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Author(s):

S. Rezakhah and A. R. Soltani

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DOMAIN OF ATTRACTION OF NORMAL LAW AND ZEROS OF RANDOM POLYNOMIALS

S. REZAKHAH* AND A. R. SOLTANI

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Abstract. Let $P_n(x) = \sum_{i=0}^{n} A_i x^i$ be a random algebraic polynomial, where $A_0, A_1, \ldots$ is a sequence of independent random variables belong to the domain of attraction of the normal law. Thus A_j's for $j = 0, 1, \ldots$ possess the characteristic functions $\exp\{-\frac{1}{2}t^2 H_j(t)\}$, where $H_j(t)$'s are complex slowly varying functions. Under the assumption that there exist a real positive slowly varying function $H(\cdot)$ and positive constants t_0, C_* and C^* that $C_* H(t) \le \mathrm{Re}[H_j(t)] \le C^* H(t),\ t \le t_0,\ j = 1, \ldots, n$, we find that while the variance of coefficients are bounded, real zeros are concentrated around ± 1, and the expected number of real zeros of $P_n(x)$ round the origin at a distance $(\log n)^{-s}$ of ± 1 are at most of order $O\left((\log n)^s \log(\log n)\right)$.

Keywords: Random algebraic polynomial, Expected number of real zeros, Slowly varying function, Domain of attraction of Normal law.
MSC(2010): Primary: 60H25; Secondary: 60E10.

1. **Introduction**

Since the fundamental paper of Kac [8], random algebraic polynomials have received tremendous attentions from researchers in theoretical and applied fields of science and engineering. Among certain features, the asymptotic behavior of the expected number of real zeros of random algebraic polynomials, as the degree n increases, have been investigated intensively. There are varieties in techniques and results depending on the statistical assumptions on the random vector $(A_0, A_1, \ldots, A_n)$ formed by the coefficients. The cases that coefficients are iid were targeted first. The iid normal case is treated by Kac [8], Sambandham [23], Wilkins [24], Farahmand [4] and others, see Farahmand [5] for a complete survey. The iid stable case is treated by Logan and Shepp [12]. The

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*Corresponding author.

case that the coefficients are iid follow a distribution in the domain of attraction of the normal law was treated by Ibragimov and Maslova [10,11]. Recently there has been much interest in cases where the coefficients of random algebraic polynomials form certain random processes. Rezakhah and Soltani [20,21] studied the expected density and asymptotic behavior of the expected number of real zeros when the coefficients are neither independent and nor identically distributed and follow a Levy and Harmonizable stable process, or form consecutive observations of Brownian motion. Rezakhah and Shemehsavar [14–17] followed the case where the coefficients are Brownian points and studied the asymptotic behavior of the expected number of level crossings, slop crossings, local maxima, and sharp crossings of such random polynomials.

Edelman and Kostlan [3], studied the asymptotic behavior of the expected number of real zeros for the case that the coefficients $A_k, \quad k = 1, 2, \dots n$, are independent centered normally distributed with $\mathrm{Var}(A_k) = \binom{n}{k}$, where the variance of the coefficients are increasing in n. They showed that the expected number of real zeros of $P_n(x)$ increases from the order of $\log(n)$ to the order $\sqrt{n}$ in compare to the case where the variance of the coefficients where not subject to the increase with n, see [5] for a review of previous studies. Their studies reveals that in such a case zeros are concentrated around zero in compare to the previous studies where zeros are concentrated around ± 1. The expected density for this case is plotted in figure 1.

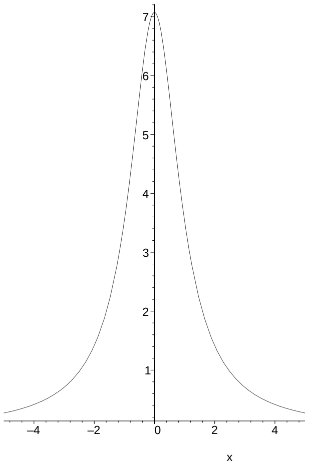

Figure 1. Expected density of $P_n(x)$

More recently Rezakhah and Shemehsavar [18, 19] considered different cases where the variance of the coefficients are increasing in n, and found that in these cases again roots of the algebraic polynomials are concentrated around zero.

In this work, we extend the work of Ibragimov and Maslova [10] to the case where the coefficients are not identically distributed. Our results illuminate that when the coefficients are independent and their variances are bounded, the roots are mainly concentrated round ±1. More precisely we study the case where the coefficients are independent and are not necessarily identically distributed, and belong to the domain of attraction of the normal law with some bounded-ness condition for the exponent of their characteristic functions. In such a case we provide an asymptotic upper bound for the expected number of the real zeros of random algebraic polynomials with degree n in the interval $(-1+\big(\log n\big)^{-s}, 1-\big(\log n\big)^{-s})$, to be of order $(\log n)^s \log\log n$ for all $0 < s \leq 1$. This result clarifies that for such wide class of distribution of the coefficients the roots of the algebraic polynomials are accumulated around ±1, and also clarifies that while the coefficients are independent the accumulation of the roots around zero just happens when the variance of the coefficients are not bounded, like the cases where considered in [3, 18, 19]. These would be an exceptional illumination in the behavior of the distribution of zeros of random algebraic polynomials. We should clarify that we follow some techniques of Ibragimov and Maslova [10] in this study.

This paper is organized as follows. In Section 2 we provide some preliminaries and some refinements on slowly varying complex functions. In Section 3 we state and prove the main result of this article.

2. **Preliminaries**

Let $X_1, X_2, \ldots$ be a sequence of independent random variables with a common distribution $F(\cdot)$, the distribution $F(\cdot)$ belongs to the domain of attraction of a distribution $G(\cdot)$ if there exist sequences of constants $a_n > 0$ and b_n such that the distribution of $a_n^{-1}(X_1 + X_2 + \ldots + X_n) - b_n$ tends to $G(\cdot)$, we take $G(\cdot)$ to be standard normal distribution $\Phi(\cdot)$.

Let X be a non-degenerate zero mean random variable with a distribution function $F(\cdot)$ in the domain of attraction of $N(0,1)$. Let $\phi(t)$ be the characteristic function of X. We cite the following fact from Chung [2]: Since $\phi(0) = 1$ and $\phi(t)$ is uniformly continuous on some interval of zero, so there is an interval $[-T, T]$ on which $\phi(t)$ is non-zero. Thus there is a unique continuous function $\Psi : [-T, T] \to \mathcal{C}$, ($\mathcal{C}$ the set of complex numbers), that $\Psi(0) = 0$ and $\phi(t) = \exp\{\Psi(t)\}$ for $-T \leq t \leq T$. Now define $H(t) = -2t^{-2}\Psi(t)$, $t \in [-T, T] - \{0\}$. Then by the fact that $\phi(-t) = \bar{\phi}(t)$ we have that:

$$(2.1)\qquad \begin{array}{ll} (i)- & H(t) \text{ is continuous on } [-T,T]-\{0\} \\ (ii)- & \phi(t)=\exp\{-\frac{1}{2}t^2H(t)\} \\ (iii)- & H(-t)=\bar{H}(t), \end{array}$$

where $\bar{H}(\cdot)$ is the complex conjugate. Thus we shall take $t>0$, and recall from Ibragimov and Linnik [9] that, as the distribution of X is in the domain of attraction of standard normal law, $H(t)$ is a *complex slowly varying at zero.* Thus $\dfrac{H(\lambda t)}{H(t)} \to 1$, as $t \downarrow 0$, for each fixed λ.

Let

$$Q_n(x)=\sum_{j=0}^{n} A_j x^j, \tag{2.2}$$

be a random algebraic polynomial of degree n, where the random coefficients $A_0, A_1, \ldots, A_n$ are assumed to be independent. We also assume each A_j, $j=0,\ldots,n$, belongs to the domain of attraction of the normal law, possessing the characteristic function

$$\varphi_j(t)=\exp\left\{-\frac{1}{2}t^2H_j(t)\right\}, \quad \text{for } t \text{ near zero,} \tag{2.3}$$

where $H_j(t)$ is a *complex slowly varying function:*

$$\lim_{t\to 0}\frac{H_j(\tau t)}{H_j(t)}=1,$$

The following lemma is a refinement on complex slowly varying functions.

Lemma 2.1. *If a distribution function belongs to the domain of attraction of the normal law, then its characteristic function possesses* (2.1) *and its corresponding complex slowly varying function* $H(t)$ *satisfies*

$$H(t)=Re[H(t)](1+o(1)), \quad \text{as } t \longrightarrow 0, \tag{2.4}$$

where $Re[z]$ *stands for the real part of a complex number* z.

Proof. Let X be non-degenerate zero mean random variable with a distribution function $F(x)$ in the domain of attraction of $N(0,1)$. Let $\phi(t)$ be the characteristic function of X. According to Feller [6, Section 17.5, Theorem 1], the necessary and sufficient condition for $F(x)$ to belong to the domain of attraction of $N(0,1)$ is that the truncated variance $\mathrm{U}(x) := \int_{(-x,x]} y^2 dF(y)$ to be slowly varying at infinity, that is

$$\frac{U(sx)}{U(s)} \to 1 \qquad \text{as} \qquad s \to \infty$$

for every positive x. Thus if $X_1, \ldots, X_n$ are i.i.d. with law F, in the domain of attraction of $N(0,1)$, then $(a_n^{-1} \sum_{i=1}^{n} X_i) \Rightarrow N(0,1)$, where it is necessary and sufficient that the constants $a_n > 0$ satisfy, see Loève [13, page 364],

$$\frac{n}{a_n^2} \mathrm{U}(a_n) \to 1 \qquad (n \to \infty).$$

It follows that the sequence (a_n) is regular varying with index 1/2, that is, the function $a_{[x]}$ is regularly varying with index 1/2. In particular $a_n \to \infty$ and $a_{n+1}/a_n \to 1$ as $n \to \infty$, see Loève [13, page 364]. In the special case where $\sigma^2 := \mathrm{Var}(X) < \infty$, clearly the truncated variance is a slowly varying function, and the requirement on a_n reduces to $a_n \sim \sigma\sqrt{n}$. But in general $F(\cdot)$ does not necessarily have a finite second moment. This special case corresponds to H being continuous at 0 with value σ^2 there, see [9, page 90].

We denote the corresponding characteristic function of $F(\cdot)$ by $\phi(\cdot)$,so

$$\phi^n(t/a_n) \to \exp\{-t^2/2\}, \qquad (n \to \infty),$$

for each $t \in R$. For any $t > 0$ we have $t/a_n \leq T$ for all large n; and so, by (2.1), the left-hand side equals $\exp\{-(n/2)(t/a_n)^2 H(t/a_n)\}$. By the uniqueness in the result of Chung [2] quoted at the beginning of this section, it follows that for each $t > 0$,

$$\frac{n}{a_n^2} H(t/a_n) \to 1 \qquad (n \to \infty).$$

Write $H(t) = \mathrm{Re}[H(t)] + i\mathrm{Im}[H(t)]$, then

$$\frac{\mathrm{Im}[H(t/a_n)]}{\mathrm{Re}[H(t/a_n)]} \to 0, \qquad (n \to \infty), \tag{2.5}$$

for each $t > 0$.

Now since X is non-degenerate, there exists $S > 0$ such that $|\phi(t)| < 1$ for $0 < t \leq S$. We may take $S \leq T$, then since $|\phi(t)| = \exp\{-(1/2)t^2\mathrm{Re}[H(t)]\}$, we deduce that $\mathrm{Re}[H(t)] > 0$ for $0 < t \leq S$. Thus by (2.1) we find that $u(t) := \mathrm{Im}[H(t)]/\mathrm{Re}[H(t)]$ is continuous on $0 < t \leq S$. We set $u(t) := u(S)$ for $t > S$ in order that $u(\cdot)$ will be defined and continuous on $(0, \infty)$. Set $c_n := \log a_n$ and $v(x) := u(\exp(-x))$ for $x \in R$, then by the fact that $a_n \sim \sigma\sqrt{n}$, so (2.5) can be written as

$$v(c_n + x) = u\left(\frac{\exp(-x)}{a_n}\right) \to 0, \qquad (n \to \infty).$$

for each fixed $x \in R$. Now v is continuous and the properties of a_n yields that $c_n \to \infty$ and $c_{n+1} - c_n \to 0$ as $n \to \infty$. This gives us the Kingman conditions needed for Theorem 1.9.1(ii) in [1] are fulfilled, giving that $v(x) \to 0$ as $x \to \infty$. Thus $u(t) \to 0$ as $t \downarrow 0$, giving that

$$\frac{H(t)}{\mathrm{Re}[H(t)]} - 1 = i\,\frac{\mathrm{Im}[H(t)]}{\mathrm{Re}[H(t)]} \to 0,$$

as $t \downarrow 0$. Thus as $t \to 0$,

$$\frac{H(t)}{\mathrm{Re}[H(t)]} - 1 = o(1) \quad \Rightarrow \quad H(t) = \mathrm{Re}[H(t)](1 + o(1)).$$

The proof of the Lemma is complete. □

Let $N_n(\alpha, \beta)$ denote the number of real zeros of the random polynomial $Q_n(x)$, given in (2.2), lying in the interval (α, β). Also let $N_n \equiv N_n(-\infty, +\infty)$. Ibragimov and Maslova [10, 11] considered the average number of real zeros when the coefficients are iid and belong to the domain of attraction of the normal law. Two cases were treated by them: (i) the coefficients possess zero mean, and (ii) the coefficients possess nonzero mean. They showed that

$$EN_n(R) \sim \begin{cases} (2/\pi)\log n, & E\{A_k\} = 0, \\ (1/\pi)\log n, & E\{A_k\} = m(\neq 0). \end{cases}$$

In the following section we will provide an upper bound for the $EN_n[-1 + (\log n)^{-s}, 1 - (\log n)^{-s}]$, $0 < s \leq 1$, under an assumption milder than "identically distributed". This will allow to deduce the relative reduction in the upper bound for the expected number of real zeros in the interval $[-1 + (\log n)^{-s}, 1 - (\log n)^{-s}]$, as it shrinks, i.e., $s \downarrow 0$.

3. An asymptotic upper bound

Let us state the main result of this article. Through out this section we assume that the coefficients $A_0, A_1, \ldots, A_n$ are:
(i) non-degenerate and centered,
(ii) belong to the domain of attraction of the normal law, possess slowly varying complex functions $H_0, H_1, \ldots, H_n$ (as justified in Lemma 2.1),
(iii) there exist constants $t_0 > 0$, C_* and C^* $(0 < C_* < C^* < \infty)$ such that

$$C_* H(t) \leq \mathrm{Re}[H_j(t)] \leq C^* H(t)$$

for all $t \leq t_0$ and all j, where $H(t)$ is a positive slowly varying function.

Theorem 3.1. *Under the conditions (i),(ii) and (iii) given above,*

$$EN_n[-1 + (\log n)^{-s}, 1 - (\log n)^{-s}] \leq C(\log n)^s \log\log n, \quad n \to \infty, \ \ 0 < s \leq 1.$$

Before processing to the proof of the theorem, Let us record that

$$Q_n(x) = x^n \sum_{j=0}^{n} A_{n-j} x^{-j} = x^n Q_n^*(y), \qquad y = \frac{1}{x}. \tag{3.1}$$

Therefore corresponding to every zero of the polynomial $Q_n(x)$ in the interval (0,1) (or (-1,0)) there is a zero of the polynomial Q_n^* on the half-line $(1, \infty)$ (or $(-\infty, -1)$). As $Q_n^*(\cdot)$ and $Q_n(\cdot)$ both satisfy the conditions of theorem, we have that

$$EN_n(0,1) = EN_n(1,\infty), \qquad EN_n(-1,0) = EN_n(-\infty,-1).$$

Hence, as it is known to the experts in this field, it will be sufficient to confine ourselves to $-1 < x < 1$.

For the proof of the theorem we needs the following lemma, that extends the formula (2) in Rudin [22, Section 15.20].

Lemma 3.2. *Let $f(z)$ be holomorphic in the disc $|z| < R$ and continuous on the closed disc $|z| \leq R$.Then the number of zeros of $f(z)$ in the disc $|z| < r$, where $0 < r < R$, does not exceed*

$$\frac{1}{\log(R/r)} \log \frac{\sup\limits_{|z|=R} |f(z)|}{|f(0)|}. \tag{3.2}$$

PROOF. Let $n(r)$ be the number of zeros, counting multiplicities, of $f(z)$ in $|z| < r$. Denote these zeros by $\alpha_1, \ldots, \alpha_{n(r)}$. Let $g(z) := \prod_{i=1}^{n} \frac{z-(\alpha_i/R)}{1-z(\overline{\alpha}_i/R)}$ be the Blashke product. Then the function $\frac{f(Rz)}{g(z)}$ is holomorphic in the open unit disc. The maximum modulus principle [22] thus gives

$$\left| \tfrac{f(0)}{g(0)} \right| \leq \sup_{|z|=1} \left| \tfrac{f(Rz)}{g(z)} \right|$$

Now it is clear that for every complex β and z with $|\beta| < 1$ and $|z| = 1$, $\left|\frac{z-\beta}{1-\overline{\beta}z}\right| = 1$. Therefore $|g(z)| = 1$ whenever $|z| = 1$. Thus

$$\left| \tfrac{f(0)}{g(0)} \right| \leq \sup_{|z|=1} |f(Rz)| = \sup_{|z|=R} |f(z)| = S_f(R).$$

So

$$|f(0)| \leq S_f(R)|g(0)| = S_f(R) \prod_{i=1}^{n} \left|\tfrac{\alpha_i}{R}\right| \leq S_f(R) \left(\tfrac{r}{R}\right)^{n(r)}$$

and the result follows.

Proof of Theorem 3.1 The variation of $H_j(t)$ is bounded, so there exist a positive number c that

$$q_j := \Pr(|A_j| < c) \quad \text{and} \quad q^* = \sup_{j \geq 1} q_j < 1. \tag{3.3}$$

This is by the fact that $C_*H(t) \leq \mathrm{Re}[H_j(t)] \leq C^*H(t)$ for all j and t, we may choose $t_0 > 0$ such that $H(t_0) \neq 0$. Then $\mathrm{Re}[H_j](t_0) \geq C_*H(t_0) > 0$ if $H(t_0) > 0$; or $\mathrm{Re}[H_j(t_0)] \leq C^*H(t_0) \leq 0$ if $H(t_0) < 0$. So there is no sequence of integers $j_n \to \infty$ such that $H_{j_n}(t_0) \to 0$, i.e. so that $\phi_{j_n}(t_0) \to 1$. So there is no sequence of integers j_n such that $A_{j_n} \overset{d}{\to} 0$. Suppose that for every $c_1 > 0$, $\sup_{n\geq 1}\Pr(|A_n| < c_1) = 1$. Let $n_1 = \min\{n : \Pr(|A_n| < 1) > 0\}$, and for $j > 1$

$$n_j = \min\{n : n > n_{j-1},\ \Pr(|A_n| < j^{-1}) > 1 - j^{-1}\}.$$

Thus as $j \to \infty$, $n_j \to \infty$, and $\Pr(|A_j| < j^{-1}) \to 1$, so $A_{j_n} \overset{d}{\to} 0$ which contradicts the above assumption. Thus there exist some $c > 0$ such that $\sup_j q_j = \sup_j \Pr(|A_j| < c) < 1$.

Now let

$$B_k = \{\omega : |A_0| \leq c, \ldots, |A_{k-1}| \leq c, |A_k| > c\} \tag{3.4}$$

where $k = 0, 1, 2, \ldots$, and let $B = \{\omega : |A_0| \leq c, \ldots, |A_{[n^s]}| \leq c\}$, so $B^c = \bigcup_0^{[n^s]} B_k$. By using the Lemma 3.2, we have that on $B_{k,}$,

$$N_n(-r,r) \leq k + \frac{1}{\log(R/r)} \log \frac{\sup_{|z|=R} |Q_n^{(k)}(z)|}{k!c}, \tag{3.5}$$

due to the facts that $N_n(-r,r) \leq k + N_n^{(k)}(-r,r)$, where $N_n^{(k)}(-r,r)$ is the number of real zeros of $Q_n^{(k)}(z)$ in $(-r,r)$, $Q_n^{(k)}(0) = k!A_k$, and on B_k, $|A_k| > c$.

Let $C_{jk} = \frac{j!}{k!(j-k)!} R^{j-k}$. Then

(3.6)
$$\sup_{|z|=R} \frac{|Q^{(k)}(z)|}{k!} = \sup_{|z|=R} \Big| \textstyle\sum_{j=k}^n \frac{j!}{(j-k)!k!} z^{j-k} A_j \Big| \leq \sum_{j=k}^{n} C_{jk}|A_j|.$$

From this, formula (3.5) and the inequality $N_n(-r,r) \leq n+1$ the validity of the following inequality follows :

$$\begin{aligned} EN_n(-r,r) &\leq \sum_{k=0}^{[n^s]} k \Pr(B_k) + \frac{1}{\log(R/r)} \sum_{k=0}^{[n^s]} \int_{B_k} \log \sum_{j=k}^{n} C_{jk}|A_j| \Pr(dw) \\ &\quad - \frac{\log c}{\log(R/r)} \sum_{k=0}^{[n^s]} \Pr(B_k) + (n+1)\Pr(B) \qquad (3.7) \\ &\leq C\left(1 + \frac{1}{\log(R/r)}\right) + \frac{1}{\log(R/r)} \sum_{k=0}^{[n^s]} \int_{B_k} \log \sum_{j=k}^{n} C_{jk}|A_j| \Pr(dw). \end{aligned}$$

This is by the fact that, using the independence of A_j's and (3.3),

$$\begin{aligned}\sum_{k=0}^{[n^s]} k\Pr(B_k) &= \sum_{k=0}^{[n^s]}\sum_{j=1}^{k}\Pr(|A_0|<c,\ldots,|A_{k-1}|<c,|A_k|\geq c)\\ &\leq \sum_{j=1}^{\infty}\sum_{k=j}^{\infty}\Pr(|A_0|<c,\ldots,|A_{k-1}|<c,|A_k|\geq c)\\ &= \sum_{j=1}^{\infty}\Pr(|A_0|<c,\ldots,|A_{j-1}|<c)\\ &= \sum_{j=1}^{\infty}q_j^j \leq \sum_{j=1}^{\infty}{q^*}^j = \frac{q^*}{1-q^*},\end{aligned}$$

so it is bounded. Also as $n\to\infty$, (3.3) implies that

$$(n+1)\Pr(B) = (n+1)\Pr(|A_0|<c,\ldots,|A_{[n^s]}|<c) = (n+1)\prod_{j=0}^{[n^s]} q_j \to 0,$$

and this is bounded too.

Let us estimate the second term in the right hand of (3.7). Let

$$T>0, \qquad Z_k \equiv Z = E\sum_{j=k}^{n}|C_{jk}A_j|,$$

$$B_{k0} = \{\omega : \sum_{j=k}^{n} C_{jk}|A_j| > TZ\}, \qquad B_{ki} = \{\omega : e^i Z < \sum_{j=k}^{n} C_{jk}|A_j| \leq e^{i+1}Z\},$$

where i assumes the values $i_0=\log T, i_0+1, i_0+2,\ldots$. Then $B_{k0}=\cup_{i=i_0}^{\infty}B_{ki}$ and

$$\int_{B_k}\log\sum_{j=k}^{n}C_{jk}|A_j|\Pr(d\omega) = \int_{B_k\cap B_{k0}^c} + \sum_{i=i_0}^{\infty}\int_{B_k\cap B_{ki}}$$

$$\leq \quad \Pr(B_k\cap B_{k0}^c)\log Z_kT + \sum_{i=i_0}^{\infty}(i+1)\Pr(B_k\cap B_{ki}) + \Pr(B_k\cap B_{k0})\log Z_k,$$

but

$$\begin{aligned}\sum_{i=i_0}^{\infty}(i+1)\Pr(B_k\cap B_{ki}) &\leq i_0\sum_{i=i_0}^{\infty}\Pr(B_k\cap B_{ki}) + \sum_{j=0}^{\infty}(j+1)\Pr(B_{ki_0+j})\\ &\leq (\log T)\Pr(B_k\cap B_{k0}) + \sum_{j=0}^{\infty}\Pr\left(\textstyle\sum_{l=k}^{n}C_{lk}|A_l| > e^{i_0+j}Z\right),\end{aligned}$$

and by the definition of z,

$$\begin{aligned}\sum_{j=0}^{\infty}\Pr\left(\textstyle\sum_{l=k}^{n} C_{lk}|A_l| > e^{i_0+j}Z\right) &\le \sum_{j=0}^{\infty}\frac{E\left|\ \sum_{l=k}^{n} C_{lk}|A_l|\ \right|}{(e^{i_0+j}Z)}\\ &\le \sum_{j=0}^{\infty}\frac{E\sum_{l=k}^{n}|C_{lk}|A_l||}{e^{(i_0+j)}Z}\\ &\le \sum_{j=0}^{\infty}\frac{1}{e^{(i_0+j)}} = Ce^{-i_0} = \frac{C}{T},\end{aligned}$$

by c_r inequality, and C is constant. Thus

$$\int_{B_k}\log\sum_{j=k}^{n} C_{jk}|A_j|\Pr(d\omega) \le \Pr(B_k)\log Z_k + \Pr(B_k)\log T + \frac{C}{T},$$

setting $T = \frac{1}{\Pr(B_k)}$ in this inequality we obtain
(3.8)

$$\int_{B_k}\log\sum_{j=k}^{n} C_{jk}|A_j|\Pr(d\omega) \le \Pr(B_k)\log Z_k + C[\Pr(B_k) - \Pr(B_k)\log\Pr(B_k)],$$

where the second term in the right of the above inequality is bounded (since $\Pr(B_k) \le \prod_{i=0}^{k-1} q_i$). Inequalities (3.7) and (3.8) imply that

$$EN_n(-r,r) \le C[1 + (\log(R/r))^{-1}] + \big(\log(R/r)\big)^{-1}\sum_{k=0}^{[n^s]}\Big(\textstyle\prod_{i=0}^{k-1} q_i\Big)\log Z_k.$$

If $|R| < 1$, then by using the definition of C_{jk},

$$\begin{aligned}\log Z_k &= \log\left\{\ E\textstyle\sum_{j=k}^{n}|C_{jk}A_j|\ \right\}\\ &\le \log(\sup E|A_j|) + \log\sum_{j=k}^{n}|C_{jk}|\\ &\le C - \log(1-R)^{k+1} = C + (k+1)\log\frac{1}{1-R},\end{aligned}$$

and hence

$$EN_n(-r,r) \le C\left[1 + (\log(R/r))^{-1}\left(1 + \log\frac{1}{1-R}\right)\right]. \tag{3.9}$$

Observe that from relation (3.9) it follows that $EN_n(-r,r) = O(1)$ as $n \longrightarrow \infty$. for any fixed r in (0,1). Setting $r = 1 - (\log n)^{-s}$ and $R = 1 - \frac{1}{2}(\log n)^{-s}$, then $\log(R/r) = \log R - \log r = \frac{1}{c}(R-r) = \frac{1}{2c}(\log n)^{-s}$ where $r < c < R$, and $\log\frac{1}{1-R} = \log[2(\log n)^s] = \log 2 + s\log\log n$, thus

$$EN_n(-1 + (\log n)^{-s}, 1 - (\log n)^{-s}) \le C(\log n)^s\log\log n.$$

The proof of the Theorem is complete.

Remark 3.3. It follows from Theorem 3.1 that if the length of the interval is reduced from $2(1-(\log n)^{-s_1})$ to $2(1-(\log n)^{-s_2})$, $s_2 < s_1$, then the relative reduction in the rate of the asymptotic upper bound will be $[(\log n)^{s_1} - (\log n)^{s_2}]/(\log n)^{s_1}$. Thus the roots of such polynomials will accumulate around ± 1.

This provides an extension of the work of Ibragimov and Maslova [10] to the non identically distributed coefficients where the variances are bounded.

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(Saeid Rezakhah) Faculty of Mathematics and Computer Sciences, Amirkabir University of Technology, Tehran, Iran.
E-mail address: `rezakhah@aut.ac.ir`

(A. R. Soltani) Department of Statistics and Operational Research, Faculty of Science, Kuwait University, State of Kuwait.
E-mail address: `soltani@kuc01.kuniv.edu.kw`