X \) such that \( a \neq 1 \). Then there is a prime \( m \)-filter \( P \) such that \( a \notin P \).
Proof. Let $\Gamma = \{ F \mid a \notin F, F \text{ is an } m - \text{filter}\}$. It follows from Zorn’s lemma that there exists a maximal element $P$ in $\Gamma$. We only show that $P$ is prime. Suppose that $P$ is not prime. There are $x, y \in X$ such that $x \lor y \in P$ but $x, y \notin P$. Since $P$ is maximal, we have $a \in [P \cup \{x\}]_m = P \lor [\forall x]$ and $a \in [P \cup \{y\}]_m = P \lor [\forall y]$. It follows from strong property that 

$$
a \in (P \lor (\forall x)) \land (P \lor (\forall y))
= P \lor ((\forall x) \land (\forall y))
= P \lor (\forall x \lor \forall y)
= P \lor (\forall (x \lor y)) = P.
$$

But this is a contradiction. Hence $P$ is prime. □

If we take the class $\{P_\lambda\}_{\lambda \in \Lambda}$ of all prime $m$-filters of $X$, then it follows from the above result above that $\bigcap_{\lambda \in \Lambda} P_\lambda = \{1\}$. It follows from the above

**Theorem 3.4.** Any strong residuated lattice with a universal quantifier is a subdirect product of a strong residuated lattice with a universal quantifier $X/P_\lambda$, where $\{P_\lambda\}_{\lambda \in \Lambda}$ is the set of all prime $m$-filters of $X$.

If $X$ satisfies the condition (p-lin): $(x \rightarrow y) \lor (y \rightarrow x) = 1$, then it is easy to show that the quotient algebra $X/P$ by a prime $m$-filter $P$ is linearly ordered monadic residuated lattice. It follows that

**Corollary 3.5.**

1. Every strong monadic MTL-algebra is a subdirect product of linearly ordered strong monadic MTL-algebras.
2. Every strong monadic BL-algebra is a subdirect product of linearly ordered strong monadic BL-algebras.
3. Every strong monadic MV-algebra is a subdirect product of linearly ordered strong monadic MV-algebras.

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**References**


(Michiro Kondo) School of Information Environment, Tokyo Denki University, P.O. Box 270-1382, Inzai, Japan

E-mail address: mkondo@mail.dendai.ac.jp