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SOLVABLE LIE ALGEBRAS WITH $N(R_n, m, r)$ NILRADICAL

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Abstract. In this paper, we classify the indecomposable non-nilpotent solvable Lie algebras with $N(R_n, m, r)$ nilradical, by using the derivation algebra and the automorphism group of $N(R_n, m, r)$. We also prove that these solvable Lie algebras are complete and unique, up to isomorphism.
Keywords: Automorphism group, Nilradical, Filiform nilpotent Lie algebra, complete Lie algebra.
MSC(2010): Primary: 17B05; Secondary: 17B30.

1. **Introduction**

The classification of Lie algebras is the most important work in Lie theory. There are two ways to get the classification of Lie algebras, by dimension or by structure. The dimension approach have got lots of useful results and have some interesting applications in general relativity [9]. However, it seems to be neither feasible, nor fruitful to proceed by dimensions in the classification of Lie algebras when their dimension are beyond 6 [3]. We then turn to the structure approach.

In 1966, M. Vergne applied the cohomology theory of a Lie algebra to the study of variety of nilpotent Lie algebras [16]. From then on, filiform Lie algebras, especially naturally graded filiform Lie algebras L_n, Q_n and their deformations have been central research objects [3]. An R_n filiform Lie algebra is an important filiform Lie algebra, which plays a role in the classification of rigid Lie algebras [3]. A quasi R_n filiform Lie algebra, denoted by $N(R_n, m, r)$, is a sum of R_n filiform Lie algebras in some sense [10, 12, 18].

It is well-known that the sum of two nilpotent ideals of a Lie algebra is again a nilpotent ideal. Therefore, a Lie algebra possesses a unique maximal nilpotent ideal, called the nilradical of the Lie algebra [4]. So there is a natural question: given a nilpotent Lie algebra $\mathfrak{n}$, can it determine the solvable Lie algebras with $\mathfrak{n}$ nilradical? Generally speaking, this is not true [1, 15]. L.

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Šnobl have obtained the classification of the indecomposable non-nilpotent solvable Lie algebras with L_n or Q_n nilradical [15]. In this paper, we study the classification of the indecomposable non-nilpotent solvable Lie algebras with $N(R_n, m, r)$ nilradical and prove that these solvable Lie algebras are unique up to isomorphism.

A derivation D of a given Lie algebra N is a linear transformation of N satisfying, for any $x, y \in N$, $D([x, y]) = [D(x), y] + [x, D(y)]$. If there exists an element $z \in N$ such that $D = \mathrm{ad}z$, that is, $D(x) = [z, x], \forall x \in N$, the derivation D is called an inner derivation. A derivation which is not inner is called an outer derivation. It is well-known that the collection $\mathrm{Der}N$ of all derivations of a Lie algebra N is a Lie subalgebra of $\mathfrak{gl}(N)$, called the derivation algebra of N. The collection $\mathrm{ad}N$ of all inner derivations is an ideal of $\mathrm{Der}N$, called the inner derivation algebra of N. A complete Lie algebra is a Lie algebra with only inner derivations and trivial center. It first appeared in 1951, in the context of Schenkman's theory of subinvariant Lie algebras [14]. In recent years, different authors have concentrated on classifications and structural properties of complete Lie algebras, but the problem of finding complete Lie algebras is still an open problem [5–7,19]. In this paper, we prove that the indecomposable non-nilpotent solvable Lie algebras with $N(R_n, m, r)$ nilradical are complete.

In this paper, all Lie algebras discussed are finite dimensional complex Lie algebras. We denote the central descending sequence of a Lie algebra N by $C^0N = N, C^iN = [N, C^{i-1}N]$, the automorphism group of N by $\mathrm{Aut}N$, the center of N by $C(N)$.

2. **Preliminaries**

In this section, we recall some elementary facts about nilpotent Lie algebras, filiform Lie algebras and complete Lie algebras.

Lemma 2.1. *[2] Let N be a nilpotent Lie algebra. Then the following two assertions are equivalent*
(1) $\{x_1, x_2, \ldots, x_n\}$ *is a minimal system of generators(msg);*
(2) $\{x_1 + C^1N, \ldots, x_2 + C^1N, x_n + C^1N\}$ *is a basis of the vector space* N/C^1N.

Definition 2.2. *[13] A torus on a Lie algebra N is a commutative subalgebra of* $\mathrm{Der}N$ *which consists of semisimple endomorphisms. A torus is said to be maximal if it is not contained in any other torus.*

Let H be a maximal tori on a Lie algebra N. Then N can be decomposed into a direct sum of root spaces with respect to H: $N = \sum_{\alpha \in H^*} N_\alpha$, where H^* is the dual of the vector space H and $N_\alpha = \{x \in N | h(x) = \alpha(h)x, \forall h \in H\}$.

Definition 2.3. *[13] Let H be a maximal tori on a Lie algebra N. One calls H-msg a minimal system of generators which consists of root vectors for H.*

Lemma 2.4. *[8] Let H_1 and H_2 be two maximal tori on a Lie algebra N. Then there exists $\theta \in \mathrm{Aut}N$, such that $H_2 = \theta H_1 \theta^{-1}$.*

Since all maximal tori on a Lie algebra N are mutually conjugated, the dimension of a maximal torus on N is an invariant of N, called the rank of N, denoted by $\mathrm{rank}(N)$.

Let N be a Lie algebra and $\mathrm{Der}N$ its derivation algebra. Define a bilinear operation in the vector space $N + \mathrm{Der}N$ by $[h_1 + n_1, h_2 + n_2] = [h_1, h_2] + h_1(n_2) - h_2(n_1) + [n_1, n_2]$, where $h_i \in H, n_i \in N$, then $N + \mathrm{Der}N$ is a Lie algebra, called the holomorph of N.

Definition 2.5. *[4] A Lie algebra N is called complete if $DerN{=}adN$ and $C(N) = 0$.*

Definition 2.6. *[16, 17] Let N be an n-dimensional Lie algebra. N is called a filiform Lie algebra if $dimC^iN = n - i - 1, 1 \le i \le n - 1$.*

Definition 2.7. *[3] An R_n filiform Lie algebra is a $(n{+}1)$-dimensional filiform Lie algebra, defined on a basis $\{e_0, e_1, \ldots, e_n\}$ by $[e_0, e_i] = e_{i+1}$, $1 \le i \le n-1$, $[e_1, e_i] = e_{i+2}$, $2 \le i \le n-2$, the undefined brackets being zero or obtained by antisymmetry.*

By the definition, R_2 is a Heisenberg algebra and R_3 is isomorphic to L_3, which has been studied in [15]. We may assume $n \ge 4$ in this paper.

Definition 2.8. *[18] If $N = N_1 + N_2 + \cdots + N_m$, where $N_i \cong R_n$, $1 \le i \le m$, and $[N_i, N_j] = 0$, $i \ne j$, then N is called a quasi R_n filiform Lie algebra. We denote the Lie algebra by $N(R_n, m, r)$, where $r{=}dimC^{n-1}N$.*

The sum $N = N_1 + N_2 + \cdots + N_m$ is not necessarily direct, so the subalgebras in the decomposition can have nontrivial intersection.

Lemma 2.9. *[18] Let $\{e_{i0}, e_{i1}, \ldots, e_{in}\}$ be the basis of N_i as in Definition 2.7. Then $\{e_{i0}, e_{i1}, \cdots, e_{i,n-1} | 1 \le i \le m\}$ is linearly independent. Suppose $\{e_{i0}, e_{i1}, \ldots, e_{i,n-1}, e_{q_1,n}, e_{q_2,n}, \ldots, e_{q_k,n} | 1 \le i \le m\}$ is a basis of $N(R_n, m, r)$. Then $\{e_{q_1,n}, e_{q_2,n} \ldots, e_{q_k,n}\}$ is a basis of $C^{n-1}N$.*

We may assume that $\Phi = \{e_{10}, e_{20}, \ldots e_{m0}, \ldots, e_{1,n-1}, e_{2,n-1}, \ldots, e_{m,n-1}, e_{1n}, e_{2n}, \ldots, e_{rn}\}$ is a basis of $N(R_n, m, r)$ and $e_{in} = \sum_{j=1}^{r} b_{ji} e_{jn}$, $i > r$. Then

$$(e_{1n}, e_{2n} \cdots e_{mn}) = (e_{1n}, e_{2n} \cdots e_{rn})(I \quad B), \tag{2.1}$$

where I is the identity matrix and $B = (b_{ij})$.

3. On DerN and AutN

In this section, we give the derivation algebra and the automorphism group of a quasi R_n filiform Lie algebra.

Let σ be a linear transformation of $N(R_n, m, r)$ such that

$$\begin{aligned}
\sigma e_{st} &= \sum_{i=1}^{m}\sum_{j=0}^{n-1} c_{ij}^{st} e_{ij} + \sum_{i=1}^{r} c_{in}^{st} e_{in}, \quad 1 \le s \le m, t = 0, 1,\\
\sigma e_{st} &= [\sigma e_{s0}, e_{s,t-1}] + [e_{s0}, \sigma e_{s,t-1}], \quad 1 \le s \le m, 2 \le t \le n-1,\\
\sigma e_{sn} &= [\sigma e_{s0}, e_{s,n-1}] + [e_{s0}, \sigma e_{s,n-1}], \quad 1 \le s \le r.
\end{aligned}$$

Theorem 3.1. *[18] Let σ be as above, $e_{in}=\sum_{j=1}^{r} b_{ji} e_{jn}$, $\lambda_{st} = (t-1)c_{s0}^{s0} + c_{s1}^{s1}$. Then σ is a derivation of $N(R_n, m, r)$ if and only if for any $1 \le s, p \le m$,*

$$b_{js}(\lambda_{sn} - \lambda_{jn}) = 0, r < s, 1 \le j \le r, \tag{3.1}$$

$$\sigma e_{s0} = c_{s0}^{s0} e_{s0} + \sum_{i=2}^{n-2} c_{si}^{s0} e_{si} + \sum_{i=1}^{m} c_{i,n-1}^{s0} e_{i,n-1} + \sum_{i=1}^{r} c_{in}^{s0} e_{in}, \tag{3.2}$$

$$\sigma e_{s1} = \sum_{i=1}^{n-1} c_{si}^{s1} e_{si} + \sum_{i=1}^{r} c_{in}^{s1} e_{in}, \tag{3.3}$$

$$c_{p,n-1}^{s0} e_{pn} = c_{s,n-1}^{p0} e_{sn}, \quad c_{s1}^{s1} = 2c_{s0}^{s0}. \tag{3.4}$$

Let ρ be a linear transformation of $N(R_n, m, r)$ such that

$$\begin{aligned}
\rho e_{st} &= \sum_{i=1}^{m}\sum_{j=0}^{n-1} d_{ij}^{st} e_{ij} + \sum_{i=1}^{r} d_{in}^{st} e_{in}, \quad 1 \le s \le m, t = 0, 1,\\
\rho e_{st} &= [\rho e_{s0}, \rho e_{s,t-1}], \quad 1 \le s \le m, 2 \le t \le n-1,\\
\rho e_{sn} &= [\rho e_{s0}, \rho e_{s,n-1}], \quad 1 \le s \le r.
\end{aligned}$$

Theorem 3.2. *[18] Let ρ be as above, $e_{in}=\sum_{j=1}^{r} b_{ji} e_{jn}$. Then ρ is an automorphism of $N(R_n, m, r)$ if and only if the following conditions hold.*
(1) *For any $1 \le s \le m$, there exists only one integer $q_s, 1 \le q_s \le m$, such that*

$$\begin{aligned}
\rho e_{s0} &= d_{q_s 0}^{s0} e_{q_s 0} + \sum_{i=2}^{n-2} d_{q_s i}^{s0} e_{q_s i} + \sum_{i=1}^{m} d_{i,n-1}^{s0} e_{i,n-1} + \sum_{i=1}^{r} d_{in}^{s0} e_{in},\\
\rho e_{s1} &= \sum_{i=1}^{n-1} d_{q_s i}^{s1} e_{q_s i} + \sum_{i=1}^{r} d_{in}^{s1} e_{in}.
\end{aligned}$$

(2) *There exists a permutation matrix T such that*

$$(q_1, q_2, \ldots, q_m) = (1, 2, \ldots, m)T.$$

(3) $d_{q_s,0}^{s0} d_{q_s,1}^{s1} \neq 0$, $d_{q_s,0}^{s0} d_{q_s,n-1}^{p0} e_{q_s,n} = d_{q_p,0}^{p0} d_{q_p,n-1}^{s0} e_{q_p,n}, 1 \le s, p \le m.$

(4) *Let* $T = (T_1 \quad T_2)$*, where* T_1 *is an* $m \times r$ *matrix,* $K_1 = \mathrm{diag}(k_1, k_2, \ldots, k_r)$*,* $K_2 = \mathrm{diag}(k_{r+1}, k_{r+2}, \cdots, k_m)$*,* $k_s = (d^{s0}_{q_s,0})^{n+1}$*, then*

$$(I \quad B)T_2K_2 = (I \quad B)T_1K_1B.$$

(5) $(d^{s0}_{q_s,0})^2 = d^{s1}_{q_s,1}$*,* $d^{s0}_{q_s,1} = d^{s1}_{q_s,0} = 0, 1 \le s \le m$*.*

A Lie algebra is called indecomposable if it cannot be decomposed into a direct sum of its proper ideals.

Lemma 3.3. *Let* $N(R_n, m, r)$ *be an indecomposable Lie algebra. If* σ *is a derivation of* $N(R_n, m, r)$*, then* $c^{i0}_{i0} = c^{j0}_{j0}$ *for any* $1 \le i, j \le m$*. If* ρ *is an automorphism of* $N(R_n, m, r)$*, then* $d^{i0}_{q_i,0} = d^{j0}_{q_j,0}$ *for any* $1 \le i, j \le m$*.*

Proof. The proof of this lemma is similar to the proof of Lemma 2.2 in [11]. If $\lambda_{1n}, \lambda_{2n}, \ldots, \lambda_{rn}$ are not all the same, we may assume $\lambda_{in} = \lambda_{1n}$, $1 \le i \le p$ or $r+1 \le i \le r+k$, $\lambda_{sn} \ne \lambda_{1n}$, $p < s \le r$ or $r+k < s \le m$. By (3.1), $b_{js} \ne 0$ implies that $b_{jt} = 0$ if $\lambda_{sn} \ne \lambda_{tn}, r+1 \le s, t \le m$. Then we have $(e_{r+1,n}, e_{r+2,n}, \cdots, e_{mn}) = (e_{1n}, e_{2n}, \cdots, e_{rn}) \begin{pmatrix} B_1 & \\ & B_2 \end{pmatrix}$, where B_1 is a $p \times k$ matrix. This implies that $N(R_n, m, r)$ is decomposable, a contradiction. So $\lambda_{1n} = \lambda_{2n} = \cdots = \lambda_{rn}$. From $e_{in} = \sum_{j=1}^r b_{ji} e_{jn}, i > r$, we have $\lambda_{in} = \lambda_{1n}$. Since $\lambda_{sn} = (n-1)c^{s0}_{s0} + c^{s1}_{s1} = (n+1)c^{s0}_{s0}$, we have $c^{i0}_{i0} = c^{j0}_{j0}, 1 \le i, j \le m$. Similarly, we have the other assertion. □

If a quasi R_n filiform Lie algebra can be decomposed into a direct sum of some ideals, then every ideal is a quasi R_n filiform Lie algebra again. Hence we will assume in this paper all quasi R_n filiform Lie algebra are indecomposable. We will sometimes denote the quasi R_n filiform Lie algebra $N(R_n, m, r)$ simply by N.

4. The classification of the indecomposable non-nilpotent solvable Lie algebras with $N(R_n, m, r)$ nilradical

In this section, we find a method to determine the indecomposable non-nilpotent solvable Lie algebras with a given nilpotent Lie algebra nilradical. Using the method, we classify the indecomposable non-nilpotent solvable Lie algebras with $N(R_n, m, r)$ nilradical.

We want to extend a given nilpotent Lie algebra $\mathfrak{n}$ to all possible indecomposable non-nilpotent solvable Lie algebra $\mathfrak{s}$ with $\mathfrak{n}$ nilradical. Hence we add further elements $\{y_1, y_2, \ldots, y_p\}$ to the basis $\{x_1, x_2, \ldots, x_n\}$ of $\mathfrak{n}$, which together with $\{x_1, x_2, \cdots, x_n\}$ form a basis of $\mathfrak{s}$. Then $\{y_1, y_2, \ldots, y_p\}$ have the following properties.

(1). $\mathrm{ad}y_1|_{\mathfrak{n}}, \mathrm{ad}y_2|_{\mathfrak{n}}, \cdots, \mathrm{ad}y_p|_{\mathfrak{n}}$ are derivations of $\mathfrak{n}$.

It is well-known that the derived algebra of a solvable Lie algebra is contained in its nilradical, that is, $[\mathfrak{s},\mathfrak{s}] \subseteq \mathfrak{n}$. So for any $1 \le i \le p$, $[y_i, \mathfrak{n}] \subseteq \mathfrak{n}$. Hence $\mathrm{ad}y_1|_{\mathfrak{n}}, \mathrm{ad}y_2|_{\mathfrak{n}}, \ldots, \mathrm{ad}y_p|_{\mathfrak{n}}$ are linear transformations of $\mathfrak{n}$. From the Jacobi identity, we have, $\forall x, y \in \mathfrak{n}$,

$$\mathrm{ad}y_i([x,y]) = [\mathrm{ad}y_i(x), y] + [x, \mathrm{ad}y_i(y)].$$

Hence $\mathrm{ad}y_1|_{\mathfrak{n}}, \mathrm{ad}y_2|_{\mathfrak{n}}, \ldots, \mathrm{ad}y_p|_{\mathfrak{n}}$ are derivations of $\mathfrak{n}$.

(2). $[\mathrm{ad}y_i, \mathrm{ad}y_j] = \mathrm{ad}([y_i, y_j]) \in \mathrm{ad}\mathfrak{n}, 1 \le i, j \le p$.

(3). No non-trivial linear combination of the linear transformations $\mathrm{ad}y_1|_{\mathfrak{n}}, \mathrm{ad}y_2|_{\mathfrak{n}}, \ldots, \mathrm{ad}y_p|_{\mathfrak{n}}$ is a nilpotent transformation of $\mathfrak{n}$.

Suppose that there exist $k_1, k_2, \ldots, k_p \in \mathbb{C}$ such that

$$\sum_{i=1}^{p} k_i \left(\mathrm{ad}y_i|_{\mathfrak{n}}\right) \neq 0$$

is a nilpotent transformation of $\mathfrak{n}$. Let $\mathfrak{a} = \mathfrak{n} + \mathbb{C}(\sum_{i=1}^{p} k_i y_i)$. Then from

$$[\mathfrak{a},\mathfrak{a}] \subseteq [\mathfrak{s},\mathfrak{s}] \subseteq \mathfrak{n} \subseteq \mathfrak{a},$$

$\mathfrak{a}$ is a Lie subalgebra of $\mathfrak{s}$. Since $\mathrm{ad}(\sum_{i=1}^{p} k_i y_i)|_{\mathfrak{n}} = \sum_{i=1}^{p} k_i \left(\mathrm{ad}y_i|_{\mathfrak{n}}\right)$ is nilpotent and every inner derivation of $\mathfrak{n}$ is nilpotent, by Engel' Theorem, we have $\mathfrak{a}$ is a nilpotent Lie algebra. $[\mathfrak{a},\mathfrak{s}] \subseteq [\mathfrak{s},\mathfrak{s}] \subseteq \mathfrak{n} \subseteq \mathfrak{a}$ means that $\mathfrak{a}$ is a nilpotent ideal of $\mathfrak{s}$, which is a contradiction to $\mathfrak{n}$ is the nilradical of $\mathfrak{s}$.

Since no non-trivial linear combination of the linear transformations $\mathrm{ad}y_1|_{\mathfrak{n}}, \mathrm{ad}y_2|_{\mathfrak{n}}, \ldots, \mathrm{ad}y_p|_{\mathfrak{n}}$ is a nilpotent transformation of $\mathfrak{n}$ and every inner derivation of $\mathfrak{n}$ is nilpotent, $\mathrm{ad}y_1|_{\mathfrak{n}}, \mathrm{ad}y_2|_{\mathfrak{n}}, \cdots, \mathrm{ad}y_p|_{\mathfrak{n}}$ are outer derivations of $\mathfrak{n}$.

A set of linear transformations $\{D_1, D_2, \ldots, D_p\}$ of a Lie algebra $\mathfrak{n}$ is called a linearly nil-independent outer derivations set if

(1). $D_1, D_2, \ldots, D_p$ are outer derivations of $\mathfrak{n}$.

(2). $[D_i, D_j] \in \mathrm{ad}\mathfrak{n}, 1 \le i, j \le p$.

(3). No non-trivial linear combination of the linear transformations $D_1, D_2, \ldots, D_p$ is a nilpotent transformation of $\mathfrak{n}$.

From the above discussion, we can get a way to find the indecomposable non-nilpotent solvable Lie algebras with $\mathfrak{n}$ nilradical. We may find all linearly nil-independent outer derivations set $\{D_1, D_2, \ldots, D_p\}$ of $\mathfrak{n}$. Define a bilinear operation in the vector space spanned by $\mathfrak{n}$ and $\{D_1, D_2, \ldots, D_p\}$ as follows

$$[x_i, x_j] = \sum_{k=1}^{n} a_{ij}^k x_k, [D_s, D_t] = \sum_{u=1}^{n} b_{st}^u \mathrm{ad}x_u, [D_s, x_i] = D_s(x_i), \tag{4.1}$$

where $x_i, x_j \in \{x_1, x_2, \ldots, x_n\}, D_s, D_t \in \{D_1, D_2, \ldots, D_p\}$ and a_{ij}^k are the structure constants of $\mathfrak{n}$. Then we determine b_{st}^u to make the bilinear operation to be Lie brackets.

In fact, this Lie algebra is a Lie subalgebra of the holomorph of $\mathfrak{n}$.

Different linearly nil-independent outer derivations sets may correspond to isomorphic Lie algebras, so redundancies should be eliminated.

Lemma 4.1. *Let $\{D_1, D_2, \ldots, D_p\}$ be a linearly nil-independent outer derivations set of $\mathfrak{n}$. Define the Lie algebra $\mathfrak{s}$ by (4.1). Then for any $\rho \in \mathrm{Aut}\mathfrak{n}$, the linear map $\tau : \mathfrak{s} \to \mathfrak{s}$ defined by*

$$\tau(x_i) = \rho(x_i), \tau(\mathrm{ad}x_i) = \rho(\mathrm{ad}x_i)\rho^{-1}, \tau(D_j) = \rho D_j \rho^{-1}, 1 \leq i \leq n, 1 \leq j \leq p$$

is an automorphism of $\mathfrak{s}$.

Proof. We can check directly,

$$\begin{aligned}
\tau[x_i, x_j] &= \sum_{k=1}^{n} a_{ij}^k \rho(x_k) = \rho(\sum_{k=1}^{n} a_{ij}^k x_k) = [\rho(x_i), \rho(x_j)] = [\tau(x_i), \tau(x_j)], \\
\tau[D_s, D_t] &= \sum_{u=1}^{n} b_{st}^u \tau(\mathrm{ad}x_u) = \sum_{u=1}^{n} b_{st}^u \rho(\mathrm{ad}x_u)\rho^{-1} = \rho[D_s, D_t]\rho^{-1} \\
&= [\rho D_s \rho^{-1}, \rho D_t \rho^{-1}] = [\tau(D_s), \tau(D_t)], \\
\tau[D_s, x_i] &= \rho(D_s(x_i)) = \rho(D_s \rho^{-1} \rho(x_i)) = (\rho D_s \rho^{-1})\rho(x_i) \\
&= [\rho D_s \rho^{-1}, \rho(x_i)] = [\tau(D_s), \tau(x_i)].
\end{aligned}$$

Hence τ is an automorphism of $\mathfrak{s}$. □

Actually, for any $\rho \in \mathrm{Aut}\mathfrak{n}$, the linea map

$$\tau(x) = \rho(x), \tau(D) = \rho D \rho^{-1}, \forall x \in \mathfrak{n}, D \in \mathrm{Der}\mathfrak{n}$$

is an automorphism of the holomorph of $\mathfrak{n}$.

From the above discussion, to eliminate the different linearly nil-independent outer derivations sets which correspond to isomorphic Lie algebras, we can

(1). Add any inner derivations of $\mathfrak{n}$ to the linearly nil-independent outer derivations set of $\mathfrak{n}$.

In fact, for any $x_1, x_2, \ldots, x_p \in \mathfrak{n}$, the Lie subalgebra of the holomorph of $\mathfrak{n}$ spanned by $\{D_1 + \mathrm{ad}x_1, D_2 + \mathrm{ad}x_2, \ldots, D_p + \mathrm{ad}x_p\}$ and $\mathfrak{n}$ is the same with the Lie subalgebra of the holomorph of $\mathfrak{n}$ spanned by $\{D_1, D_2, \ldots, D_p\}$ and $\mathfrak{n}$. Hence adding any inner derivation of $\mathfrak{n}$ to the linearly nil-independent outer derivations set of $\mathfrak{n}$ will not change the indecomposable non-nilpotent Lie algebra $\mathfrak{s}$.

(2). Use the automorphism of $\mathfrak{n}$ to simplify the linearly nil-independent outer derivations set of $\mathfrak{n}$.

Now we return to determine the indecomposable non-nilpotent solvable Lie algebras with $N(R_n, m, r)$ nilradical.

Lemma 4.2. *Let N be an indecomposable R_n filiform Lie algebra. Then any non-empty linearly nil-independent outer derivations set of N contains exactly one element.*

Proof. Suppose that there are two elements D_1, D_2 in a linearly nil-independent outer derivations set of N. From Theorem 3.1 and Lemma 3.3, we have the matrices of D_1 and D_2 with respect to the basis Φ are lower triangular matrices and the diagonal entries are

$$(c_{s0}^{s0}(D_1), \dots, c_{s0}^{s0}(D_1), 2c_{s0}^{s0}(D_1), \dots, 2c_{s0}^{s0}(D_1), \dots, \dots, (n+1)c_{s0}^{s0}(D_1), \dots, (n+1)c_{s0}^{s0}(D_1))$$

and

$$(c_{s0}^{s0}(D_2), \dots, c_{s0}^{s0}(D_2), 2c_{s0}^{s0}(D_2), \dots, 2c_{s0}^{s0}(D_2), \dots, \dots, (n+1)c_{s0}^{s0}(D_2), \dots, (n+1)c_{s0}^{s0}(D_2))$$

respectively. Hence we have $c_{s0}^{s0}(D_1)D_2 - c_{s0}^{s0}(D_2)D_1$ is a nilpotent derivation, a contradiction. So there exists only one element in the linearly nil-independent outer derivations set of N. $\square$

We will denote

$$\begin{array}{llll}
A_{si}e_{p0} = 0, & 1 \le p \le m, & A_{si}e_{pt} = \delta_{sp}e_{s,t+i}, & 1 \le p \le m, 1 \le t \le n, \\
B_{si}e_{p0} = \delta_{sp}e_{i,n-1}, & 1 \le p \le m, & B_{si}e_{pt} = 0, & 1 \le p \le m, 1 \le t \le n, \\
C_{si}e_{p0} = \delta_{sp}e_{in}, & 1 \le p \le m, & C_{si}e_{pt} = 0, & 1 \le p \le m, 1 \le t \le n, \\
D_{si}e_{p1} = \delta_{sp}e_{in}, & 1 \le p \le m, & D_{si}e_{pt} = 0, & 1 \le p \le m, t \ne 1 \\
h_1(e_{st}) = (t+1)e_{st}, & 1 \le s \le m, 0 \le t \le r. & &
\end{array}$$

Lemma 4.3. *Let N be an indecomposable R_n filiform Lie algebra and $\{D_1\}$ be a linearly nil-independent outer derivations set of N. Then by adding an inner derivation of N to D_1, the linearly nil-independent outer derivations set becomes $\{D\}$, where*

$$D = h_1 + \sum_{s=1}^{m}\sum_{i=2}^{n-1} a_{si}A_{si} + \sum_{s=1}^{m}\sum_{i=1, i\ne s}^{m} b_{si}B_{si} + \sum_{s=1}^{m}\sum_{i=1, i\ne s}^{r} c_{si}C_{si} + \sum_{s=1}^{m}\sum_{i=1}^{r} d_{si}D_{si}$$

and $a_{si}, b_{si} = b_{is}, c_{si}, d_{si} \in \mathbb{C}$.

Proof. Since D_1 is a derivation of N, from Theorem 3.1,

$$\begin{aligned}
D_1(e_{s0}) &= c_{s0}^{s0}e_{s0} + \sum_{i=2}^{n-2} c_{si}^{s0}e_{si} + \sum_{i=1}^{m} c_{i,n-1}^{s0}e_{i,n-1} + \sum_{i=1}^{r} c_{in}^{s0}e_{in}, \\
D_1(e_{s1}) &= \sum_{i=1}^{n-1} c_{si}^{s1}e_{si} + \sum_{i=1}^{r} c_{in}^{s1}e_{in}.
\end{aligned}$$

Adding D_1 by the inner derivation

$$\pi = \sum_{s=1}^{m}\sum_{i=1}^{n-1} c_{s,i+1}^{s0}\mathrm{ad}e_{si} - c_{s2}^{s1}\mathrm{ad}e_{s0},$$

we have

$$(D_1+\pi)e_{s0} = c_{s0}^{s0}e_{s0} + \sum_{i=1, i\ne s}^{m} c_{i,n-1}^{s0}e_{i,n-1} + \sum_{i=1, i\ne s}^{r} c_{in}^{s0}e_{in},$$

$$(D_1+\pi)e_{s1}=\sum_{i=1,i\neq 2}^{n-1}c_{si}^{s1}e_{si}-\sum_{i=2}^{n-2}c_{s,i+1}^{s0}e_{s,i+2}+\sum_{i=1}^{r}c_{in}^{s1}e_{in}.$$

Since $D_1+\pi$ is a derivation of N, using

$$(D_1+\pi)e_{st}=[(D_1+\pi)e_{s0},e_{s,t-1}]+[e_{s0},(D_1+\pi)e_{s,t-1}],2\leq t\leq n,$$

we have

$$(D_1+\pi)e_{st}=(t+1)c_{s0}^{s0}e_{st}-\sum_{j=3}^{n-t}c_{sj}^{s0}e_{s,j+t}+\sum_{j=3}^{n-t+1}c_{sj}^{s1}e_{s,j+t-1},2\leq t\leq n.$$

Hence

$$\begin{aligned}D_1+\pi=&c_{s0}^{s0}h_1+\sum_{s=1}^{m}\sum_{i=2}^{n-1}(c_{s,i+1}^{s1}-(1-\delta_{i2})c_{si}^{s0})A_{si}\\&+\sum_{s=1}^{m}\sum_{i=1,i\neq s}^{m}c_{i,n-1}^{s0}B_{si}+\sum_{s=1}^{m}\sum_{i=1,i\neq s}^{r}c_{in}^{s0}C_{si}+\sum_{s=1}^{m}\sum_{i=1}^{r}c_{in}^{s1}D_{si}.\end{aligned}$$

Since $D_1+\pi$ is a derivation, we have $c_{p,n-1}^{s0}e_{pn}=c_{s,n-1}^{p0}e_{sn}$. Let $D=\frac{D_1+\pi}{c_{s0}^{s0}}$. Then D satisfies the lemma. □

Lemma 4.4. *Let N be an indecomposable R_n filiform Lie algebra and $\{D\}$ be a linearly nil-independent outer derivations set of N defined in Lemma 4.3. Then there exists $\rho\in\mathrm{Aut}N$ such that*

$$\rho D\rho^{-1}=h_1+\sum_{s=1}^{m}\sum_{i=1,i\neq s}^{m}b_{si}^{*}B_{si}+\sum_{s=1}^{m}\sum_{i=1,i\neq s}^{r}c_{si}^{*}C_{si}+\sum_{s=1}^{m}\sum_{i=1}^{r}d_{si}^{*}D_{si},$$

where $a_{si}^{*},a_{ti}^{*},b_{si}^{*}=b_{is}^{*},c_{si}^{*},d_{si}^{*}\in\mathbb{C}$.

Proof. From Theorem 3.2, we have for any $1\leq s\leq m$, $\rho_1=I+\frac{1}{2}a_{s2}A_{s2}\in\mathrm{Aut}N$ (which is the case $T=I,d_{s0}^{s0}=d_{s1}^{s1}=1$, $d_{s3}^{s1}=\frac{1}{2}a_{s2}$ and the other coefficients are all 0 in Theorem 3.2). Because $\frac{1}{2}a_{s2}A_{s2}$ is nilpotent, we may assume that

$$(\frac{1}{2}a_{s2}A_{s2})^k\neq 0,(\frac{1}{2}a_{s2}A_{s2})^{k+1}=0.$$

Then we have

$$\rho_1^{-1}=\sum_{i=0}^{k}(-\frac{1}{2}a_{s2}A_{s2})^i.$$

For any $2\leq j,1\leq p\leq m$, we have $h_1A_{ij}e_{p0}=0$ and for any $t\neq 0$,

$$h_1A_{ij}e_{pt}=\delta_{ip}h_1e_{i,j+t}=(j+t+1)\delta_{ip}e_{i,j+t}=j\delta_{ip}e_{i,j+t}+(t+1)\delta_{ip}e_{i,j+t}.$$

Hence

$$h_1A_{ij}=jA_{ij}+A_{ij}h_1.$$

Using this equation, we have

$$
\begin{aligned}
&\rho_1(h_1 + a_{s2}A_{s2})\rho_1^{-1} \\
=&(I + \frac{1}{2}a_{s2}A_{s2})\,(h_1 + a_{s2}A_{s2})\sum_{i=0}^{k}(-\frac{1}{2}a_{s2}A_{s2})^i \\
=&(I + \frac{1}{2}a_{s2}A_{s2})\left(\sum_{i=1}^{k}2i(-\frac{1}{2}a_{s2}A_{s2})^i + \sum_{i=0}^{k}(-\frac{1}{2}a_{s2}A_{s2})^i h_1\right. \\
&\left.+a_{s2}A_{s2}\sum_{i=0}^{k}(-\frac{1}{2}a_{s2}A_{s2})^i\right) \\
=&(I + \frac{1}{2}a_{s2}A_{s2})\left(\sum_{i=1}^{k}2i(-\frac{1}{2}a_{s2}A_{s2})^i\right) \\
&+\left((I + \frac{1}{2}a_{s2}A_{s2})\sum_{i=0}^{k}(-\frac{1}{2}a_{s2}A_{s2})^i\right)(h_1 + a_{s2}A_{s2}) \\
=&(I + \frac{1}{2}a_{s2}A_{s2})\left(\sum_{i=1}^{k}2i(-\frac{1}{2}a_{s2}A_{s2})^i\right) + (h_1 + a_{s2}A_{s2}) \\
=&(I + \frac{1}{2}a_{s2}A_{s2})\left(\sum_{i=2}^{k}2i(-\frac{1}{2}a_{s2}A_{s2})^i\right) + \left(h_1 - \frac{1}{2}a_{s2}^2A_{s4}\right).
\end{aligned}
$$

Therefore,

$$
\rho_1(h_1 + a_{s2}A_{s2})\rho_1^{-1} \in h_1 + \sum_{i=3}^{n-1}\mathbb{C}A_{si}.
$$

By direct computation, we have for any $1 \leq p \leq m, 1 \leq q \leq n$,

$$
\begin{aligned}
(I + \frac{1}{2}a_{s2}A_{s2})A_{pq}\sum_{i=0}^{k}(-\frac{1}{2}a_{s2}A_{s2})^i =&A_{pq} - \delta_{sp}(-\frac{1}{2}a_{s2})^{(k+1)}A_{p,q+2k+2}, \\
(I + \frac{1}{2}a_{s2}A_{s2})x\sum_{i=0}^{k}(-\frac{1}{2}a_{s2}A_{s2})^i =&x, \quad \forall x = B_{pq}, C_{pq}, D_{pq}.
\end{aligned}
$$

Hence

$$\begin{aligned}
&\rho_1 D\rho_1^{-1}\\
=&(I+\frac{1}{2}a_{s2}A_{s2})\left(h_1+\sum_{s=1}^{m}\sum_{i=2}^{n-1}a_{si}A_{si}+\sum_{s=1}^{m}\sum_{i=1,i\neq s}^{m}b_{si}B_{si}\right.\\
&\left.+\sum_{s=1}^{m}\sum_{i=1,i\neq s}^{r}c_{si}C_{si}+\sum_{s=1}^{m}\sum_{i=1}^{r}d_{si}D_{si}\right)\sum_{i=0}^{k}(-\frac{1}{2}a_{s2}A_{s2})^i\\
=&(I+\frac{1}{2}a_{s2}A_{s2})\left(h_1+a_{s2}A_{s2}+\sum_{i=3}^{n-1}a_{si}A_{si}+\sum_{t=1,t\neq s}^{m}\sum_{i=2}^{n-1}a_{ti}A_{ti}\right)\cdot\\
&\sum_{i=0}^{k}(-\frac{1}{2}a_{s2}A_{s2})^i+(I+\frac{1}{2}a_{s2}A_{s2})\left(\sum_{s=1}^{m}\sum_{i=1,i\neq s}^{m}b_{si}B_{si}+\sum_{s=1}^{m}\sum_{i=1,i\neq s}^{r}c_{si}C_{si}\right.\\
&\left.+\sum_{s=1}^{m}\sum_{i=1}^{r}d_{si}D_{si}\right)\sum_{i=0}^{k}(-\frac{1}{2}a_{s2}A_{s2})^i\\
=&h_1+\sum_{i=3}^{n-1}a^*_{si}A_{si}+\sum_{t=1,t\neq s}^{m}\sum_{i=2}^{n-1}a^*_{ti}A_{ti}\\
&+\sum_{s=1}^{m}\sum_{i=1,i\neq s}^{m}b^*_{si}B_{si}+\sum_{s=1}^{m}\sum_{i=1,i\neq s}^{r}c^*_{si}C_{si}+\sum_{s=1}^{m}\sum_{i=1}^{r}d^*_{si}D_{si},
\end{aligned}$$

where $b^*_{si}=b^*_{is}, c^*_{si}, d^*_{si}\in\mathbb{C}$. Similarly, using these automorphisms $\rho_{sj}=I+\frac{1}{j}a^*_{sj}A_{sj}$, we can put to zero first a^*_{s3}, then a^*_{s4} etc. up to $a^*_{s,n-1}$. Let

$$\rho=\prod_{s=1}^{m}(I+\frac{1}{n-1}a^*_{s,n-1}A_{s,n-1})(I+\frac{1}{n-2}a^*_{s,n-2}A_{s,n-2})\cdots(I+\frac{1}{2}a_{s2}A_{s2}).$$

Then ρ satisfies the lemma. $\square$

Since the product of any two of B_{ij}, C_{ij}, D_{ij} is 0 and

$$\left(I+\frac{b^*_{ij}B_{ij}+b^*_{ji}B_{ji}}{n-1}\right)(h_1+b^*_{ij}B_{ij}+b^*_{ji}B_{ji})\left(I-\frac{b^*_{ij}B_{ij}+b^*_{ji}B_{ji}}{n-1}\right)=h_1,$$

$$\left(I+\frac{c^*_{ij}C_{ij}}{n}\right)(h_1+c^*_{ji}C_{ji})\left(I-\frac{c^*_{ij}C_{ij}}{n}\right)=h_1,$$

$$\left(I+\frac{d^*_{ij}D_{ij}}{n-1}\right)(h_1+d^*_{ji}D_{ji})\left(I-\frac{d^*_{ij}D_{ij}}{n-1}\right)=h_1,$$

we can use the automorphisms $I+\frac{b^*_{ij}B_{ij}+b^*_{ji}B_{ji}}{n-1}$, $I+\frac{c^*_{ij}C_{ij}}{n}$, $I+\frac{d^*_{ij}D_{ij}}{n-1}$ to put to zero $b^*_{ij}, c^*_{ij}, d^*_{ij}$. Hence we have

Theorem 4.5. *Let $N(R_n, m, r)$ be an indecomposable R_n filiform Lie algebra. Then all indecomposable non-nilpotent solvable Lie algebras with $N(R_n, m, r)$ nilradical are isomorphic to $\mathbb{C}h_1 + N(R_n, m, r)$, where the brackets are*

$$[h_1, e_{ij}] = (j+1)e_{ij}, 1 \le i \le m, 0 \le j \le n,$$
$$[e_{s0}, e_{si}] = e_{s,i+1}, 1 \le i \le n-1,$$
$$[e_{s1}, e_{si}] = e_{s,i+2}, 2 \le i \le n-2,$$

the undefined brackets being zero or obtained by antisymmetry.

5. The indecomposable non-nilpotent solvable Lie algebras with $N(R_n, m, r)$ nilradical are complete Lie algebras

In this section, we get a decomposition of the derivation algebra of a Lie algebra and prove the indecomposable non-nilpotent solvable Lie algebras with $N(R_n, m, r)$ nilradical are complete Lie algebras.

Lemma 5.1. *Let $\mathfrak{h}$ be a maximal torus on a nilpotent Lie algebra $\mathfrak{n}$ and the decomposition of $\mathfrak{n}$ with respect to $\mathfrak{h}$ be*

$$\mathfrak{n} = \sum_{\alpha \in \Delta} \mathfrak{n}_\alpha,$$

where $\Delta = \{\alpha \in \mathfrak{h}^* | \mathfrak{n}_\alpha \ne 0\}$. Define a bilinear operation in $\mathfrak{g} = \mathfrak{h} + \mathfrak{n}$ by $[h_1 + n_1, h_2 + n_2] = h_1(n_2) - h_2(n_1) + [n_1, n_2]$, where $h_i \in \mathfrak{h}, n_i \in \mathfrak{n}, i = 1, 2$. Then $\mathfrak{g} = \mathfrak{h} + \mathfrak{n}$ is a solvable Lie algebra. If $0 \notin \Delta$ and $C(\mathfrak{g}) = 0$, then

$$\mathrm{Der}\mathfrak{g} = \mathrm{ad}\mathfrak{g} + \mathfrak{D}_0$$

where $\mathfrak{D}_0 = \{\phi \in \mathrm{Der}\mathfrak{g} | \phi(\mathfrak{h}) = 0\}$.

Proof. Let $h \in \mathfrak{h}, D \in \mathrm{Der}\mathfrak{g}, \beta \in \Delta, 0 \ne y_\beta \in \mathfrak{g}_\beta$. Suppose

$$D(h) = h' + \sum_{\alpha \in \Delta} x_\alpha, \quad x_\alpha \in \mathfrak{n}_\alpha,$$

$$D(y_\beta) = h'' + \sum_{\alpha \in \Delta} y'_\alpha, \quad y'_\alpha \in \mathfrak{n}_\alpha.$$

From $D([h, y_\beta]) = \beta(h)D(y_\beta) = [D(h), y_\beta] + [h, D(y_\beta)]$, comparing the vectors in $\mathfrak{n}_\beta$, we have

$$\beta(h)y'_\beta = \beta(h')y_\beta + \beta(h)y'_\beta.$$

So $\beta(h') = 0, \forall \beta \in \Delta$. By $C(\mathfrak{g}) = 0$, we have $h' = 0$, which means

$$D(h) = \sum_{\alpha \in \Delta} x_\alpha, \quad x_\alpha \in \mathfrak{n}_\alpha.$$

Let

$$D' = D + \sum_{\alpha(h) \ne 0} \frac{1}{\alpha(h)} \mathrm{ad} x_\alpha.$$

Then

$$D^{'}(h) = D(h) + \sum_{\alpha(h)\neq 0} \frac{1}{\alpha(h)}[x_\alpha, h] = \sum_{\alpha(h)=0} x_\alpha.$$

For any $h_1 \in \mathfrak{h}$, from the above proof, we can assume

$$D^{'}(h_1) = \sum_{\alpha\in\Delta} z_\alpha = \sum_{\alpha(h)=0} z_\alpha + \sum_{\alpha(h)\neq 0} z_\alpha.$$

By $[h, h_1] = 0$, we have $[D^{'}(h), h_1] = [D^{'}(h_1), h]$, which implies that

$$-\sum_{\alpha(h)=0} \alpha(h_1)x_\alpha = -\sum_{\alpha(h)=0} \alpha(h)z_\alpha - \sum_{\alpha(h)\neq 0} \alpha(h)z_\alpha = -\sum_{\alpha(h)\neq 0} \alpha(h)z_\alpha.$$

Hence

$$\sum_{\alpha(h)=0} \alpha(h_1)x_\alpha = \sum_{\alpha(h)\neq 0} \alpha(h)z_\alpha = 0.$$

We have

$$[D^{'}(h), h_1] = 0, \quad \forall h_1 \in \mathfrak{h}.$$

By $0 \notin \Delta$, we get $D^{'}(h) \in \mathfrak{h}$. But

$$D^{'}(h) = \sum_{\alpha(h)=0} x_\alpha,$$

so $D^{'}(h) = 0$. Hence $\mathrm{Der}\mathfrak{g} = \mathrm{ad}\mathfrak{g} + \mathfrak{D}_0$. □

Lemma 5.2. *Let N be an indecomposable quasi R_n filiform Lie algebra and h_1 be a linear transformation of N, whose matrix with respect to Φ is diag$(1, 1, \ldots, 1, 2, 2, \ldots, 2, \ldots, n+1, n+1, \ldots, n+1)$. Then rank$(N) = 1$ and $H = \mathbb{C}h_1$ is a maximal torus on N.*

Proof. From Theorem 3.1, $h_1 \in \mathrm{Der}N$. Let H be a maximal torus on N such that $h_1 \in H$. $\forall h \in H$, since $[h_1, h] = 0$, the matrix of h with respect to Φ is $\mathrm{diag}(A_1, A_2, \ldots, A_{n+1})$, where A_i, $1 \le i \le n$ are $m \times m$ matrices and A_{n+1} is a $r \times r$ matrix. This means in the expansion of he_{s0} with respect to the basis Φ, all the coefficients of e_{ij}, $j > 0$ are zero. But on the other hand, since $h \in H \subseteq \mathrm{Der}N$, by (3.2),

$$he_{s0} = c_{s0}^{s0}(h)e_{s0} + \sum_{i=2}^{n-2} c_{si}^{s0}(h)e_{si} + \sum_{i=1}^{m} c_{i,n-1}^{s0}(h)e_{i,n-1} + \sum_{i=1}^{r} c_{in}^{s0}(h)e_{in}.$$

Therefore, $he_{s0} = c_{s0}^{s0}(h)e_{s0}$. Similarly, $he_{s1} = c_{s1}^{s1}(h)e_{s1}$. Since $c_{s1}^{s1}(h) = 2c_{s0}^{s0}(h)$ and $c_{i0}^{i0}=c_{j0}^{j0}$, $1 \le i, j \le m$, the matrix of h with respect to Φ is

$$c_{s0}^{s0}(h)\mathrm{diag}(1, 1, \cdots, 1, 2, 2, \cdots, 2, \cdots, n+1, n+1, \cdots, n+1).$$

Hence $\dim H$=1 and rank(N)=1, $H = \mathbb{C}h_1$ is a maximal torus on N. □

Theorem 5.3. *Let $N(R_n, m, r)$ be an indecomposable quasi R_n filiform Lie algebra. Then the indecomposable non-nilpotent solvable Lie algebras with $N(R_n, m, r)$ nilradical are complete Lie algebras.*

Proof. Denote by $\mathfrak{g}$ the indecomposable non-nilpotent solvable Lie algebras with $N(R_n, m, r)$ nilradical in Theorem 4.5. Let $H = \mathbb{C}h_1$ be the maximal torus in Lemma 5.2. Suppose the decomposition of $N(R_n, m, r)$ with respect to H is

$$N = \sum_{\alpha \in \Delta} N_\alpha.$$

It is obvious that $0 \notin \Delta$ and $C(\mathfrak{g}) = 0$. Hence we have,

$$\mathrm{Der}\mathfrak{g} \subseteq \mathrm{ad}\mathfrak{g} + \mathfrak{D}_0.$$

For any $D_0 \in \mathfrak{D}_0$ and $0 \neq x_\alpha \in N_\alpha$, set

$$D_0(x_\alpha) = h^{'} + \sum_{\beta \in \Delta} x_\beta,$$

where $h^{'} \in H, x_\beta \in N_\beta$. $\forall h \in H$, from

$$D_0[h, x_\alpha] = [D_0(h), x_\alpha] + [h, D_0(x_\alpha)] = [h, D_0(x_\alpha)],$$

we have

$$\alpha(h)h^{'} + \sum_{\beta \in \Delta} \alpha(h)x_\beta = \left[h, h^{'} + \sum_{\beta \in \Delta} x_\beta\right] = \sum_{\beta \in \Delta} \beta(h)x_\beta.$$

By $0 \notin \Delta$, we have $\alpha \neq 0$. So $h^{'} = 0$, and if $\beta \neq \alpha$, $x_\beta = 0$. Thus $D_0(N_\alpha) \subseteq N_\alpha$. Since $D_0 \in \mathrm{Der}N$, by (3.2),

$$D_0(e_{s0}) = c_{s0}^{s0}(D_0)e_{s0} + \sum_{i=2}^{n-2} c_{si}^{s0}(D_0)e_{si} + \sum_{i=1}^{m} c_{i,n-1}^{s0}(D_0)e_{i,n-1} + \sum_{i=1}^{r} c_{in}^{s0}(D_0)e_{in}.$$

Therefore, $D_0(e_{s0}) = c_{s0}^{s0}(D_0)e_{s0}$. Similarly, $D_0(e_{s1}) = c_{s1}^{s1}(D_0)e_{s1}$. It means that $D_0|_{\mathfrak{n}}$ is a semisimple derivation of N. Thus $D_0|_N$ generates a torus on N and $[D_0|_N, H] = 0$. Because H is a maximal torus on N, we have $D_0|_N \in H$. It means that there exists $h_0 \in H$ such that $D_0|_N = (\mathrm{ad}h_0)|_N$. Noting that $D_0(H) = 0$, we immediately have $D_0 = \mathrm{ad}h_0$. Hence $\mathfrak{D}_0 \subseteq \mathrm{ad}\mathfrak{g}$ and $\mathfrak{g}$ is a complete Lie algebra. □

When $m = 1$, we have the following corollary.

Corollary 5.4. *The indecomposable non-nilpotent solvable Lie algebras with R_n nilradical are isomorphic to $\mathbb{C}h_1 + R_n$, where the brackets are*

$$[h_1, e_i] = (i+1)e_i, 0 \leq i \leq n,$$
$$[e_0, e_i] = e_{i+1}, 1 \leq i \leq n-1,$$
$$[e_1, e_i] = e_{i+2}, 2 \leq i \leq n-2,$$

the undefined brackets being zero or obtained by antisymmetry. Moreover, the indecomposable non-nilpotent solvable Lie algebras with R_n nilradical are complete Lie algebras.

Remark 5.5. *Classifications and structural properties of complete Lie algebras have been studied in a lot of papers, such as [5–7, 19]. But these papers discussed the complete Lie algebra theoretically and gave few explicit examples. So there are still few examples of complete Lie algebras and the problem of finding complete Lie algebras is still an open problem. The Lie algebras in Theorem 4.5 and Corollary 5.4 are complete Lie algebras different from the examples given in [5–7, 19]. Hence they can be viewed as new explicit examples of complete Lie algebras.*

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