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ON THE DUAL OF CERTAIN LOCALLY CONVEX FUNCTION SPACES

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Abstract. In this paper, we first introduce some function spaces, with certain locally convex topologies, closely related to the space of real-valued continuous functions on X, where X is a C-distinguished topological space. Then, we show that their dual spaces can be identified in a natural way with certain spaces of Radon measures.
Keywords: C-distinguished space; Function spaces; Radon measure; Uniformly τ_k-smooth; Uniformly τ_s-smooth.
MSC(2010): Primary: 46E10; Secondary: 46E27, 46A40, 46A03.

1. **Introduction**

A common problem in functional analysis is whether a given linear functional defined on a vector lattice of real-valued functions is representable as an integral with respect to some suitable regular measure. It is known in literature as Riesz-Radon problem on description of Radon integrals as linear functionals. There is an extensive literature on this problem; see [5, 8, 17–20] and the references therein. Let $C_b(X)$ denote the space of all real-valued bounded continuous functions on a topological space X. A Hausdorff space X is said to be C-distinguished when $C_b(X)$ separates the points of X. In this paper, we first introduce locally convex function spaces $\widetilde{C}_b(X)$ and $\widehat{C}_b(X)$ whose duals can be identified with a topological vector space of (not necessarily bounded) Radon measures. Then, we show that a Riesz type representation theorem holds for these function spaces.

The paper is organized as follows. In Section 2, we give the definition of the topological vector space $M(X)$ and prove some of its basic properties. We define locally convex space $\widetilde{C}_b(X)$ in Section 3, and show that the dual space

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of $\widetilde{C}_b(X)$ can be identified with $M(X)$ in a natural way. Finally, in Section 4, we consider the same study for the space $\widehat{C}_b(X)$.

2. **Topological vector space of unbounded Radon measures**

In this section we fix some terminologies and notations. All terminologies concerning the theory of measure and integration will be as in [2]. We also introduce and study topological vector space $M(X)$, where X is a Hausdorff topological space.

The σ-algebra generated by the open subsets of X is called the σ-algebra of Borel sets and is denoted by $\mathcal{B} = \mathcal{B}(X)$. We denote by $\mathcal{K} = \mathcal{K}(X)$ the family of all compact subsets of X. A (positive) Radon measure is a Borel measure $\mu : \mathcal{B}(X) \longrightarrow [0, \infty]$ such that $\mu(C)$ is finite for each $C \in \mathcal{K}(X)$, and μ is inner-regular; that is,

$$\mu(B) = \sup\{ \mu(C) : C \subseteq B, C \in \mathcal{K}(X) \}.$$

The set of all positive Radon measures on X is denoted by $M^+(X)$. We also let

$$\begin{aligned} M_b^+(X) &= \{\mu \in M^+(X) : \mu(X) < \infty\}, \\ M_b(X) &= \{\mu - \nu : \mu, \nu \in M_b^+(X)\}. \end{aligned}$$

As usual we denote the total variation measure associated with μ by $|\mu|$, and we let $\|\mu\| = |\mu|(X)$. Then $M_b(X)$ with the norm $\|.\|$ is a Banach space.

Let $\sim$ be the binary relation on $M^+(X) \times M^+(X)$ defined by

$$(\mu, \nu) \sim (\mu', \nu') \quad \text{if and only if} \quad \mu + \nu' = \nu + \mu'$$

for each $(\mu, \nu), (\mu', \nu') \in M^+(X) \times M^+(X)$. The inner-regularity and the finiteness on compacta of the elements in $M^+(X)$ show that $\sim$ is actually an equivalence relation on $M^+(X) \times M^+(X)$. We denote by $[\mu, \nu]$ the equivalence class of $(\mu, \nu) \in M^+(X) \times M^+(X)$. Then we define $M(X)$ as the set of all these equivalence classes. For each $[\mu, \nu], [\mu', \nu'] \in M(X)$ and $\lambda \in \mathbb{R}$, let us define

$$[\mu, \nu] + [\mu', \nu'] = [\mu + \mu', \nu + \nu']$$

and

$$\lambda[\mu, \nu] = \begin{cases} [\lambda\mu, \lambda\nu] & \text{if } \lambda \geq 0, \\ [\lambda\nu, \lambda\mu] & \text{otherwise.} \end{cases}$$

It is clear that these operations are well-defined and turn $M(X)$ into a vector space over $\mathbb{R}$, which contains $M_b(X)$ as a subspace. This vector space has been introduced by the second named author in [13, 14]; see also [20] in which it is called bimeasure.

Finally, let $P_K : M(X) \longrightarrow M_b(X)$ be defined as

$$P_K([\mu, \nu]) = \|\mu\chi_K - \nu\chi_K\|$$

for each $[\mu,\nu] \in M(X)$ and each $K \in \mathcal{K}(X)$, where $(\mu\chi_K)(B) = \mu(K \cap B)$ for $B \in \mathcal{B}(X)$. We always equip $M(X)$ with the weakest topology on $M(X)$ for which each P_K is continuous.

The following lemma is an immediate consequence of our definitions.

Lemma 2.1. *Let X be a Hausdorff space. Then the following hold.*

(i) *A net $([\mu_\alpha,\nu_\alpha])_\alpha$ in $M(X)$ converges to $[\mu,\nu]$ if and only if for each $K \in \mathcal{K}$, the net $(\mu_\alpha\chi_K - \nu_\alpha\chi_K)_\alpha$ converges to $\mu\chi_K - \nu\chi_K$ in the Banach space $M_b(X)$.*

(ii) *$M(X)$ is a real Hausdorff topological vector space.*

Let us recall the definition of the projective limit of a family of locally convex spaces. Let $(\Lambda,\leq)$ be a partially ordered set and $\{X_\alpha : \alpha \in \Lambda\}$ be a family of locally convex spaces, and for $\alpha \leq \beta$, denote by $f_{\alpha,\beta}$ a continuous linear map of X_β into X_α. Further suppose that $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$ for all $\alpha \leq \beta \leq \gamma$ and $f_{\alpha\alpha}$ be the identity map on X_α for all $\alpha \in \Lambda$. Then the projective limit of the family $(X_\alpha, f_{\alpha,\beta})$ is defined as

$$\lim_{\alpha}(X_\alpha, f_{\alpha,\beta}) = \{\,(x_\alpha) \in \Pi_{\alpha\in\Lambda} X_\alpha \;:\; x_\alpha = f_{\alpha,\beta}(x_\beta),\; \text{ whenever } \alpha \leq \beta\,\};$$

for more details see for example [15].

We also need the definition of a content. A Radon content is a set function $\lambda : \mathcal{K}(X) \longrightarrow [0,\infty)$ for which, for all $C_1, C_2 \in \mathcal{K}(X)$ with $C_1 \subseteq C_2$, we have

$$\lambda(C_2) - \lambda(C_1) = \sup\{\lambda(C) \;:\; C \in \mathcal{K}(X), C \subseteq C_2 \setminus C_1\}.$$

Let λ be a Radon content. Then the set function $\lambda_t : \mathcal{B}(X) \longrightarrow [0,\infty]$ defined for each $B \in \mathcal{B}(X)$ by

$$\lambda_t(B) = \sup\{\lambda(C) \;:\; C \subseteq B,\, C \in \mathcal{K}(X)\}$$

is called the Radon part of λ. For more details see [2, 7].

In the sequel, whenever we need, we suppose that the family $\mathcal{K}$ has been partially ordered by the inclusion.

Lemma 2.2. *Let X be a Hausdorff space, and let $\phi_{K,L} : M_b(L) \to M_b(K)$ be the restriction map for compact subsets K and L with $K \subseteq L$. Then the mapping $\Omega : \mu \mapsto (\mu\chi_K)$ is a bijection from $M^+(X)$ onto $\lim_K(M_b^+(K), \phi_{K,L})$.*

Proof. The mapping Ω is injective because the elements of $M^+(X)$ are inner-regular. Next we show that Ω is surjective. Let $(\mu^K) \in \lim_K(M_b^+(K), \phi_{K,L})$ be arbitrary and define

$$\lambda(F) = \sup\{\mu^K(F \cap K) \;:\; K \in \mathcal{K}\}$$

for all $F \in \mathcal{K}$. It is not so hard to verify that λ is a Radon content. Now Theorem 2.1.4 in [2] or Theorem 2.1 in [7] shows that λ has a unique extension to a Radon measure, say μ. It is clear that $\Omega(\mu) = (\mu^K)$. □

For a subset K of X, we let

$$M_K(X) = \{\mu \in M_b(X) \ : \ \text{supp}(\mu) \subseteq K\}$$

and

$$C_K(X) = \{f\chi_K : f \in C_b(X)\},$$

where supp(μ) denotes the support of the measure μ. When X is a locally compact space we also let

$$C_K^c(X) = \{f \in C_c(X) \ : \ \text{supp}(f) \subseteq K\},$$

where $C_c(X)$ is the Banach space of all real-valued continuous functions on X with compact support. We endow $M_K(X)$ and $C_K(X)$ with their usual norms.

Lemma 2.3. *Let X be a C-distinguished space, and let $K \in \mathcal{K}$. Then*

(i) $M_K(X) = \{\mu \in M_b(X) : \forall f \in C_b(X), \text{supp}(f) \subseteq X \setminus K, \int_X f \, d\mu = 0\}$.
(ii) $M_K(X) = C_K(X)^*$ *as Banach spaces.*

Proof. To prove (i), let $\mu \in M_b(X) \setminus M_K(X)$. Then $\text{supp}(\mu) \cap (X \setminus K) \neq \emptyset$. Put $\nu := |\mu|\chi_{X \setminus K}$, since $X \setminus K$ is an open set, then $\nu \neq 0$ and $\text{supp}(\nu) \subseteq X \setminus K$. Thus there exists a compact subset $L \subseteq X \setminus K$ such that $\nu(L) > 0$. But $K \cap L = \emptyset$, and since by our assumption X is C-distinguished, then there exists a real-valued function $f \in C_b(X)$ such that $f(X) \subseteq [0,1], f(K) = \{0\}$ and $f(L) = \{1\}$; see Lemma 2.2.1 in [2]. Hence

$$\begin{aligned} \int_X f \, d|\mu| &\geq \int_{X \setminus K} f \, d|\mu| \\ &\geq \int_L f \, d\nu = \nu(L) > 0. \end{aligned}$$

Therefore (i) is immediate.

(ii) It is immediate from Theorem 16.10 in [7]. □

Proposition 2.4. *Let X be a C-distinguished space. For compact subsets $K, L \subseteq X$ with $K \subseteq L$, let $\phi_{K,L} : M_L(X) \longrightarrow M_K(X)$ and $\psi_{K,L} : C_L(X)^* \longrightarrow C_K(X)^*$ be the restriction maps. Then*

$$M(X) = \lim_K (M_K(X), \phi_{K,L}) = \lim_K (C_K(X)^*, \psi_{K,L}),$$

as topological vector spaces. If, further, X is a locally compact space, then

$$M(X) = \lim_K (C_K^c(X)^*, \psi_{K,L}).$$

Proof. Let $\Phi : M(X) \longrightarrow \lim_K (M_K(X), \phi_{K,L})$ be defined as $[\mu, \nu] \mapsto (\mu\chi_K - \nu\chi_K)$. A similar argument to the proof of Lemma 2.2 shows that Φ is bijective. This together with Lemma 2.1 clearly shows that Φ is a linear topological isomorphism.

Define $\Psi : \lim_K(M_K(X), \phi_{K,L}) \longrightarrow \lim_K(C_K(X)^*, \psi_{K,L})$ by $(\mu^K) \mapsto (I_{\mu^K})$, where $I_{\mu^K}(f) = \int_X f\, d\mu^K$. Notice by Lemma 2.3, that Ψ is well-defined. Now it is easy to see that Ψ is a linear topological isomorphism.

Finally, to prove $M(X) = \lim_K(C_K^c(X)^*, \psi_{K,L})$ in the case where X is locally compact, notice first that $C_K^c(X)^* = M_K(X)$ as Banach spaces and then apply the first part. □

3. **Locally convex space** $\widetilde{C}_b(X)$

In this section, we first introduce a locally convex space of functions closely related to the space $C_b(X)$, and then show that its dual can be identified with the topological vector space $M(X)$.

By $\widetilde{C}_b(X)$ we shall denote the vector space generated by the set $\{f\chi_K : f \in C_b(X),\, K \in \mathcal{K}(X)\}$. A subset $\mathcal{F}$ of linear functionals on $\widetilde{C}_b(X)$ is called uniformly τ_k-smooth, if for each $K \in \mathcal{K}(X)$, each $\varepsilon > 0$ and for any net $(f_\alpha)_\alpha$ in $C_b(X)$, with $f_\alpha \searrow f$ and $f \in \widetilde{C}_b(X)^+$, there is α_0 such that

$$|F(f_\alpha\chi_K) - F(f\chi_K)| < \varepsilon$$

for all $F \in \mathcal{F}$ and $\alpha \geq \alpha_0$. Here, $f_\alpha \searrow f$ means $(f_\alpha)_\alpha$ is a decreasing net and $f_\alpha(x) \to f(x)$ for each x in X. A single linear functional F is τ_k-smooth if the one point set $\{F\}$ is uniformly τ_k-smooth.

The following lemma enables us to define a locally convex topology on $\widetilde{C}_b(X)$.

Lemma 3.1. *Let $\mathcal{F}$ be a uniformly τ_k-smooth set of linear functionals on $\widetilde{C}_b(X)$. Then the mapping $\mathcal{P}_\mathcal{F} : \widetilde{C}_b(X) \longrightarrow [0, \infty)$ defined by*

$$\mathcal{P}_\mathcal{F}(g) = \sup\{|F(g)| \,:\, F \in \mathcal{F}\} \quad (g \in \widetilde{C}_b(X))$$

is a seminorm on $\widetilde{C}_b(X)$.

Proof. Let $g = f\chi_K \in \widetilde{C}_b(X)$ where $f \in C_b(X)^+$ and $K \in \mathcal{K}(X)$. Since the sequence $(1/nf)_n$ tends to zero, then the uniformly τ_k-smoothness of $\mathcal{F}$ implies that there exists a $n_0 \in \mathbb{N}$ such that

$$|F(\frac{1}{n} f\chi_K)| < 1$$

for each $F \in \mathcal{F}$ and all $n \geq n_0$. Hence

$$\mathcal{P}_\mathcal{F}(f\chi_K) = \sup\{|F(f\chi_K)| \,:\, F \in \mathcal{F}\} \leq n_0,$$

so clearly $\mathcal{P}_\mathcal{F}$ is a seminorm on $\widetilde{C}_b(X)$. □

We denote by γ the locally convex topology on $\widetilde{C}_b(X)$ generated by the family of all seminorms $\{\mathcal{P}_\mathcal{F}\}_\mathcal{F}$ where $\mathcal{F}$ is a uniformly τ_k-smooth set of linear functionals on $\widetilde{C}_b(X)$, i.e., the topology of uniform convergence on the uniformly τ_k-smooth sets of the linear functionals on $\widetilde{C}_b(X)$. By $\widetilde{C}_\gamma(X)$ we mean the space $\widetilde{C}_b(X)$ equipped with the γ topology.

Proposition 3.2. *Let $\mathcal{F}$ be a γ-equicontinuous set of linear functionals on $\widetilde{C}_b(X)$. Then $\mathcal{F}$ is uniformly τ_k-smooth.*

Proof. Let $\mathcal{F}$ be a γ-equicontinuous set of linear functionals, then there is a uniformly τ_k-smooth set $\mathcal{F}_1$ of linear functionals such that $\mathcal{F} \subseteqq \mathcal{F}_1^{\circ\circ}$, where $\mathcal{F}_1^{\circ}$ denotes the polar of the set $\mathcal{F}_1$. Let $(f_\alpha)_\alpha$ be a net in $C_b(X)$ with $f_\alpha \searrow 0$ and $\varepsilon > 0$, then $\frac{1}{\varepsilon} f_\alpha \searrow 0$. So, for each $K \in \mathcal{K}$, there exists α_0 such that $|H(\frac{1}{\varepsilon} f_\alpha \chi_K)| \leq 1$ for all $\alpha \geq \alpha_0$ and all $H \in \mathcal{F}_1$. But this means that $\frac{1}{\varepsilon} f_\alpha \chi_K \in \mathcal{F}_1^{\circ}$ and so $|F(\frac{1}{\varepsilon} f_\alpha \chi_K)| \leq 1$ for all $\alpha \geq \alpha_0$ and all $F \in \mathcal{F}$. Hence $|F(f_\alpha \chi_K)| \leq \varepsilon$ for all $\alpha \geq \alpha_0$ and all $F \in \mathcal{F}$, and so $\mathcal{F}$ is uniformly τ_k-smooth. □

By a linear functional on $\widetilde{C}_b(X)$ is γ-continuous if and only if it is a τ_k-smooth linear functional.

A subset A of a vector lattice is called solid if whenever $x \in A$ and $|y| \leq |x|$ then $y \in A$. The solid hull of A is the smallest solid set containing A. A vector space topology on a vector lattice is locally solid if there is a base of solid neighborhood of zero. For a convenient account of locally convex vector lattices, see [1].

Proposition 3.3. *The solid hull of a uniformly τ_k-smooth set of linear functionals on $\widetilde{C}_b(X)$ is also uniformly τ_k-smooth.*

Proof. Suppose that $\mathcal{F}$ is a uniformly τ_k-smooth set of linear functionals on $\widetilde{C}_b(X)$. It is sufficient to show that $\{F^+ : F \in \mathcal{F}\}$ is uniformly τ_k-smooth; see [11]. On the contrary, suppose that there exist $\varepsilon > 0$, $K \in \mathcal{K}$ and a net $(f_\alpha)_\alpha$ in $C_b(X)$ such that $f_\alpha \searrow 0$ and $\sup\{F^+(f_\alpha \chi_K) : F \in \mathcal{F}\} > \varepsilon$ for all α. Let α_0 be fixed, then there exists $F \in \mathcal{F}$ such that $F^+(f_{\alpha_0}\chi_K) > \varepsilon$. Now $F^+(f_{\alpha_0}\chi_K) = \sup\{F(g) : 0 \leq g \leq f_{\alpha_0}\chi_K\}$, and so there exists $g \in \widetilde{C}_b(X)$ such that $0 \leq g \leq f_{\alpha_0}\chi_K$ and $F(g) > \varepsilon$. By the τ_k-smoothness of F we can find a $g_1 \in C_b(X)$ such that $0 \leq g_1\chi_K \leq f_{\alpha_0}\chi_K$ and $F(g_1\chi_K) > F(g) - \varepsilon$. Let $\varepsilon_1 = F(g_1\chi_K) - F(g) + \varepsilon > 0$. Now $((f_\alpha - g_1)^+)_\alpha \searrow 0$, where $\alpha \geq \alpha_0$. So, by the τ_k-smoothness of F, there exists $\alpha_1 \geq \alpha_0$ such that $|F((f_{\alpha_1} - g_1)^+\chi_K)| < F(g_1\chi_K) - \varepsilon_1$. Let $h = \max(f_{\alpha_1}, g_1)$. Then

$$\begin{aligned} F(h\chi_K) &= F(g_1\chi_K + (f_{\alpha_1} - g_1)^+\chi_K) \\ &= F(g_1\chi_K) + F((f_{\alpha_1} - g_1)^+\chi_K) \\ &> \varepsilon_1. \end{aligned}$$

Thus, for each α_0, there exist $h \in C_b(X)$, $F \in \mathcal{F}$ and $\alpha_1 \geq \alpha_0$ such that $f_{\alpha_1} \leq h \leq f_{\alpha_0}$ and $F(h\chi_K) > \varepsilon_1$. Let D be the set of all functions h such that there exist $\alpha \leq \beta$ and $F \in \mathcal{F}$ such that $f_\beta \leq h \leq f_\alpha$ and $F(h\chi_K) > \varepsilon_1$. Then the set D can be directed downward to zero. Since $\mathcal{F}$ is a τ_k-smooth set, there exists h in D with $|F(h\chi_K)| < \varepsilon_1$ for all $F \in \mathcal{F}$. This contradiction proves the assertion. □

From Proposition 3.3 it follows that the γ topology is locally solid.

Let us recall that the strict topology β on $C_b(X)$ is defined as the locally convex topology generated by the seminorms

$$\mathcal{P}_\varphi(g) = \sup\{\, \varphi(x)|g(x)| : x \in X \,\},$$

as φ varies through the set of all the positive bounded Borel measurable functions on X vanishing at infinity. It is well-known that the dual space $(C_b(X), \beta)$ can be identified with $M_b(X)$, the Banach space of all finite regular Borel measures on X. For more details see [4, 6].

Lemma 3.4. *Let X be a C-distinguished space. Suppose that F is a positive linear functional on $\widetilde{C}_b(X)$. Then, for any $K \in \mathcal{K}(X)$, the functional F_K defined on $C_b(X)$ as $F_K(f) = F(f\chi_K)$ is β-continuous.*

Proof. Assume that $(f_\alpha)_\alpha$ is a uniformly bounded net in $C_b(X)$ such that converges to zero uniformly on compact subsets. Then, for each compact set K and $\varepsilon > 0$, there exists α_0 such that

$$||f_\alpha \chi_K||_\infty < \varepsilon$$

for all $\alpha \geq \alpha_0$. Hence $-\varepsilon\chi_K \leq f_\alpha\chi_K \leq \varepsilon\chi_K$ and by the positivity of F, we have $F(f_\alpha) \longrightarrow F(\chi_K)$. By Proposition 2.8 in [10], it follows that F_K is β-continuous. □

Lemma 3.4 enables us to define the map $q_K : \widetilde{C}_\gamma(X)^* \longrightarrow C_\beta(X)^*$ by $q_K(F) = F_K$, for each $K \in \mathcal{K}(X)$. We endow $\widetilde{C}_\gamma(X)^*$ with the weakest topology for which each q_K is continuous.

We need the following easy lemma.

Lemma 3.5. *Let X be a C-distinguished space. If f is a positive Borel measurable function on X and $\mu \in M^+(X)$ then*

$$\int_X f\, d\mu = \sup\left\{ \int_X \varphi\, d\mu : 0 \leq \varphi \leq f, \varphi \in \mathcal{C} \right\},$$

where $\mathcal{C} = \{\varphi : \varphi = \sum_{i=1}^n a_i\chi_{K_i}, K_i \in \mathcal{K}, a_i \in \mathbb{R}, i = 1, \ldots, n\}$.

The following result gives a representation for positive γ-continuous linear functionals on $\widetilde{C}_b(X)$.

Proposition 3.6. *Let X be a C-distinguished space. Let F be a positive γ-continuous linear functional on $\widetilde{C}_b(X)$. Then there exists a unique measure $\mu \in M^+(X)$ such that*

$$F(g) = \int_X g\, d\mu$$

for all $g \in \widetilde{C}_b(X)$.

Proof. For each $K \in \mathcal{K}$, define $F_K(f) = F(f\chi_K)$ for any $f \in C_b(X)$. By Lemma 3.4, F_K is β-continuous on $C_b(X)$. Hence there exists a unique measure $\mu_K \in M_b^+(X)$ such that

$$F_K(f) = \int_X f d\mu_K$$

for each $f \in C_b(X)$; see Theorem 6.3 in [4]. Moreover, by Lemma 2.5 of [3], we have

$$\mu_K(C) = \inf\{F_K(f) \ : \ f \in C_b(X),\, f \geq \chi_K\} \quad (C \in \mathcal{K}).$$

Now, let C be a compact subset of X with $C \subseteq K$. Then the set $A = \{f \in C_b(X) : f_{|C} = 1\}$ can be directed downward to χ_C. Indeed, $A = \{f_\alpha\}_\Lambda$ so that $\alpha > \beta$ if and only if $f_\alpha \leq f_\beta$ pointwise, where Λ is a directed set. If $\alpha, \beta \in \Lambda$, $\min(f_\alpha, f_\beta) = f_\gamma$ for some $\gamma \in \Lambda$ and $\gamma \geq \alpha, \beta$. Thus $A = \{f_\alpha\}_\Lambda$ is a net of continuous functions with $f_\alpha \searrow \chi_C$. Now γ-continuity of F implies that $F_K(f_\alpha) \longrightarrow F_K(\chi_C)$. We conclude that $\mu_K(C) = F(\chi_C)$ for all compact subsets C, K with $C \subseteq K$.

For $K \in \mathcal{K}$, define $\lambda(K) = F(\chi_K)$. We will show that λ is a Radon content. To show this, suppose that K_1, K_2 are in $\mathcal{K}$ with $K_1 \subseteq K_2$. Then

$$\begin{aligned}
\lambda(K_2) - \lambda(K_1) &= F(\chi_{K_2}) - F(\chi_{K_1}) \\
&= \mu_{K_1 \cup K_2}(K_2) - \mu_{K_1 \cup K_2}(K_1) \\
&= \mu_{K_2 \cup K_1}(K_2 \setminus K_1) \\
&= \sup\{\mu_{K_1 \cup K_2}(C) : C \subseteq K_2 \setminus K_1\,, C \in \mathcal{K}\} \\
&= \sup\{F(\chi_C) : C \subseteq K_2 \setminus K_1, C \in \mathcal{K}\} \\
&= \sup\{\lambda(C) : C \in \mathcal{K}, C \subseteq K_2 \setminus K_1\},
\end{aligned}$$

as required.

Let μ be the Radon part of λ. Then $\mu \in M^+(X)$ is a unique Radon measure on X such that $\mu(K) = \lambda(K)$ for all compact subsets K; see Theorem 2.1 in [7] or Theorem 2.1.4 in [2].

Now we show that $F(g) = \int_X g\, d\mu$ for all $g \in \widetilde{C}_b(X)$. On the one hand, by applying Lemma 3.5, for each $f \in C_b^+(X)$ and $K \in \mathcal{K}$, we obtain

$$\begin{aligned}
\int_X f\chi_K\, d\mu &= \sup\left\{\int_X \varphi\, d\mu : \varphi \in \mathcal{C}, \varphi \leq f\chi_K\right\} \\
&= \sup\{F(\varphi) : \varphi \in \mathcal{C}, \varphi \leq f\chi_K\} \\
&\leq F(f\chi_K).
\end{aligned}$$

On the other hand,

$$\begin{aligned} F(\|f\|\chi_K) - \int_X f\chi_K \, d\mu &= \|f\|\lambda(K) - \int_X f\chi_K \, d\mu \\ &= \int_X (\|f\|\chi_K - f\chi_K) \, d\mu \\ &\leq F(\|f\|\chi_K - f\chi_K) \\ &= F(\|f\|\chi_K) - F(f\chi_K), \end{aligned}$$

and hence

$$F(f\chi_K) \leq \int_X f\chi_K \, d\mu.$$

From the above, we conclude that

$$F(f\chi_K) = \int_X f\chi_K \, d\mu.$$

Hence $F(g) = \int_X g \, d\mu$ for all $g \in \widetilde{C}_b(X)$. The uniqueness of μ easily follows from the inner-regularity of μ as usual. □

Example 3.7. Let $\mathcal{F}$ be a nonprincipal ultrafilter on $\mathbb{N}$. Then for any bounded sequence $(a_n)_n$ of real numbers there is a unique $a = \lim_{n\in\mathcal{F}} a_n$ which is defined so that for any $\varepsilon > 0$ the set $\{n \in \mathbb{N} : |a_n - a| < \varepsilon\}$ is in $\mathcal{F}$. Consider X = [0,1]. Then $F(g) = \lim_{n\to\mathcal{F}} g(1/n)$ for $g \in \widetilde{C}_b([0,1])$ defines a positive linear functional. The functional F is not represented by any Radon measures, in particular not by δ_0 since $h = \chi_{\{0\}} \in \widetilde{C}_b([0,1])$ and $F(h) = 0$, which is impossible. Note that F is norm-continuous on $\widetilde{C}_b([0,1])$ but not γ-continuous. In fact, $f_n \to \chi_{[1/2,1]}$ in the γ-topology, but $(F(f_n))_n$ does not converge to $F(\chi_{[1/2,1]}) = 0$, where f_n is defined as

$$f_n(x) = \begin{cases} 1/n & \text{if } 1/2 \leq x \leq 1; \\ 2(1/n - 1)x + 1 & \text{if } 0 \leq x < 1/2. \end{cases}$$

We now state the main theorem of this section.

Theorem 3.8. *Let X be a C-distinguished space. Then the mapping Φ : $[\mu,\nu] \mapsto F_\mu - F_\nu$ is a topological linear isomorphism from $M(X)$ onto $\widetilde{C}_\gamma(X)^*$, where*

$$F_\mu(g) = \int_X g \, d\mu \quad (g \in \widetilde{C}_b(X)).$$

Proof. To show that Φ is well-defined let $\mu \in M^+(X)$ and let $(f_\alpha)_\alpha$ be a net in $C_b(X)$ such that $f_\alpha \searrow 0$. Fix α_0 and put ${f'}_\alpha := f_{\alpha_0} - f_\alpha$ for $\alpha \geq \alpha_0$. Thus $f'_\alpha \nearrow f_{\alpha_0}$ in $C_b(X)$. From Theorem 2.1.5 of [2], it follows that $(\mu\chi_K)({f'}_\alpha) \nearrow (\mu\chi_K)(f_{\alpha_0})$ for each $K \in \mathcal{K}$. That is F_μ is τ_k-smooth. Hence, by Proposition 3.3, $F_\mu \in \widetilde{C}_\gamma(X)^*$.

It is easy to see that Φ is linear and injective. The mapping Φ is also surjective. Indeed, let $F \in \widetilde{C}_\gamma(X)^*$ be arbitrary. In view of Lemma 4.3. in [9] and Proposition 3.3, we can write $F = F^+ - F^-$, where F^+ and F^- are positive functionals in $\widetilde{C}_\gamma(X)^*$. From Proposition 3.6, it follows that there are unique measures $\mu_1, \mu_2 \in M^+(X)$ such that

$$F^+(g) = \int_X g\, d\mu_1,\; F^-(g) = \int_X g\, d\mu_2 \quad (g \in \widetilde{C}_b(X)).$$

Thus

$$\Phi(\mu) = F^+ - F^- = F,$$

where $\mu := [\mu_1, \mu_2] \in M(X)$. Finally, we show that Φ and its inverse are continuous. Suppose that $([\mu_\alpha, \nu_\alpha])_\alpha$ is a net in $M(X)$, then, by Lemma 2.1, $([\mu_\alpha, \nu_\alpha])_\alpha$ converges to $[\mu, \nu]$ in $M(X)$ if and only if

$$\mu_\alpha \chi_K - \nu_\alpha \chi_K \to \mu \chi_K - \nu \chi_K$$

in $M_b(X)$ for each compact subset K. This is also equivalent to that

$$F_{\mu_\alpha \chi_K} - F_{\nu_\alpha \chi_K} \to F_{\mu \chi_K} - F_{\nu \chi_K}$$

in $\widetilde{C}_\gamma(X)^*$. This happens if and only if

$$F_{\mu_\alpha} - F_{\nu_\alpha} \to F_\mu - F_\nu.$$

Whence Φ is homeomorphism. $\square$

In the following, $C_\gamma^c(X)$ will denote the space $C_c(X)$ be equipped with the γ topology.

Corollary 3.9. *Let X be a locally compact space. Then*

$$M(X) = C_\gamma^c(X)^*,$$

as topological vector spaces.

Proof. In view of Theorem 3.8, it is sufficient to show that the restriction map

$$\Gamma : \widetilde{C}_\gamma(X)^* \longrightarrow C_\gamma^c(X)^*$$

is a linear topological isomorphism. For this, the only thing remains to prove is that Γ is injective, then the rest is straightforward. Let $f \in C_c(X)$ and $K = \mathrm{Supp}(f)$. Then, from the Urysohn Lemma it follows that

$$f\chi_K = \inf\{g \in C_c^+(X) : \; g \geq f\chi_K\}.$$

Now, by Theorem 2.1.5 in [2] and Theorem 3.8, we infer that

$$F(f\chi_K) = \sup\{F(g) : g \in C_c^+(X),\, g \geq f\chi_K\}.$$

From which we conclude that Γ is injective. $\square$

We conclude this section by an example which shows that there exists a linear functional on $\widetilde{C}_b(X)$ which is continuous if $\widetilde{C}_b(X)$ has the supremum norm, but not continuous for the γ topology.

Example 3.10. Let $V = \{g \in \widetilde{C}_b(\mathbb{R}) : \lim_{x\to 0^+} g(x) \text{ exists}\}$ and $F : V \longrightarrow \mathbb{R}$ be defined by $F(g) = \lim_{x\to 0^+} g(x)$ for each $g \in V$. Let V and $\widetilde{C}_b(\mathbb{R})$ equipped with the supremum norm. Then $V \subset \widetilde{C}_b(\mathbb{R})$ and $F \in V^*$, so there is a positive functional $\bar{F} \in \widetilde{C}_b(\mathbb{R})^*$ such that $\bar{F}|_V = F$, $||\bar{F}|| = 1$ and $\bar{F} \notin \widetilde{C}_\gamma(\mathbb{R})^*$, which shows that $\widetilde{C}_\gamma(\mathbb{R})^* \subsetneqq \widetilde{C}_b(\mathbb{R})^*$. In fact, let C be the Cantor set. Then $\chi_C \in \widetilde{C}_b(\mathbb{R})\backslash V$, which proves the first assertion. For the second assertion, suppose, on the contrary, $\bar{F} \in \widetilde{C}_\gamma(\mathbb{R})^*$. Thus, by Proposition 3.8, there exist $\mu, \nu \in M^+(\mathbb{R})$ such that $\bar{F}(g) = \int_{\mathbb{R}} g d\mu - \int_{\mathbb{R}} g d\nu$ for $g \in \widetilde{C}_b(\mathbb{R})$. Let $K \subseteq (0,\infty)$ be a compact subset. Then

$$\mu(K) - \nu(K) = \bar{F}(\chi_K) = F(\chi_K) = 0.$$

Similarly, $\mu(K) = \nu(K)$ for each compact set $K \subseteq (-\infty, 0)$. Also $\mu(\{0\}) - \nu(\{0\}) = \bar{F}(\chi_{\{0\}}) = 0$. Therefore, $\mu = \nu$ and $\bar{F} = 0$, which is a contradiction.

4. Locally convex space $\widehat{C}_b(X)$

We begin with some definitions. By a sood we mean a bounded Borel measurable function $\psi : X \longrightarrow [0,\infty)$ with compact support. We denote by $Sd(X)$ the set of all soods on X. We also denote by $\widehat{C}_b(X)$ the vector space generated by the set $\{f\psi \ : \ f \in C_b(X),\ \psi \in Sd(X)\}$. That space actually consists of all bounded Borel measurable functions on X with compact support.

Let $\mathcal{I}$ be a subset of linear functionals on $\widehat{C}_b(X)$. Then $\mathcal{I}$ is said to be uniformly τ_s-smooth, if for any net $(f_\alpha)_\alpha$ in $C_b(X)$ such that $f_\alpha \searrow f$ with $f \in \widehat{C}_b(X)^+$ and $\psi \in Sd(X)$, then for each $\varepsilon > 0$ there is α_0 such that

$$|I(f_\alpha\psi) - I(f\psi)| < \varepsilon$$

for all $I \in \mathcal{I}$ and $\alpha \geq \alpha_0$. A linear functional F on $\widehat{C}_b(X)$ is called τ_s-smooth whenever the set $\{F\}$ is uniformly τ_s-smooth.

Let $\mathcal{I}$ be a uniformly τ_s-smooth set of linear functionals on $\widehat{C}_b(X)$, and let $\mathcal{P}_{\mathcal{I}} : \widehat{C}_b(X) \longrightarrow [0,\infty)$ be defined by

$$\mathcal{P}_{\mathcal{I}}(g) = \sup\{|I(g)| \ : \ I \in \mathcal{I}\}$$

for each $g \in \widehat{C}_b(X)$. It is straightforward to see that $\mathcal{P}_{\mathcal{I}}$ is a seminorm on $\widehat{C}_b(X)$. Then by σ topology on $\widehat{C}_b(X)$ we mean the locally convex topology generated by the seminorms $\mathcal{P}_{\mathcal{I}}$. We write $\widehat{C}_\sigma(X)$ for the space $\widehat{C}_b(X)$ equipped with the σ topology.

We omit the proof of the following proposition because it is similar to the proof of Proposition 3.2.

Proposition 4.1. *A linear functional on $\widehat{C}_b(X)$ is σ-continuous if and only if it is τ_s-smooth.*

Thus the dual of $\widehat{C}_\sigma(X)$ consists of τ_s-smooth linear functionals.

Lemma 4.2. *Let X be a C-distinguished space, and let $\psi \in Sd(X)$ and I be a positive linear functional on $\widehat{C}_b(X)$. Then $I_\psi(f) := I(f\psi)$ is a β-continuous functional on $C_b(X)$.*

Proof. Suppose that I is a positive linear functional on $\widehat{C}_b(X)$, and $(f_\alpha)_\alpha$ be a net in $C_b(X)$ such that $f_\alpha \to 0$ in β-topology. Then, by definition, $\|f_\alpha\psi\| \to 0$ for each $\psi \in Sd(X)$. Let K be a compact subset such that $\psi = \psi\chi_K$, then the positivity of I gives

$$|I_\psi(f_\alpha)| = |I(\psi f_\alpha)| = |I(f_\alpha\psi\chi_K)| \leq \|f_\alpha\psi\| I(\chi_K),$$

for all α. Hence I_ψ is β-continuous on $C_b(X)$. □

The Lemma 4.2 allows us to define a topology on $\widehat{C}_\sigma(X)^*$. In fact, we equip $\widehat{C}_\sigma(X)^*$ with the weakest topology which makes $q_\psi : \widehat{C}_\sigma(X)^* \longrightarrow C_\beta(X)^*$ continuous for each $\psi \in Sd(X)$, where $q_\psi(I) = I_\psi$.

The next theorem is our main result in this section.

Theorem 4.3. *Let X be a C-distinguished space. Then the mapping $\Psi : M(X) \to \widehat{C}_\sigma(X)^*$ defined by $\Psi([\mu,\nu]) = I_\mu - I_\nu$, where*

$$I_\mu(g) = \int_X g d\mu, \quad (g \in \widehat{C}_b(X))$$

is a linear topological isomorphism.

Proof. Let us first prove that Ψ is well-defined. Given $\mu \in M^+(X)$, let $I_\mu(g) = \int_X g d\mu$ for each $g \in \widehat{C}_b(X)$. Suppose that $(f_\alpha)_\alpha$ be a net in $C_b(X)$ with $f_\alpha \searrow 0$. Set ${f'}_\alpha := f_{\alpha_0} - f_\alpha$ for $\alpha \geq \alpha_0$, where α_0 is an arbitrary but fixed index. Thus $f'_\alpha \nearrow f_{\alpha_0}$. Since $\mu\psi$ is a Radon measure for $\psi \in Sd(X)$, by Theorem 2.1.5 of [2], we deduce that

$$\int_X f'_\alpha \, d(\psi\mu) \to \int_X f_{\alpha_0} \, d(\psi\mu),$$

which means that $I_\mu(\psi f'_\alpha) \to 0$. Consequently, $I_\mu(\psi f_\alpha) \to 0$. That is I is τ_s-smooth and, by Proposition 4.1, $I \in \widehat{C}_\sigma(X)^*$, as required.

It is easy to verify that Ψ is injective. Now, we are going to prove that Ψ is surjective. Let $I \in \widehat{C}_\sigma(X)^*$ be an arbitrary element. A similar argument as in Proposition 3.3 shows that the σ topology is locally solid. Now invoke Lemma 4.3. in [9] to decompose I as $I = I^+ - I^-$, where $I^+, I^- \in \widehat{C}_\sigma(X)^*$ are positive functionals. Since the restrictions of I^+ and I^- to $\widetilde{C}_b(X)$ belong to $\widetilde{C}_\gamma(X)^*$, Proposition 3.6 implies that there exist unique measures $\mu_1, \mu_2 \in M^+(X)$ such that

$$I^+(g) = \int_X g\, d\mu_1,\ I^-(g) = \int_X g\, d\mu_2 \quad (g \in \widetilde{C}_b(X)).$$

In particular,

$$I^+(\chi_K) = \mu_1(K),\ I^-(\chi_K) = \mu_2(K) \quad (K \in \mathcal{K}).$$

We will show that $I = I_{\mu_1} - I_{\mu_2}$. To show this, we first assume that $g \in \widehat{C}_b(X)$ be a positive function. Then, by Lemma 3.5, we get

$$\begin{aligned} \int_X g\, d\mu_1 &= \sup\left\{\sum_{i=1}^n a_i\mu_1(K_i) :\ 0 \le \varphi := \sum_{i=1}^n a_i\chi_{K_i} \le g, \varphi \in \mathcal{C}\right\} \\ &= \sup_{\varphi\in\mathcal{C}}\{I^+(\varphi) : 0 \le \varphi \le g\} \\ &\le I^+(g). \end{aligned}$$

Let $g \in \widehat{C}_b(X)$ and $K = \mathrm{supp}(g)$. Then $g' = \|g\|\chi_K - g$ is a positive function in $\widehat{C}_b(X)$, and so

$$\int_X g'\, d\mu_1 \le I^+(g').$$

From the above, it follows that

$$\int_X g\, d\mu_1 \ge I^+(g) \quad (g \in \widehat{C}_b(X)).$$

By replacing g with $-g$ in the above inequality, we obtain

$$I^+(g) = \int_X g\, d\mu_1.$$

Similarly,

$$I^-(g) = \int_X g\, d\mu_1 \quad (g \in \widehat{C}_b(X)).$$

Thus, we proved that $I = I_{\mu_1} - I_{\mu_2}$.

Finally, we are required to prove that Ψ is continuous. To prove this, suppose $([\mu_\alpha, \nu_\alpha])_\alpha$ be a net in $M(X)$ which converges to zero. Then the net $(\mu_\alpha\chi_K - \nu_\alpha\chi_K)_\alpha$ converges to zero in $M_b(X)$ for each compact subset K in X. Given $\psi \in Sd(X)$ and $f \in C_b(X)$ with $\|f\| \le 1$ and $K = \mathrm{Supp}(\psi)$, we have

$$\begin{aligned} |I_{\mu_\alpha}(f\psi) - I_{\nu_\alpha}(f\psi)| &= \left|\int_X \psi f\, d\mu_\alpha - \int_X \psi f\, d\nu_\alpha\right| \\ &= \left|\int_X \psi f\, d(\mu_\alpha\chi_K) - \int_X \psi f\, d(\nu_\alpha\chi_K)\right| \\ &\le \int_X |\psi f|\, d|\mu_\alpha\chi_K - \nu_\alpha\chi_K| \\ &\le \|\psi\|\, \|\mu_\alpha\chi_K - \nu_\alpha\chi_K\|. \end{aligned}$$

From which, we conclude that the net $\Psi([\mu_\alpha, \nu_\alpha]) = I_{\mu_\alpha} - I_{\mu_\alpha}$ converges to zero in $\widehat{C}_\sigma(X)^*$; i.e., Ψ is continuous.

Conversely, assume $(I_\alpha)_\alpha$, where $I_\alpha = I_{\mu_\alpha} - I_{\nu_\alpha}$, be a net converges to $I := I_\mu - I_\nu$ in $\widehat{C}_\sigma(X)^*$. Then for each compact set K in X we have

$$\|(\mu_\alpha \chi_K - \nu_\alpha \chi_K) - (\mu \chi_K - \nu \chi_K)\| = \|(I_{\mu_\alpha \chi_K} - I_{\nu_\alpha \chi_K}) - (I_{\mu \chi_K} - I_{\nu \chi_K})\|,$$

so that $[\mu_\alpha, \nu_\alpha] \to [\mu, \nu]$ in $M(X)$. This completes the proof. □

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