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SUZUKI-TYPE FIXED POINT THEOREMS FOR GENERALIZED CONTRACTIVE MAPPINGS THAT CHARACTERIZE METRIC COMPLETENESS

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Abstract. Inspired by the work of Suzuki in [T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, *Proc. Amer. Math. Soc.*, **136** (2008), 1861–1869], we prove a fixed point theorem for contractive mappings that generalizes a theorem of Geraghty in [M.A. Geraghty, On contractive mappings, *Proc. Amer. Math. Soc.*, **40** (1973), 604–608] and characterizes metric completeness. We introduce the family **A** of all nonnegative functions ϕ with the property that, given a metric space (X, d) and a mapping $T : X \to X$, the condition

$$x, y \in X,\ x \neq y,\ d(x, Tx) \le d(x, y) \implies d(Tx, Ty) < \phi(d(x, y)),$$

implies that the iterations $x_n = T^n x$, for any choice of initial point $x \in X$, form a Cauchy sequence in X. We show that the family of L-functions, introduced by Lim in [T.C. Lim, On characterizations of Meir-Keeler contractive maps, *Nonlinear Anal.*, **46** (2001), 113–120], and the family of test functions, introduced by Geraghty, belong to **A**. We also prove a Suzuki-type fixed point theorem for nonlinear contractions.
Keywords: Banach contraction principle, Contractive mappings, Fixed points, Suzuki-type fixed point theorem, Metric completeness.
MSC(2010): Primary: 54H25; Secondary: 54E50, 11Y50.

1. Introduction

Throughout this paper, $\mathbb{R}^+$ denotes the set of nonnegative real numbers, $\mathbb{Z}^+$ denotes the set of nonnegative integers, and $\mathbb{N}$ denotes the set of positive integers. Given a set X and a mapping $T : X \to X$, the nth iterate of T is denoted by T^n so that $T^2 x = T(Tx)$, $T^3 x = T(T^2 x)$ and so on. A point $x_0 \in X$ is called a *fixed point* of T if $Tx_0 = x_0$.

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Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be *contractive* if $d(Tx, Ty) < d(x, y)$, for all $x, y \in X$ with $x \neq y$, and a *contraction* if there is $r \in [0, 1)$ such that $d(Tx, Ty) \leq rd(x, y)$, for all $x, y \in X$. The following famous theorem is referred to as the Banach contraction principle.

Theorem 1.1 (Banach, [1]). *If X is a complete metric space, then every contraction T on X has a unique fixed point.*

The Banach contraction principle is very simple and powerful. It became a classical tool in nonlinear analysis with many generalizations; see [2–4, 7, 12, 14, 15, 20–23, 25, 26]. For instance, the following result, due to D.W. Boyd and J.S. Wong, is a great generalization of Theorem 1.1.

Theorem 1.2 (Boyd and Wong, [2]). *Let (X, d) be a complete metric space, and let T be a mapping on X. Suppose there exists a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\phi(0) = 0$, $\phi(s) < s$ for $s > 0$ and that ϕ is right upper semicontinuous such that*

$$\forall\, x, y \in X, \quad d(Tx, Ty) \leq \phi(d(x, y)). \tag{1.1}$$

Then T has a unique fixed point.

Another interesting generalization of Banach contraction principle was given, in [14], by A. Meir and E. Keeler:

Definition 1.3 ([14]). *A mapping $T : X \to X$ on a metric space (X, d) is called a* Meir-Keeler contraction *if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\forall\, x, y \in X, \quad \Big(\varepsilon \leq d(x, y) < \varepsilon + \delta \implies d(Tx, Ty) < \varepsilon\Big). \tag{1.2}$$

Theorem 1.4 (Meir and Keeler, [14]). *If X is a complete metric space, then every Meir-Keeler contraction T on X has a unique fixed point.*

In [13], T.C. Lim introduced the notion of L-functions and gave a characterization of Meir-Keeler contractions; see Theorem 1.6 below. Lim's characterization reveals that Meir-Keeler's Theorem 1.4 is a very strong generalization of Boyd-Wong's Theorem 1.2.

Definition 1.5 ([13]). *A function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an* L-function *if $\phi(0) = 0$, $\phi(s) > 0$ for $s > 0$, and, for every $s > 0$, there exists $\delta > 0$ such that $\phi(t) \leq s$ for all $t \in [s, s + \delta]$.*

The family of L-functions is denoted by **L**. Note that every L-function ϕ satisfies $\phi(s) \leq s$, for all $s \geq 0$.

Theorem 1.6 (Lim [13], see also [24]). *Let (X, d) be a metric space. A mapping $T : X \to X$ is a Meir-Keeler contraction if and only if there exists an L-function ϕ such that*

$$\forall\, x, y \in X, \quad d(Tx, Ty) < \phi(d(x, y)).$$

There is an example, in [5], of an incomplete metric space X on which every contraction has a fixed point. This means that Theorem 1.1 cannot characterize the metric completeness of X. Recently, in [25], Suzuki proved the following remarkable generalization of the classical Banach contraction principle that characterizes the metric completeness of X.

Define a function $\theta : [0,1) \to (1/2,1]$ by

$$\theta(r) = \begin{cases} 1, & \text{if } 0 \le r \le (\sqrt{5}-1)/2; \\ (1-r)r^{-2}, & \text{if } (\sqrt{5}-1)/2 \le r \le 1/\sqrt{2}; \\ (1+r)^{-1}, & \text{if } 1/\sqrt{2} \le r < 1. \end{cases} \tag{1.3}$$

Theorem 1.7 (Suzuki, [25]). *Let (X,d) be a metric space. Then X is complete if and only if every mapping T on X satisfying the following has a fixed point:*

- *There exists $r \in [0,1)$ such that*

$$\forall\, x,y \in X, \quad \Big(\theta(r)d(x,Tx) \le d(x,y) \implies d(Tx,Ty) \le rd(x,y)\Big). \tag{1.4}$$

The above Suzuki's generalized version of Banach contraction principle initiated a lot of work in this direction and led to some important contribution in metric fixed point theory. Several authors obtained variations and refinements of Suzuki's result; see [8, 10, 11, 16, 18, 19].

The situation for contractive mappings is different. A contractive mapping on a complete metric space need not have a fixed point. Edelstein, in [6], proved that if the metric space is compact then every contractive mapping possesses a unique fixed point. Then, in [26], Suzuki generalized Edelstein's result as follows.

Theorem 1.8 (Suzuki, [26]). *Let X be a compact metric space and let $T : X \to X$ satisfy the following condition:*

$$\forall\, x,y \in X, \quad \Big(\frac{1}{2}d(x,Tx) < d(x,y) \implies d(Tx,Ty) < d(x,y)\Big). \tag{1.5}$$

Then T has a unique fixed point.

It is interesting to note that, although the above Suzuki's theorem generalizes Edelstein's theorem in [6], these two theorems, as Suzuki mentioned in [26], are not of the same type.

Let T be contractive, fix a point $x \in X$, and set $x_n = T^n x$, for $n \in \mathbb{N}$. Criteria for the sequence of iterates $\{x_n\}$ to be Cauchy are of interest, for if it is Cauchy then it converges to a unique fixed point of T, [9]. Many papers have presented such criteria, especially since the important paper of Rakotch [17]. For example, Geraghty in [9] proved the following theorem that gives a necessary and sufficient condition for a sequence of iterates to be convergent. Here, and in the sequel, the following notation is used: for any pair of subsequences

$\{x_{p_n}\}$ and $\{x_{q_n}\}$ of a given sequence $\{x_n\}$ in X, we let $\delta_n = d(x_{p_n}, x_{q_n})$ and

$$\Delta_n = \begin{cases} 0, & \delta_n = 0; \\ d(Tx_{p_n}, Tx_{q_n})/\delta_n, & \delta_n > 0. \end{cases}$$

Theorem 1.9 (Geraghty, [9]). *Let T be a contractive mapping on a complete metric space X, let $x \in X$, and set $x_n = T^n x$, $n \in \mathbb{N}$. Then $\{x_n\}$ converges to a unique fixed point of T if and only if, for any two subsequences $\{x_{p_n}\}$ and $\{x_{q_n}\}$, with $x_{p_n} \neq x_{q_n}$, if $\Delta_n \to 1$ then $\delta_n \to 0$.*

2. A fixed point theorem for generalized contractive mappings

Definition 2.1. *Let (X, d) be a metric space and $T : X \to X$ be a mapping. We call T a* generalized contractive mapping *if*

$$\forall\, x, y \in X, \quad \Big(x \neq y,\ d(x, Tx) \le d(x, y) \implies d(Tx, Ty) < d(x, y)\Big). \tag{2.1}$$

Theorem 2.2. *Let $T : X \to X$ be a generalized contractive mapping on a metric space X. Given $x \in X$, the following statements for the sequence $x_n = T^n x$, $n \in \mathbb{N}$, are equivalent:*

(1) *$\{x_n\}$ is a Cauchy sequence.*
(2) *For any two subsequences $\{x_{p_n}\}$ and $\{x_{q_n}\}$, with $d(x_{p_n}, Tx_{p_n}) \le d(x_{p_n}, x_{q_n})$ for all n, if $\Delta_n \to 1$ then $\delta_n \to 0$.*

Proof. The implication (1) ⇒ (2) is clear, because if $\{x_n\}$ is Cauchy then, for any two subsequences $\{x_{p_n}\}$ and $\{x_{q_n}\}$, we have $\delta_n \to 0$.

To prove (2) ⇒ (1), first we assume that $x_m = x_{m+1}$, for some m. Then $x_n = x_m$, for $n \ge m$, and particularly $\{x_n\}$ is a Cauchy sequence. Next, assume that $x_n \neq x_{n+1}$ for all n. Since $d(x_n, Tx_n) \le d(x_n, Tx_n)$, condition (2.1) implies that the sequence $\delta_n = d(x_n, x_{n+1})$ is strictly decreasing. Thus $\delta_n \to \delta$ for some nonnegative number δ. If $\delta > 0$, take $p_n = n$ and $q_n = n + 1$. Then $d(x_{p_n}, Tx_{p_n}) \le d(x_{p_n}, x_{q_n})$, for all n, and $\Delta_n \to 1$ while $\delta_n \to \delta \neq 0$. This is a contradiction and hence $d(x_n, x_{n+1}) \to 0$.

For every $n \in \mathbb{N}$, choose $k_n \in \mathbb{N}$ such that $d(x_m, x_{m+1}) < 1/n$ for $m \ge k_n$. If $\{x_n\}$ is not a Cauchy sequence, there exist $\varepsilon > 0$ and sequences $\{p_n\}$ and $\{q_n\}$ of positive integers such that $q_n > p_n \ge k_n$ and $d(x_{p_n}, x_{q_n}) \ge \varepsilon$. We also assume that q_n is the least such integer so that $d(x_{p_n}, x_{q_n - 1}) < \varepsilon$. Therefore,

$$\varepsilon \le d(x_{p_n}, x_{q_n}) \le d(x_{p_n}, x_{q_n-1}) + d(x_{q_n-1}, x_{q_n}) < \varepsilon + 1/n.$$

This shows that $\delta_n \to \varepsilon$. Since we have, for every $n \in \mathbb{N}$,

$$d(x_{p_n}, Tx_{p_n}) \le d(x_{p_n}, x_{q_n}),$$

condition (2.1) shows that $d(Tx_{p_n}, Tx_{q_n}) < \delta_n$. So

$$\frac{\delta_n - 2/n}{\delta_n} \le \frac{d(Tx_{p_n}, Tx_{q_n})}{\delta_n} = \Delta_n < 1.$$

It shows that $\Delta_n \to 1$ and thus $\delta_n \to 0$. This is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. □

The following is a Susuki-type generalization of Theorem 1.9.

Theorem 2.3. *Let X be a complete metric space and let T be a mapping on X satisfying the following condition:*

$$\forall\, x, y \in X, \quad \Big(\frac{1}{2} d(x, Tx) < d(x, y) \implies d(Tx, Ty) < d(x, y)\Big). \tag{2.2}$$

Given $x \in X$, the following statements for the sequence $x_n = T^n x$, $n \in \mathbb{N}$, are equivalent:

(1) *$x_n \to z$ in X, with z a unique fixed point of T;*
(2) *for any two subsequences $\{x_{p_n}\}$ and $\{x_{q_n}\}$, with $d(x_{p_n}, Tx_{p_n}) \le d(x_{p_n}, x_{q_n})$ for all n, if $\Delta_n \to 1$ then $\delta_n \to 0$.*

Proof. Let us first prove that T has at most one fixed point. If z is a fixed point of T and $z \ne y$ then $(1/2)d(z, Tz) < d(z, y)$ and condition (2.2) implies that $d(Tz, Ty) < d(z, y)$. Since $Tz = z$, we must have $Ty \ne y$, i.e., y is not a fixed point of T.

The implication (1) ⇒ (2) is clear. We prove (2) ⇒ (1). By Theorem 2.2, the sequence $\{x_n\}$ is Cauchy and, since the metric space X is complete, $x_n \to z$ for some $z \in X$. We show that $Tz = z$. First note that,

$$\forall n \quad \big(d(x_n, x_{n+1}) < 2d(x_n, z) \text{ or } d(x_{n+1}, x_{n+2}) < 2d(x_{n+1}, z)\big). \tag{2.3}$$

For, if $2d(x_n, z) \le d(x_n, x_{n+1})$ and $2d(x_{n+1}, z) \le d(x_{n+1}, x_{n+2})$ hold, for some n, then

$$\begin{aligned} 2d(x_n, x_{n+1}) &\le 2d(x_n, z) + 2d(x_{n+1}, z) \\ &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \\ &< d(x_n, x_{n+1}) + d(x_n, x_{n+1}) = 2d(x_n, x_{n+1}). \end{aligned}$$

This is absurd and thus we have (2.3). Now condition (2.2) together with (2.3) imply that

$$\forall n \quad \big(d(x_{n+1}, Tz) < d(x_n, z) \text{ or } d(x_{n+2}, Tz) < d(x_{n+1}, z)\big). \tag{2.4}$$

Since $x_n \to z$, condition (2.4) implies the existence of a subsequence of $\{x_n\}$ that converges to Tz. This shows that $Tz = z$. □

Next, we prove that the constant $1/2$ in Theorem 2.3 is the best.

Theorem 2.4. *For every $\eta > 1/2$, there exist a complete metric space (X, d) and a mapping $T : X \to X$ with the following properties:*

(1) *the mapping T has no fixed point in X;*
(2) *$\eta d(x, Tx) \le d(x, y)$ implies $d(Tx, Ty) < d(x, y)$, for all $x, y \in X$;*
(3) *condition (2) of Theorem 2.3 holds for any choice of initial point.*

Proof. Take $\eta > 1/2$ and choose $r \in (1/\sqrt{2}, 1)$ such that $(1+r)^{-1} < \eta$. As in [25, Theorem 4], for $n \in \mathbb{Z}^+$, let $u_n = (1-r)(-r)^n$, and then set $X = \{0,1\} \cup \{u_n : n \in \mathbb{Z}^+\}$. Define a mapping T on X by $T0 = 1$, $T1 = u_0$ and $Tu_n = u_{n+1}$ for $n \in \mathbb{Z}^+$. Obviously T has no fixed point in X and thus (1) is proved. We now prove part (2). In [25], Suzuki showed the following

$$\forall\, x, y \in X, \quad \Big((1+r)^{-1} d(x,Tx) < d(x,y) \implies d(Tx,Ty) \le r d(x,y)\Big).$$

Now, if $\eta d(x,Tx) \le d(x,y)$ then $(1+r)^{-1} d(x,Tx) < d(x,y)$ and thus $d(Tx,Ty) \le rd(x,y) < d(x,y)$. This proves part (2). Finally, we show that, in this setting, condition (2) of Theorem 2.3 holds. Take an arbitrary element $x \in X$ as an initial point and set $x_n = T^n x$, $n \in \mathbb{N}$. Then $\{x_n : n \ge 2\}$ is a subsequence of $\{u_n\}$ and since $u_n \to 0$ the sequence $\{x_n\}$ is Cauchy. Hence if $\{x_{p_n}\}$ and $\{x_{q_n}\}$ are two subsequences of $\{x_n\}$ we have $d(x_{p_n}, x_{q_n}) \to 0$. □

3. A fixed point theorem for generalized ϕ-contractions

Definition 3.1. *Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that $\phi(s) \le s$, for all s. A mapping $T : X \to X$ on a metric space X, is called a* generalized ϕ-contraction *if*

$$(3.1)\quad \forall\, x, y \in X, \quad \Big(x \ne y,\ d(x,Tx) \le d(x,y) \implies d(Tx,Ty) < \phi(d(x,y))\Big).$$

We call ϕ admissible *if, for every choice of initial point $x \in X$, the iterations $x_n = T^n x$, $n \in \mathbb{N}$, form a Cauchy sequence.*

Notation. The family of admissible functions is denoted by $\mathbf{A}$. We denote by $\mathbf{A}_0$ the set of those admissible functions $\phi \in \mathbf{A}$ for which the function $\alpha(s) = \phi(s)/s$ is decreasing near zero, i.e., there exists $\delta > 0$ such that

$$(3.2)\qquad 0 < s < t < \delta \implies \alpha(t) \le \alpha(s).$$

We denote by $\mathbf{A}_0^+$ the set of those functions $\phi \in \mathbf{A}_0$ for which

$$(3.3)\qquad \alpha_0 = \liminf_{s \to 0+} \alpha(s) = \liminf_{s \to 0+} \frac{\phi(s)}{s} > 0.$$

For simplicity, given two distinct points x, y in X, we will write $\alpha(x,y)$ to mean $\alpha(d(x,y))$.

Proposition 3.2. *Let X be a complete metric space and let $T : X \to X$ be a mapping. Assume that, for some admissible function $\phi \in \mathbf{A}$, we have*

$$(3.4)\qquad \forall x, y \in X, \quad \Big(\frac{1}{2} d(x,Tx) < d(x,y) \implies d(Tx,Ty) < \phi(d(x,y))\Big).$$

Then T has a unique fixed point.

Proof. The proof is similar to that of Theorem 2.3. □

Theorem 3.3. *Every L-function is admissible, that is,* $\mathbf{L} \subset \mathbf{A}$.

Proof. Let ϕ be an L-function and let T be a generalized ϕ-contraction on a metric space X. Fix $x \in X$ and let $x_n = T^n x$, $n \in \mathbb{N}$. If $d(x_m, x_{m+1}) = 0$, for some m, then $x_n = x_m$ for $n \geq m$ and there is nothing to prove. Assume that $d(x_n, x_{n+1}) > 0$ for all n. Since $d(x_n, Tx_n) \leq d(x_n, Tx_n)$ and $x_n \neq x_{n+1}$, condition (3.1) implies that, for every $n \in \mathbb{N}$,

$$d(x_{n+1}, x_{n+2}) < \phi(d(x_n, x_{n+1})) \leq d(x_n, x_{n+1}).$$

This shows that the sequence $\{d(x_n, x_{n+1})\}$ is strictly decreasing and thus it converges to some point $s \geq 0$. If $s > 0$, since ϕ is an L-function, there is $\delta > 0$ such that $\phi(t) \leq s$ for $s \leq t \leq s + \delta$. Take $n \in \mathbb{N}$ large enough so that $s \leq d(x_n, x_{n+1}) \leq s + \delta$. Then

$$d(x_{n+1}, x_{n+2}) < \phi(d(x_n, x_{n+1})) \leq s,$$

which is a contradiction. Hence $d(x_n, x_{n+1}) \to 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. To this end, we adopt the same method used by Suzuki in [24]. Fix $\varepsilon > 0$ and let $s = \varepsilon/2$. Since ϕ is an L-function, there exists $\delta \in (0, s)$ such that $\phi(t) \leq s$ for $s \leq t \leq s + \delta$. Since $d(x_n, x_{n+1}) \to 0$, there is $N \in \mathbb{N}$ such that $d(x_n, x_{n+1}) < \delta$ for $n \geq N$. We show that

$$d(x_n, x_{n+m}) < \delta + s \leq \varepsilon, \qquad (n \geq N,\ m \in \mathbb{N}). \tag{3.5}$$

For every $n \geq N$, we prove (3.5) by induction on m. It is obvious that (3.5) holds for $m = 1$. Assume that (3.5) holds for some $m \in \mathbb{N}$. Then $\phi(d(x_n, x_{n+m})) \leq s$. Now, if $d(x_n, Tx_n) \leq d(x_n, x_{n+m})$ then (3.1) shows that $d(x_{n+1}, x_{n+m+1}) < \phi(d(x_n, x_{n+m}))$ and thus

$$d(x_n, , x_{n+m+1}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+m+1}) < \delta + s \leq \varepsilon.$$

If $d(x_n, x_{n+m}) < d(x_n, Tx_n)$ then $d(x_n, x_{n+m}) < \delta$ and thus

$$d(x_n, , x_{n+m+1}) \leq d(x_n, x_{n+m}) + d(x_{n+m}, x_{n+m+1}) < \delta + \delta \leq \delta + s \leq \varepsilon.$$

Therefore (3.5) is verified and $\{x_n\}$ is a Cauchy sequence. □

As in [9], we take $\mathbf{S}$ as the class of all functions $\alpha : \mathbb{R}^+ \to [0, 1]$ such that, for any sequence $\{s_n\}$ of positive numbers, if $\alpha(s_n) \to 1$ then $s_n \to 0$.

Theorem 3.4. *If $\alpha \in \mathbf{S}$, the function $\phi(s) = \alpha(s)s$ is admissible.*

Proof. Let $\alpha \in \mathbf{S}$ and define $\phi(s) = \alpha(s)s$. Let T be a generalized ϕ-contraction on a metric space X, let $x \in X$ and let $x_n = T^n x$, $n \in \mathbb{N}$. Let $s_n = d(x_n, x_{n+1})$. As in the proof of Theorem 3.3, we assume that $s_n > 0$ for all n. Then $s_{n+1} < \alpha(s_n)s_n$ and thus $s_n \to s$ for some $s \geq 0$. If $s > 0$ then $s_{n+1}/s_n \to 1$ and thus $\alpha(s_n) \to 1$. Since $\alpha \in \mathbf{S}$, we must have $s = 0$ which is a contradiction. Hence $s = 0$ and $d(x_n, x_{n+1}) \to 0$.

For every $n \in \mathbb{N}$, choose $k_n \in \mathbb{N}$ such that $d(x_m, x_{m+1}) < 1/n$ for $m \geq k_n$. If $\{x_n\}$ is not a Cauchy sequence, there exist $\varepsilon > 0$ and sequences $\{p_n\}$ and

$\{q_n\}$ of positive integers such that $q_n > p_n \geq k_n$ and $d(x_{p_n}, x_{q_n}) \geq \varepsilon$, and $d(x_{p_n}, x_{q_n-1}) < \varepsilon$. Therefore,

$$\varepsilon \leq d(x_{p_n}, x_{q_n}) \leq d(x_{p_n}, x_{q_n-1}) + d(x_{q_n-1}, x_{q_n}) < \varepsilon + 1/n.$$

This shows that $s_n \to \varepsilon$. Since $d(x_{p_n}, Tx_{p_n}) \leq d(x_{p_n}, x_{q_n})$, for every $n \in \mathbb{N}$, condition (3.1) shows that $d(x_{p_n+1}, x_{q_n+1}) < \alpha(s_n)s_n$. Hence we have

$$\begin{aligned} s_n = d(x_{p_n}, x_{q_n}) &\leq d(x_{p_n}, x_{p_n+1}) + d(x_{p_n+1}, x_{q_n+1}) + d(x_{q_n+1}, x_{q_n}) \\ &< 2/n + \alpha(s_n)s_n. \end{aligned}$$

Dividing the above inequality by s_n, since $\alpha(s_n) \leq 1$, we get $\alpha(s_n) \to 1$ and thus $s_n \to 0$ which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. □

We now state and prove a Suzuki-type fixed point theorem for ϕ-contractions.

Theorem 3.5. *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping. Suppose, for some $\phi \in \mathbf{A}_0$ and $\alpha(s) = \phi(s)/s$, we have*

$$\forall x, y \in X, \quad \Big(\frac{d(x, Tx)}{1 + \alpha(x, Tx)} < d(x, y) \implies d(Tx, Ty) < \phi(d(x, y))\Big), \tag{3.6}$$

Then T has a unique fixed point.

Proof. If $z \in X$ is a fixed point of T and $y \neq z$ then

$$\big(1 + \alpha(z, Tz)\big)^{-1} d(z, Tz) < d(z, y),$$

and thus by (3.6) we have $d(Tz, Ty) < d(z, y)$. Since $Tz = z$, we must have $Ty \neq y$, i.e., y is not a fixed point of T.

Now, we prove the existence of the fixed point. Take two points $x, y \in X$ with $x \neq y$. If $d(x, Tx) \leq d(x, y)$ then $\big(1 + \alpha(x, Tx)\big)^{-1} d(x, Tx) < d(x, y)$, because $\alpha(x, Tx) > 0$ and $d(x, y) > 0$. Hence T satisfies condition (3.1) with $\phi(s) = \alpha(s)s$. Fix $x \in X$ and define $x_n = T^n x$, $n \in \mathbb{N}$. Since the function $\phi(s) = \alpha(s)s$ is admissible, the sequence $\{x_n\}$ is Cauchy. Since X is complete, there is $z \in X$ such that $x_n \to z$. Next, we show that $Tz = z$.

If $x_m = Tx_m$ for some m, the $x_n = z$ for $n \geq m$ and $Tz = z$. We assume that $x_n \neq Tx_n$ for all n. Since $\phi \in \mathbf{A}_0$, condition (3.2) holds for some $\delta > 0$. Take a positive number N such that $d(x_n, Tx_n) < \delta$ for $n \geq N$. Then

$$0 < d(Tx_n, T^2x_n) < \phi(d(x_n, Tx_n)) \leq d(x_n, Tx_n),$$

and condition (3.2) shows that $\alpha(x_n, Tx_n) \leq \alpha(Tx_n, T^2x_n)$, for $n \geq N$, so that

$$\frac{1}{1 + \alpha(x_n, Tx_n)} + \frac{\alpha(x_n, Tx_n)}{1 + \alpha(Tx_n, T^2x_n)} \leq 1. \tag{3.7}$$

We claim that

$$(3.8)\qquad \forall\, n \geq N, \quad \begin{cases} \big(1+\alpha(x_n,Tx_n)\big)^{-1} d(x_n,Tx_n) < d(x_n,z), \\ \quad \text{or} \\ \big(1+\alpha(Tx_n,T^2x_n)\big)^{-1} d(Tx_n,T^2x_n) < d(x_{n+1},z). \end{cases}$$

If (3.8) fails to hold, then, for some $n \geq N$, we have

$$d(x_n,z) \leq \big(1+\alpha(x_n,Tx_n)\big)^{-1} d(x_n,Tx_n),$$
$$d(x_{n+1},z) \leq \big(1+\alpha(Tx_n,T^2x_n)\big)^{-1} d(Tx_n,T^2x_n).$$

Using (3.7), we have

$$\begin{aligned} d(x_n,Tx_n) &\leq d(x_n,z) + d(Tx_n,z) \\ &\leq \big(1+\alpha(x_n,Tx_n)\big)^{-1} d(x_n,Tx_n) + \big(1+\alpha(Tx_n,T^2x_n)\big)^{-1} d(Tx_n,T^2x_n) \\ &< \Big[\big(1+\alpha(x_n,Tx_n)\big)^{-1} + \big(1+\alpha(Tx_n,T^2x_n)\big)^{-1}\alpha(x_n,Tx_n)\Big] d(x_n,Tx_n) \\ &\leq d(x_n,Tx_n). \end{aligned}$$

This is absurd and thus (3.8) must hold. Now condition (3.6) together with (3.8) imply that

$$(3.9)\quad \forall n \geq N, \quad d(x_{n+1},Tz) < \phi(d(x_n,z)) \text{ or } d(x_{n+2},Tz) < \phi(d(x_{n+1},z)).$$

Since $x_n \to z$ and $\phi(s) \leq s$, condition (3.9) implies the existence of a subsequence of $\{x_n\}$ that converges to Tz. This shows that $Tz = z$. □

The following theorem states that, for a certain family of functions $\phi \in \mathbf{A}$, the coefficient $1/(1+\alpha)$, in Theorem 3.5, is the best.

Theorem 3.6. *For $\phi \in \mathbf{A}$ and $\alpha(s) = \phi(s)/s$, suppose*

$$(3.10)\qquad \alpha_0 = \liminf_{s\to 0+} \alpha(s) = \liminf_{s \to 0+} \frac{\phi(s)}{s} > 1/\sqrt{2}.$$

Then, for every constant $\eta > 1/(1+\alpha_0)$, there exist a complete metric space (X,d) and a mapping $T : X \to X$ such that T does not have a fixed point and

$$\forall\, x,y \in X, \quad \Big(\eta d(x,Tx) < d(x,y) \implies d(Tx,Ty) < \phi(d(x,y))\Big).$$

Proof. Take a number $r \in (1/\sqrt{2}, \alpha_0)$ such that $(1+r)^{-1} < \eta$. The proof of Theorem 3 in [25] shows that there exist a closed and bounded subset X of $\mathbb{R}$ and a mapping $T : X \to X$ such that T does not have a fixed point and

$$(3.11)\ \forall\, x,y \in X, \quad \Big((1+r)^{-1}|x-Tx| < |x-y| \implies |Tx-Ty| \leq r|x-y|\Big).$$

Since $r < \liminf_{s\to 0+} \alpha(s)$, there exists $\delta > 0$ such that $r < \alpha(s)$ for $s \in (0, \delta)$. Since X is bounded, there is a constant M such that $|x - y| < M\delta$, for all $x, y \in X$. Now, define a metric d on X by

$$d(x,y) = \frac{1}{M}|x-y|, \qquad (x, y \in X).$$

For $x, y \in X$, if $\eta d(x, Tx) < d(x, y)$ then $(1 + r)^{-1}d(x, Tx) < d(x, y)$. Now, condition (3.11) and the fact that $d(x, y) < \delta$ shows that

$$d(Tx, Ty) \leq rd(x, y) < \alpha(d(x, y))d(x, y). \qquad \square$$

4. Metric completeness

In this section, we discuss the metric completeness.

Theorem 4.1. *Let (X, d) be a metric space. Then X is complete if and only if every mapping $T : X \to X$ satisfying the following two conditions has a fixed point in X;*

(1) *There exists a constant $\eta \in (0, 1/2]$ such that $\eta d(x, Tx) < d(x, y)$ implies $d(Tx, Ty) < d(x, y)$, for all $x, y \in X$.*
(2) *There exists a point $x \in X$ such that condition (2) of Theorem 2.3* holds.

Proof. If the metric space (X, d) is complete, then every mapping T satisfying conditions (1) and (2) possesses a unique fixed point by Theorem 2.3.

Suppose the metric space (X, d) is not complete and let $(\tilde{X}, \tilde{d})$ be its completion. There exists a sequence $\{u_n\}$ in X which converges to a point $u \in \tilde{X} \setminus X$. Define a mapping $T : X \to X$ as follows: For each $x \in X$, since $\tilde{d}(x, u) > 0$ and $\tilde{d}(u_n, u) \to 0$, there exists $m \in \mathbb{N}$ such that

$$\tilde{d}(u_n, u) < \frac{\tilde{d}(x, u)}{7}, \qquad (n \geq m). \tag{4.1}$$

Put $T(x) = u_m$. In case $x = u_k$, for some k, we choose m large enough such that $m > k$ and (4.1) holds. It is obvious that $\tilde{d}(Tx, u) < \tilde{d}(x, u)$ so that $Tx \neq x$, for every $x \in X$. That is, T does not have a fixed point. Let us prove that T satisfies (2.2). Fix $x, y \in X$ with $(1/2)d(x, Tx) < d(x, y)$. In the case where $2\tilde{d}(x, u) \leq \tilde{d}(y, u)$, we have

$$\begin{aligned} d(Tx, Ty) &\leq \tilde{d}(Tx, u) + \tilde{d}(Ty, u) < \frac{1}{7}\big(\tilde{d}(x, u) + \tilde{d}(y, u)\big) \\ &\leq \frac{1}{7}\Big(\tilde{d}(x, u) + \tilde{d}(y, u) + 2(\tilde{d}(y, u) - 2\tilde{d}(x, u))\Big) \\ &\leq \tilde{d}(y, u) - \tilde{d}(x, u) \leq d(x, y). \end{aligned}$$

In the other case, where $\tilde{d}(y,u) < 2\tilde{d}(x,u)$, we have

$$d(x,y) > \frac{1}{2}d(x,Tx) \geq \frac{1}{2}\big(\tilde{d}(x,u) - \tilde{d}(Tx,u)\big) \geq \frac{1}{2}\Big(1 - \frac{1}{7}\Big)\tilde{d}(x,u) = \frac{3}{7}\tilde{d}(x,u).$$

Therefore,

$$\begin{aligned} d(Tx,Ty) &\leq \tilde{d}(Tx,u) + \tilde{d}(Ty,u) < \frac{1}{7}\big(\tilde{d}(x,u) + \tilde{d}(y,u)\big) \\ &\leq \frac{1}{7}\big(\tilde{d}(x,u) + 2\tilde{d}(x,u)\big) = \frac{3}{7}\tilde{d}(x,u) \leq d(x,y). \end{aligned}$$

Finally, we show that, for any initial point x, condition (2) of Theorem 2.3 holds for the iteration sequence $x_n = T^n x$, $n \in \mathbb{N}$. The definition of T shows that there exists a sequence $\{m_n\}$ of positive integers such that $m_n < m_{n+1}$ and $x_n = u_{m_n}$. Hence $\{x_n\}$ is a subsequence of $\{u_n\}$. Now, if $\{x_{p_n}\}$ and $\{x_{q_n}\}$ are subsequences of $\{x_n\}$, they are also subsequences of $\{u_n\}$ and thus $d(x_{p_n}, x_{q_n}) \to 0$ because $\{u_n\}$ is a Cauchy sequence. This shows that condition (2) of Theorem 2.3 holds for the sequence $\{x_n\}$. This is a contradiction since condition (1) of Theorem 2.3 does not hold for the sequence $\{x_n\}$. □

We say that two metrics d and ρ on X are equivalent if they generate the same topology and the same Cauchy sequences. Given a metric ρ on X, we denote the family of all metrics d on X equivalent to ρ by $\mathcal{E}_\rho$. It is obvious that (X,ρ) is complete if and only if (X,d), for some $d \in \mathcal{E}_\rho$, is complete if and only if (X,d), for all $d \in \mathcal{E}_\rho$, is complete.

Theorem 4.2. *For a metric space (X,ρ) the following are equivalent:*

(1) *The space (X,ρ) is complete.*
(2) *For every $\phi \in \mathbf{A}_0$ and $d \in \mathcal{E}_\rho$, every mapping T satisfying* (3.6) *has a fixed point.*
(3) *For some $\phi \in \mathbf{A}_0^+$ and $\eta \in (0,1/2]$, and for all $d \in \mathcal{E}_\rho$, every mapping T satisfying the following condition has a fixed point;*

$$\forall\, x,y \in X, \quad \Big(\eta d(x,Tx) < d(x,y) \implies d(Tx,Ty) < \phi(d(x,y))\Big). \tag{4.2}$$

Proof. The implication (1) ⇒ (2) follows from Theorem 3.5. The implication (2) ⇒ (3) is clear because $\mathbf{A}_0^+ \subset \mathbf{A}_0$ and, for $\eta \leq 1/2$, condition (4.2) implies condition (3.6).

To prove (3) ⇒ (1), towards a contradiction, assume that the metric space (X,ρ) is not complete. Define α_0 as in (3.3). Then $\alpha_0 > 0$ since $\phi \in \mathbf{A}_0^+$. Take a number $r \in (0,\alpha_0)$ and let δ be a positive number such that $r < \phi(s)/s$ for all $s \in (0,\delta)$. Define a metric d on X as follows:

$$d(x,y) = \delta\frac{\rho(x,y)}{1+\rho(x,y)}, \qquad (x,y \in X).$$

Then $d \in \mathcal{E}_\rho$ and thus (X, d) is not complete. The proof of Theorem 4 in [25] shows that there exists a mapping $T : X \to X$ with no fixed point such that

$$\forall\, x, y \in X,\ \big(\eta d(x, Tx) < d(x, y) \implies d(Tx, Ty) \le r d(x, y)\big).$$

Since $d(x, y) < \delta$, we have $rd(x, y) < \phi(d(x, y))$ and thus T satisfies (4.2). This is a contradiction. □

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