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BINOMIAL EDGE IDEALS AND RATIONAL NORMAL SCROLLS

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Abstract. Let $X = \begin{pmatrix} x_1 & \dots & x_{n-1} & x_n \\ x_2 & \dots & x_n & x_{n+1} \end{pmatrix}$ be the Hankel matrix of size $2 \times n$ and let G be a closed graph on the vertex set $[n]$. We study the binomial ideal $I_G \subset K[x_1, \dots, x_{n+1}]$ which is generated by all the 2-minors of X which correspond to the edges of G. We show that I_G is Cohen-Macaulay. We find the minimal primes of I_G and show that I_G is a set theoretical complete intersection. Moreover, a sharp upper bound for the regularity of I_G is given.

Keywords: Rational normal scroll, closed graph, set-theoretic complete intersection, Cohen-Macaulay.
MSC(2010): Primary: 13H10; Secondary: 13P10.

Introduction

Let K be a field and $S = K[x_1, \dots, x_{n+1}]$ the polynomial ring in $n+1$ variables over the field K. The 2-minors of the matrix $X = \begin{pmatrix} x_1 & \dots & x_{n-1} & x_n \\ x_2 & \dots & x_n & x_{n+1} \end{pmatrix}$ generate the ideal I_X of the rational normal curve $\mathcal{X} \subset \mathbb{P}^n$. It is well-known that S/I_X is Cohen- Macaulay and has an S–linear resolution. We refer the reader to [5], [4], [1] for properties of the ideal of the rational normal scroll.

On the other hand, in the last few years, the so-called binomial edge ideals have been intensively studied. They are defined as follows. Given a simple graph G on the vertex set $[n]$ with edge set $E(G)$, one considers the ideal J_G generated by all the minors $f_{ij} = x_i y_j - x_j y_i$ of the matrix $\begin{pmatrix} x_1 & \dots & x_{n-1} & x_n \\ y_1 & \dots & y_{n-1} & y_n \end{pmatrix}$ in the polynomial ring $R = K[x_1, \dots, x_n, y_1, \dots, y_n]$. Binomial edge ideals were defined in [8] and [9].

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In analogy to this construction, in this paper we consider the following ideals in S. For a simple graph G on the vertex set $[n]$, let I_G be the ideal generated by the 2-minors $g_{ij} = x_i x_{j+1} - x_j x_{i+1}$ of X with $i < j$ and $\{i, j\} \in E(G)$. We call I_G the *binomial edge ideal of* X.

It is clear already from the beginning that unlike the case of classical binomial edge ideals, the ideal I_G strongly depends on the labeling of the graph G. For example, if G is the graph displayed in Figure 2, we get $\dim(S/I_G) = 3$ for the labeling given in Figure 2 (a) and $\dim(S/I_G) = 4$ for the labeling of G given in Figure 2 (b).

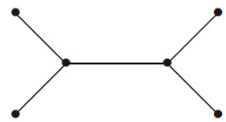

FIGURE 1

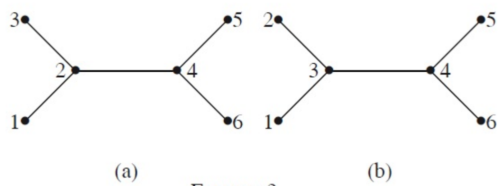

FIGURE 2

However, for some classes of graphs G which admit a natural labeling, we may associate with G a unique ideal I_G and study its properties. This is the case, for instance, for closed graphs. We recall from [8] that G is closed if it has a labeling with respect to which is closed. A graph G is called closed with respect to its given labeling if for all edges $\{i, j\}$ and $\{i, k\}$ with $j > i < k$ or $j < i > k$, one has $\{j, k\} \in E(G)$. A closed graph G is chordal and, therefore, by Dirac's Theorem, its clique complex $\Delta(G)$ is a quasi-forest. We recall that the clique complex $\Delta(G)$ of G is a simplicial complex whose facets are the maximal cliques of G, that is, the maximal complete subgraphs of G. $\Delta(G)$ is a quasi-forest if the facets $F_1, \ldots, F_r$ of $\Delta(G)$ have a leaf order, that is, F_i is a leaf of the simplicial complex generated by $F_1, \ldots, F_i$ for all $i > 1$. For a simplicial complex Δ, a facet F is called a leaf if there is another facet G of Δ such that for any facet $H \neq F$, one has $H \cap F \subseteq G \cap F$. It was shown in [6] that if G is closed, then we may label the vertices of G such that the facets of $\Delta(G)$, say $F_1, \ldots, F_r$, are intervals, $F_i = [a_i, b_i] \subset [n]$ and if we order $F_1, \ldots, F_r$ such that $a_1 < a_2 < \cdots < a_r$, then this is a leaf order.

The paper is structured as follows. In Section 1, we show that the generators of I_G form a Gröbner basis with respect to the reverse lexicographic order if and only if G is closed with the given labeling. As a consequence of this theorem, we derive that for a closed graph G, the ideal I_G is Cohen-Macaulay of dimension $1+c$, where c is the number of connected components of G.

In Section 2, we study the properties of I_G for a closed graph G. We compute the minimal prime ideals of I_G in Theorem 2.2. By using this theorem, we characterize those connected closed graphs G for which I_G is a radical ideal (Proposition 2.3). In addition, we show in Corollary 2.4, that I_G is a set-theoretic complete intersection if G is connected and closed. In the last part of Section 2, we a give sharp upper bound for the regularity of I_G (Theorem 2.7) and we show that I_G has a linear resolution if and only if G is a complete graph.

1. Gröbner bases

Let G be a graph on the vertex set $[n]$ and $I_G \subset S = K[x_1, \ldots, x_n]$ its associated ideal. The main result of this section is the following.

thm 1.1. *The generators of I_G form the reduced Gröbner basis of I_G with respect to the reverse lexicographic order induced by $x_1 > \cdots > x_n > x_{n+1}$ if and only if G is closed with respect to its given labeling.*

Proof. Let us first assume that the generators form a Gröbner basis of I_G. This implies that for any pair of generators $g_{ij} = x_i x_{j+1} - x_j x_{i+1}$ and $g_{k\ell} = x_k x_{\ell+1} - x_\ell x_{k+1}$ of I_G, the S–polynomial $S_{\text{rev}}(g_{ij}, g_{k\ell})$ reduces to zero. Now let $1 \le i < j < k \le n$ such that $\{i,j\}, \{i,k\} \in E(G)$. We have to show that $\{j,k\}$ is also an edge of G. We have

$$S_{\text{rev}}(g_{ij}, g_{ik}) = x_i x_{j+1} x_k - x_i x_j x_{k+1}.$$

Since its initial monomial is $x_i x_{j+1} x_k$, g_{jk} must be a generator of I_G, thus $\{j,k\}$ is an edge of G. In a similar way we argue if $n \ge i > j > k \ge 1$.

For the converse, let us assume that G is closed. We show that the S–polynomial $S_{\text{rev}}(g_{ij}, g_{k\ell})$ reduces to zero with respect to the generators of I_G for any two generators $g_{ij}, g_{k\ell}$ of I_G. Note that $\text{in}_{\text{rev}}(g_{ij}) = x_j x_{i+1}$ and $\text{in}_{\text{rev}}(g_{k\ell}) = x_\ell x_{k+1}$. If these two monomials have disjoint supports we know that $S_{\text{rev}}(g_{ij}, g_{k\ell})$ reduces to zero with respect to $g_{ij}, g_{k\ell}$. Assuming that, for instance, $i < k$, we have to consider the following remaining cases.

Case 1. $\ell = j$. Then one may check that $S_{\text{rev}}(g_{ij}, g_{k\ell}) = x_{j+1} g_{ik}$ which is obviously a standard representation.

Case 2. $j = k+1$. If $\ell = k+1$ we get $S_{\text{rev}}(g_{ij}, g_{k\ell}) = x_{k+2} g_{ik}$. If $\ell > k+1$, we obtain $S_{\text{rev}}(g_{ij}, g_{k\ell}) = x_i g_{k+1,\ell} - x_{\ell+1} g_{ik}$ which is again a standard representation.

Therefore, in all cases, the S-polynomials $S_{\text{rev}}(g_{ij}, g_{k\ell})$ reduce to zero with respect to the generators of I_G. □

As in the case of classical binomial edge ideals associated with graphs, the ideal I_G where G is the line graph on n vertices has nice properties.

Let G be a line graph on $[n]$ with $E(G) = \{\{i, i+1\} : 1 \leq i \leq n-1\}$. Then I_G is minimally generated by $\{g_{i,i+1} = x_{i+1}^2 - x_i x_{i+2} : 1 \leq i \leq n\}$ and $\mathrm{in}_{\mathrm{rev}}(I_G) = (x_2^2, x_3^2, \ldots, x_n^2)$. As $x_2^2, x_3^2, \ldots, x_n^2$ is a regular sequence in S, it follows that the generators of I_G form a regular sequence as well. Consequently, the Koszul complex of the generators of I_G gives the minimal free resolution of S/I_G over S.

The following proposition shows that, for a closed graph G, the initial ideal of I_G with respect to the reverse lexicographic order has a simple structure.

Proposition 1.2. *Let G be a closed graph on $[n]$ with $\Delta(G) = \langle F_1, \ldots, F_r \rangle$ where $F_i = [a_i, b_i]$ for $1 \leq i \leq r$, and $1 = a_1 < \cdots < a_r < b_r = n$. Then $\mathrm{in}_{\mathrm{rev}}(I_G)$ is a primary monomial ideal, hence it is Cohen-Macaulay.*

Proof. We only need to observe that I_F, where $F = [a, b]$ is a clique, has the initial ideal $\mathrm{in}_{\mathrm{rev}}(I_F) = (x_{a+1}, \ldots, x_b)^2$. Then, as $\mathrm{in}_{\mathrm{rev}}(I_G) = \mathrm{in}_{\mathrm{rev}}(I_{F_1}) + \cdots + \mathrm{in}_{\mathrm{rev}}(I_{F_r})$, the conclusion follows. □

Corollary 1.3. *Let G be a closed graph. Then I_G is a Cohen-Macaulay ideal of $\dim(S/I_G) = 1 + c$ where c is the number of connected components of G.*

Proof. I_G is a Cohen-Macaulay ideal by [7, Corollary 3.3.5] and

$$\dim(S/I_G) = \dim(S/\,\mathrm{in}_{\mathrm{rev}}(I_G)) = 1 + c,$$

the last equality being obvious by the form of $\mathrm{in}_{\mathrm{rev}}(I_G)$. □

2. Properties of the scroll binomial edge ideals of closed graphs

In this section we study several algebraic and homological properties of the ideal I_G where G is a closed graph on the vertex set $[n]$.

2.1. **Associated primes.**

We recall that I_X denotes the binomial edge ideal associated with the complete graph K_n. It is well known that I_X is a prime ideal.

Proposition 2.1. *Let G be an arbitrary connected graph on the vertex set $[n]$. Then I_X is a minimal prime of I_G. If P is a minimal prime ideal of I_G which contains no variable, then $P = I_X$.*

Proof. Let $x = \prod_{i=1}^{n+1} x_i$. We claim that I_X is equal to the saturation of I_G with respect to x, that is, $I_X = I_G : x^\infty$. This will be enough to prove the statement of our proposition. Indeed, if P is a minimal prime ideal of I_G which does not contain any variable, then $P \supset I_G : x^\infty = I_X \supset I_G$. Since I_X is a prime ideal, it follows that $P = I_X$.

To prove our claim we first observe that $I_G \subset I_X$ implies that $I_G : x^\infty \subset I_X : x^\infty = I_X$. For the other inclusion, we show that any minimal generator $\delta_{ij} = x_i x_{j+1} - x_j x_{i+1}$ belongs to $I_G : x^\infty$. Let $1 \leq i < j \leq n$. Since G is connected, there exists a path in G from i to j. We prove that $\delta_{ij} \in I_G : x^\infty$

by induction on the length r of the path. If $\{i,j\} \in E(G)$, there is nothing to prove. Let $r > 1$ and let $i, i_1, \ldots, i_{r-1}, i_r = j$ be a path from i to j. By induction, $\delta_{i,i_{r-1}} \in I_G : x^\infty$. We also have $\delta_{i_{r-1}j} \in I_G : x^\infty$. Then $x_{i_{r-1}+1}\delta_{ij} = x_{j+1}\delta_{ii_{r-1}} + x_{i+1}\delta_{i_{r-1}j} \in I_G : x^\infty$, therefore, $\delta_{ij} \in I_G : x^\infty$. □

Now we restrict our study to ideals associated with connected closed graphs.

thm 2.2. *Let $G \neq K_n$ be a connected closed graph on the vertex set $[n]$ and I_G its associated ideal. Then*

$$\mathrm{Ass}(S/I_G) = \mathrm{Min}(I_G) = \{I_X, (x_2, \ldots, x_n)\}.$$

Proof. By Corollary 1.3 and Proposition 2.1, we only need to show that if P is a minimal prime of I_G which contains at least one variable, then $P = (x_2, \ldots, x_n)$. Let $P \in \mathrm{Min}(I_G)$ such that $x_i \in P$ for some $2 \leq i \leq n$. Let $i < n$. Then, as $\{i, i+1\} \in E(G)$, we get $x_{i+1} \in P$. Thus, $(x_i, \ldots, x_n) \subset P$. If $i = 2$, we get $P \supset (x_2, \ldots, x_n) \supset I_G$, thus we have $P = (x_2, \ldots, x_n)$. Let now $i > 2$. Since $\{i-2, i-1\} \in E(G)$, we get $x_{i-1} \in P$. Thus, for $i > 2$, we get as well $P = (x_2, \ldots, x_n)$.

Let us now assume that $P \in \mathrm{Min}(I_G)$ and $x_1 \in P$. Since $\{i, i+1\} \in E(G)$ for all i, we get $(x_1, \ldots, x_n) \subset P$, which is impossible since P is minimal. A similar argument shows that P cannot contain x_{n+1}. □

As a consequence of the above theorem, we may characterize the radical ideals I_G.

Proposition 2.3. *Let G be a connected closed graph on the vertex set $[n]$. Then I_G is a radical ideal if and only if $G = K_n$ or $\Delta(G) = \langle [1, n-1], [2, n] \rangle$.*

Proof. The claim is evident if $G = K_n$. Let now $G \neq K_n$. Then, by the above theorem, we have $\sqrt{I_G} = I_X \cap (x_2, \ldots, x_n)$. We claim that $I_X \cap (x_2, \ldots, x_n) = I_H$ where H is the closed graph on $[n]$ whose clique complex is generated by the intervals $[1, n-1]$ and $[2, n]$. We obviously have $I_H \subset I_X \cap (x_2, \ldots, x_n)$. Let $f \in I_X \cap (x_2, \ldots, x_n)$. Then $f = \sum_{1 \leq i < j \leq n} h_{ij}\delta_{ij}$ where δ_{ij} are the generators of I_X and h_{ij} are polynomials in S. We have to show that $h_{1n}\delta_{1n} \in I_H$ because $\delta_{ij} \in I_H$ for all $i < j$ with $(i,j) \neq (1,n)$. Since $\delta_{ij} \in (x_2, \ldots x_n)$ for all $i < j$ such that $(i,j) \neq (1,n)$, it follows that $h_{1n}\delta_{1n} \in (x_2, \ldots, x_n)$ which implies that $x_1 x_{n+1} h_{1n} \in (x_2, \ldots, x_n)$. But $x_1 x_{n+1}$ is regular on $S/(x_2, \ldots, x_n)$. Thus $h_{1n} \in (x_2, \ldots, x_n)$. We show that for all $2 \leq j \leq n$, we have $x_j \delta_{1n} \in I_H$ which will end our proof. For $j = 2$ we have $x_j \delta_{1n} = x_2(x_1 x_{n+1} - x_2 x_n) = x_1 \delta_{2n} + x_n \delta_{12} \in I_H$. For $j \geq 3$, we obtain $x_j \delta_{1n} = x_{n+1}\delta_{1,j-1} + x_2 \delta_{j-1,n} \in I_H$. □

Theorem 2.2 has the following nice consequence.

Corollary 2.4. *Let G be a connected closed graph. Then I_G is a set-theoretic complete intersection.*

Proof. The statement is known for $G = K_n$ [1]. Let now $G \neq K_n$ and let P_n be the line graph on n vertices. Obviously, the generators of I_{P_n} are generators for I_G as well. By Theorem 2.2, we have $\sqrt{I_G} = \sqrt{I_{P_n}}$. The ideal I_{P_n} is generated by $n-1 = \text{height}(I_G)$ polynomials. Therefore, I_G is a set-theoretic complete intersection. □

2.2. **Regularity.** Let G be a closed graph on the vertex set $[n]$ and $I_G \subset S$ its associated ideal. The first question we may ask is under which conditions on the graph G the ideal I_G has a linear resolution. The next proposition answers this question. We first need the following known statement.

Lemma 2.5. [3, Exercise 4.1.17 (c)] *Let $R = K[x_1, \dots, x_n]/I$ be a homogeneous Cohen-Macaulay ring. The ring R has an m-linear resolution if and only if $I_j = 0$ for $j < m$ and $\dim_K I_m = \binom{m+g-1}{m}$ where $g = \text{height}\, I$.*

Proposition 2.6. *Let G be a closed graph on $[n]$. Then the following are equivalent:*

(a) *G is a complete graph;*
(b) *I_G has a linear resolution;*
(c) *All powers of I_G have a linear resolution.*

Proof. (a)⇒(b) is well known. Let us prove (b)⇒(a). Let G be closed with c connected components, say $G_1, \dots, G_c$. Since I_G has a 2–linear resolution, by Lemma 2.5 and Corollary 1.3, it follows that $\dim_K(I_G)_2 = \binom{n-c+1}{2}$. Hence, we get

$$\binom{n-c+1}{2} = \sum_{i=1}^{c} \dim_K(I_{G_i})_2 \leq \sum_{i=1}^{c} \binom{n_i}{2}$$

where $n_i = |V(G_i)|$ for $1 \leq i \leq c$. The above inequality is equivalent to

$$(n-c)(n-c+1) \leq \sum_{i=1}^{c} n_i(n_i - 1).$$

Set $m_i = n_i - 1$ for $1 \leq i \leq c$. Then we get the equivalent inequality

$$(\sum_{i=1}^{c} m_i)(\sum_{i=1}^{c} m_i + 1) \leq \sum_{i=1}^{c} m_i(m_i + 1)$$

or

$$(\sum_{i=1}^{c} m_i)^2 \leq \sum_{i=1}^{c} m_i^2.$$

This inequality holds if and only if $c = 1$, thus G must be connected. Moreover, in this case, since I_G has a linear resolution, we must have $\dim_K(I_G)_2 = \binom{n}{2} = \dim_K(I_{K_n})_2$, hence $G = K_n$.

The implication $(c) \Rightarrow (b)$ is trivial, and $(a) \Rightarrow (c)$ is known; see, for example, [4, Theorem 1] and [2, Corollary 3.9]. □

In the next theorem we give an upper bound for the regularity of I_G when G is a closed graph.

thm 2.7. *Let G be a closed graph on the vertex set $[n]$. Then* $\operatorname{reg}(S/I_G) \leq r$ *where r is the number of maximal cliques of G.*

Proof. Let $H_{S/I_G}(t)$ be the Hilbert series of S/I_G. Then, since $\dim(S/I_G) = 1+c$, where c is the number of connected components of G, we have

$$H_{S/I_G}(t) = \frac{P(t)}{(1-t)^{1+c}}$$

where $P(t) \in \mathbb{Z}[t]$ with $P(1) \neq 0$. Since I_G is Cohen-Macaulay, we have $\operatorname{reg}(S/I_G) = \deg(P)$.

On the other hand, we have

$$H_{S/I_G}(t) = H_{S/\operatorname{in}_{rev}(I_G)}(t).$$

Let us first suppose that G is connected and let $F_1, \ldots F_r$ be the maximal cliques of G where $F_i = [a_i, b_i]$ for $1 \leq i \leq r$ with $1 = a_1 < a_2 < \cdots < a_r < b_r = n$. Then

$$\begin{aligned} \operatorname{in}_{\mathrm{rev}}(I_G) &= \operatorname{in}_{\mathrm{rev}}(I_{F_1}) + \cdots + \operatorname{in}_{\mathrm{rev}}(I_{F_r}) \\ &= (x_2, \ldots, x_{b_1})^2 + (x_{a_2+1}, \ldots, x_{b_2})^2 + \cdots + (x_{a_{r-1}+1}, \ldots, x_n)^2 \end{aligned}$$

Then, as x_1 and x_{n+1} are regular on $S/\operatorname{in}_{\mathrm{rev}}(I_G)$, we get

$$P(t) = H_{S/(\operatorname{in}_{\mathrm{rev}}(I_G), x_1, x_{n+1})}(t) = h_0 + h_1 t + \cdots + h_q t^q$$

where $q = \deg(P)$ and $h_i = \dim(S/(\operatorname{in}_{\mathrm{rev}}(I_G), x_1, x_{n+1}))_i$ for $0 \leq i \leq q$.

In order to prove our statement, it is enough to show that $q \leq r$. Let $i > r$. We have to show that $\dim(S/(\operatorname{in}_{\mathrm{rev}}(I_G), x_1, x_{n+1}))_i = 0$. But $\dim(S/(\operatorname{in}_{\mathrm{rev}}(I_G), x_1, x_{n+1}))_i$ is equal to the number of squarefree monomials $w = x_F$ in the variables $x_2, \ldots, x_n$ such that $x_F \notin \operatorname{in}_<(I_G)$ and $\deg x_F = i$. Let $F = \{j_1, \ldots, j_i\}$ with $2 \leq j_1 < \cdots < j_i \leq n$. Since $\deg x_F \geq r+1$, there exists $1 \leq p < q \leq i$ such that j_p and j_q belong to the same clique F_ℓ of G. This implies that $x_F \in \operatorname{in}_<(I_G)$. Therefore, $\dim(S/(\operatorname{in}_<(I_G), x_1, x_{n+1}))_i = 0$ and, consequently, $\operatorname{reg}(S/I_G) = \deg(P) \leq r$.

Now, let $G_1, \ldots, G_c$ be the connected components of G and let r_i the number of cliques of G_i for $1 \leq i \leq c$. We may assume that $V(G_i) = [n_i + 1, n_{i+1}]$ for some integers $0 = n_1 < \cdots < n_c < n_{c+1} = n$. We set $S_i = K[\{x_j : n_i + 1 \leq j \leq n_{i+1}\}]$ for $1 \leq i \leq c$. Let M_i be the set of minimal monomial generators of $\operatorname{in}_{\mathrm{rev}}(J_{G_i})$ for all i. One observes that any two monomials $u \in M_i$, $v \in M_j$ with $i \neq j$, have disjoint supports. This implies that

$$S/\operatorname{in}_{\mathrm{rev}}(J_G) \cong \bigotimes_{i=1}^{c} S_i/\operatorname{in}_{\mathrm{rev}}(J_{G_i}).$$

Consequently,

$$\operatorname{reg}(S/J_G) = \operatorname{reg}(S/\operatorname{in}_{\mathrm{rev}}(J_G)) = \sum_{i=1}^{c} \operatorname{reg}(S_i/\operatorname{in}_{\mathrm{rev}}(J_{G_i})) \leq \sum_{i=1}^{c} r_i = r$$

□

Remark 2.8. The upper bound given in the above theorem is sharp. Indeed, let G be a closed graph with the maximal cliques $F_i = [a_i, a_{i+1}]$ where $1 = a_1 < a_2 < \cdots < a_r < a_{r+1} = n$. In this case, we have

$$\operatorname{in}_{\mathrm{rev}}(I_G) = (x_2, \ldots, x_{a_2})^2 + (x_{a_2+1}, \ldots, x_{a_3})^2 + \cdots + (x_{a_r+1}, \ldots, x_n)^2.$$

Therefore,

$$S/(\operatorname{in}_{\mathrm{rev}}(I_G), x_1, x_{n+1}) \cong (S_1/(x_2, \ldots, x_{a_2})^2) \otimes_K \cdots \otimes_K (S_r/(x_{a_r+1}, \ldots, x_n)^2)$$

where $S_i = K[x_{a_i+1}, \ldots, x_{a_{i+1}}]$ for all i, which implies that

$$H_{S/(\operatorname{in}_{\mathrm{rev}}(I_G), x_1, x_{n+1})}(t) = \prod_{i=1}^{r} (1 + (a_{i+1} - a_i)t).$$

This shows that $\operatorname{reg}(S/I_G) = r$.

From Proposition 2.6 and Theorem 2.7, we derive the following consequence.

Corollary 2.9. *Let G be a closed graph with two maximal cliques. Then* $\operatorname{reg}(S/I_G) = 2$.

The following example shows that the inequality given in Theorem 2.7 may be also strict.

Example 2.10. Let G be the closed graph on the vertex set $[6]$ with the maximal cliques $F_1 = [1, 4],\ F_2 = [3, 5]$, and $F_3 = [4, 6]$. We have $\operatorname{reg}(S/I_G) = 2 < 3$.

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