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MATRIX REPRESENTATION OF A SIXTH ORDER STURM-LIOUVILLE PROBLEM AND RELATED INVERSE PROBLEM WITH FINITE SPECTRUM

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Abstract. In this paper, we find matrix representation of a class of sixth order Sturm-Liouville problem (SLP) with separated, self-adjoint boundary conditions and we show that such SLP have finite spectrum. Also for a given matrix eigenvalue problem $HX = \lambda VX$, where H is a block tridiagonal matrix and V is a block diagonal matrix, we find a sixth order boundary value problem of Atkinson type that is equivalent to matrix eigenvalue problem.
Keywords: Matrix representation, Sixth order Sturm-Liouville, Finite spectrum.
MSC(2010): Primary: 34B24; Secondary: 47A75.

1. Introduction

We consider a sixth-order Sturm-Liouville equation of the form

$$(p(x)y'''(x))''' + q(x)y(x) = \lambda w(x)y(x),\ x \in (a,b). \tag{1.1}$$

General, separated, self adjoint boundary conditions for (1.1) are of the form:

$$A_1u(a) + A_2v(a) = 0,\ B_1u(b) + B_2v(b) = 0, \tag{1.2}$$

where

$$u = [y, y', y''],\ v = [(py''')'', (py''')', py'''],$$

and A_1, A_2, B_1, B_2 are real matrices of order 3, such that

$$A_1A_2^T = A_2A_1^T,\ B_1B_2^T = B_2B_1^T,$$

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and the matrices $(A_1 : A_2)$ and $(B_1 : B_2)$ have rank 3 see [4]. In this paper, we consider separated, self-adjoint boundary conditions as follows

$$\begin{aligned} &\cos\alpha y(a) - \sin\alpha(py''')''(a) = 0, \ \cos\alpha y'(a) - \sin\alpha(py''')'(a) = 0 \\ &\cos\alpha y''(a) - \sin\alpha(py''')(a) = 0, \ \cos\beta y(b) - \sin\beta(py''')''(b) = 0 \\ &\cos\beta y'(b) - \sin\beta(py''')'(b) = 0, \ \cos\beta y''(b) - \sin\beta(py''')(b) = 0. \end{aligned} \tag{1.3}$$

Equation (1.1) with boundary conditions (1.3) is called sixth-order Sturm-Liouville problem (SLP). If λ is such that the SLP has a nontrivial solution, then λ is called an eigenvalue and nontrivial solution for that λ is called an eigenfunction. The set of all eigenvalues of SLP is called the spectrum. We assume that the interval (a, b) is finite and the coefficient functions $r = \frac{1}{p}, w, q$ are real and in $L^1(a, b)$. The classical results of self-adjoint Sturm-Liouville problem states that under this assumptions the eigenvalues are bounded below and can be ordered as

$$\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots,$$

where $\lim_{k\to\infty} \lambda_k = \infty$ see [3, 4, 5]. Spectral problems for differential equations arise in many different physical applications. Sixth-order Sturm-Liouville problems arise in astrophysics, i.e., the narrow convecting layers bounded by stable layers which are believed to surround A-type stars may be modeled by sixth-order boundary value problems, also this problem arise in hydrodynamic and magnetohydrodynamic stability theory [4, 6, 9]. Recently [7] studied a typical Sturm-Liouville problems having finite spectrum. More recently [8, 1, 2] studied typical Sturm-Liouville problems of second and fourth order having matrix representation. In this paper, we introduce a class of sixth order self-adjoint Sturm-Liouville problems with finite spectrum and we find matrix representation of these problems. Also for a given matrix eigenvalue problem $HX = \lambda VX$, we find a sixth order boundary value problem equivalent to matrix eigenvalue problem.

2. Matrix representation of SLP

Since, $q, w, r \in L^1(a, b)$, they can be identically zero on subintervals of (a, b). Using this fact and following quasi-derivatives we obtain the system formulation of equation (1.1) and matrix representation of SLP.
Let

$$u_1 = y, \ u_2 = y', \ u_3 = y'', \ u_4 = py''', \ u_5 = (py''')', \ u_6 = (py''')''. \tag{2.1}$$

Then, we obtain the system formulation of equation (1.1) as follows

$$\begin{array}{l} u_1' = u_2, \\ u_2' = u_3, \\ u_3' = ru_4, \\ u_4' = u_5, \\ u_5' = u_6, \\ u_6' = (\lambda w - q)u_1 \end{array}, \; x \in (a,b). \tag{2.2}$$

The functions $u_1, \cdots, u_6$ are called quasi-derivatives.

Definition 2.1. A differential equation of the form (1.1) is of Atkinson type if for some integer number $n > 2$, there exists a partition of the interval (a, b)

$$a = a_0 < b_0 < a_1 < b_1 < \cdots < a_n < b_n = b, \tag{2.3}$$

such that

$$r = 0, \; on \; [a_k, b_k], \; \int_{a_k}^{b_k} x^j w(x)dx \neq 0, j = 0, 1, 2, 3, 4, \; k = 0, 1, \cdots, n. \tag{2.4}$$

(2.5)

$$q = w = 0, \; on \; [b_{k-1}, a_k], \; \int_{b_{k-1}}^{a_k} x^j r(x)dx \neq 0, j = 0, 1, 2, 3, 4, \; k = 1, \ldots, n.$$

Definition 2.2. A SLP of Atkinson type and a matrix eigenvalue problem are equivalent if they have the same spectrum. Such matrix eigenvalue problem is called a matrix representation of SLP.

For matrix representation of SLP we need the following notations:

$$\begin{array}{ll} r_{jk} = \int_{b_{k-1}}^{a_k} x^j r(x)dx, & w_{jk} = \int_{a_k}^{b_k} x^j w(x)dx, \\ q_{jk} = \int_{a_k}^{b_k} x^j q(x)dx, & j = 0, 1, ..., 4 \end{array} \tag{2.6}$$

$$A_k = \begin{pmatrix} \frac{1}{4}r_{4k} & \frac{1}{2}r_{3k} & \frac{1}{2}r_{2k} \\ -\frac{1}{2}r_{3k} & -r_{2k} & -r_{1k} \\ \frac{1}{2}r_{2k} & r_{1k} & r_k \end{pmatrix}, \quad D_k = \det(A_k), \; r_k = r_{0k},$$

and A_{ij}^k denote the (i, j)th minor of matrix A_k.

In the subintervals $[a_k, b_k]$, r is zero, thus from the system formulation (2.2) we find that u_3 is constant, u_2 and u_1 are polynomials of order one and two, respectively. In subintervals $[b_{k-1}, a_k]$, $w = q = 0$, thus u_6 is constant, u_5 and u_6 are polynomials of order one and two, respectively. Thus we have

$$\begin{array}{l} u_3(x) = c_k, \qquad u_2(x) - xu_3(x) = d_k \\ u_1(x) + \frac{1}{2}x^2 u_3(x) - xu_2(x) = e_k, \; k = 0, 1, \ldots, n, \; x \in [a_k, b_k]. \end{array} \tag{2.7}$$

$$\begin{array}{l} u_6(x) = f_k, \qquad u_5(x) - xu_6(x) = g_k \\ u_4(x) + \frac{1}{2}x^2 u_6(x) - xu_5(x) = h_k, \; k = 1, \ldots, n, \; x \in [b_{k-1}, a_k]. \end{array} \tag{2.8}$$

Also we assume that

(2.9)

$$u_6(a) = f_0, \quad u_5(a) - au_6(a) = g_0, \quad u_4(a) + \tfrac{1}{2}a^2u_6(a) - au_5(a) = h_0,$$
$$u_6(b) = f_{n+1}, \quad u_5(b) - bu_6(b) = g_{n+1}, \quad u_4(b) + \tfrac{1}{2}b^2u_6(b) - bu_5(b) = h_{n+1}.$$

Lemma 2.3. *Suppose that the differential equation* (1.1) *is of Atkinson type, then for any solution y of SLP and $(u_1, u_2, u_3, u_4, u_5, u_6)$ of the system* (2.2) *we have:*

$$\begin{array}{l} e_k - e_{k-1} = \tfrac{1}{4}r_{4k}f_k + \tfrac{1}{2}r_{3k}g_k + \tfrac{1}{2}r_{2k}h_k \\ d_k - d_{k-1} = -\tfrac{1}{2}r_{3k}f_k - r_{2k}g_k - r_{1k}h_k \\ c_k - c_{k-1} = \tfrac{1}{2}r_{2k}f_k + r_{1k}g_k + r_kh_k \end{array}, \quad k = 1, 2, \ldots, n. \tag{2.10}$$

(2.11)

$$\begin{array}{l} f_{k+1} - f_k = e_k(\lambda w_k - q_k) + d_k(\lambda w_{1k} - q_{1k}) + \tfrac{1}{2}c_k(\lambda w_{2k} - q_{2k}) \\ g_{k+1} - g_k = -e_k(\lambda w_{1k} - q_{1k}) - d_k(\lambda w_{2k} - q_{2k}) - \tfrac{1}{2}c_k(\lambda w_{3k} - q_{3k}) \\ h_{k+1} - h_k = \tfrac{1}{2}e_k(\lambda w_{2k} - q_{2k}) + \tfrac{1}{2}d_k(\lambda w_{3k} - q_{3k}) + \tfrac{1}{4}c_k(\lambda w_{4k} - q_{4k}) \end{array},$$

for $k = 0, 1, 2, \ldots, n$.

Proof.

$$\begin{aligned} c_k - c_{k-1} &= u_3(a_k) - u_3(b_{k-1}) = \int_{b_{k-1}}^{a_k} u_3'(x)dx = \int_{b_{k-1}}^{a_k} ru_4(x)dx \\ &= \int_{b_{k-1}}^{a_k} r(\frac{1}{2}x^2f_k + xg_k + h_k)dx = r_kh_k + r_{1k}g_k + \frac{1}{2}r_{2k}f_k, \end{aligned}$$

$$\begin{aligned} d_k - d_{k-1} &= [u_2(a_k) - a_ku_3(a_k)] - [u_2(b_{k-1}) - b_{k-1}u_3(b_{k-1})] \\ &= u_2(a_k) - u_2(b_{k-1}) - (a_kc_k - c_{k-1}b_{k-1}) \\ &= \int_{b_{k-1}}^{a_k} u_2'(x)dx - (a_kc_k - c_{k-1}b_{k-1}) \\ &= \int_{b_{k-1}}^{a_k} u_3(x)dx - (a_kc_k - c_{k-1}b_{k-1}) \\ &= xu_3(x)|_{b_{k-1}}^{a_k} - \int_{b_{k-1}}^{a_k} xu_3'(x)dx - (a_kc_k - c_{k-1}b_{k-1}) \\ &= -\int_{b_{k-1}}^{a_k} xu_3'(x)dx = -\int_{b_{k-1}}^{a_k} xru_4(x)dx \\ &= -r_{1k}h_k - r_{2k}g_k - \frac{1}{2}r_{3k}f_k, \end{aligned}$$

$$
\begin{aligned}
e_k - e_{k-1} &= [u_1(a_k) + \frac{1}{2}a_k^2 u_3(a_k) - a_k u_2(a_k)] \\
&- [u_1(b_{k-1}) + \frac{1}{2}b_{k-1}^2 u_3(b_{k-1}) - b_{k-1}u_2(b_{k-1})] \\
&= \int_{b_{k-1}}^{a_k} u_1'(x)dx - \frac{1}{2}a_k^2 c_k + \frac{1}{2}b_{k-1}^2 c_{k-1} - a_k d_k + b_{k-1}d_{k-1} \\
&= xu_2|_{b_{k-1}}^{a_k} - \int_{b_{k-1}}^{a_k} xu_2'(x)dx - \frac{1}{2}a_k^2 c_k + \frac{1}{2}b_{k-1}^2 c_{k-1} - a_k d_k \\
&+ b_{k-1}d_{k-1} = \int_{b_{k-1}}^{a_k} \frac{1}{2}x^2 r u_4(x)dx \\
&= \int_{b_{k-1}}^{a_k} \frac{1}{2}rx^2(h_k + xg_k + \frac{1}{2}x^2 f_k)dx = \frac{1}{2}r_{2k}h_k + \frac{1}{2}r_{3k}g_k + \frac{1}{4}r_{4k}f_k.
\end{aligned}
$$

Relations (2.11) may be proved similarly. □

We assume that $\det(A_k) \neq 0,\ k = 1, 2, \ldots, n$. Thus, we can find f_k, g_k and h_k from (2.10) uniquely as follows

$$
\begin{aligned}
h_k &= \tfrac{1}{D_k}[(c_k - c_{k-1})A_{11}^k + (d_k - d_{k-1})A_{21}^k + (e_k - e_{k-1})A_{31}^k] \\
g_k &= \tfrac{1}{D_k}[(c_k - c_{k-1})A_{12}^k + (d_k - d_{k-1})A_{22}^k + (e_k - e_{k-1})A_{32}^k] \\
f_k &= \tfrac{1}{D_k}[(c_k - c_{k-1})A_{13}^k + (d_k - d_{k-1})A_{23}^k + (e_k - e_{k-1})A_{33}^k].
\end{aligned} \tag{2.12}
$$

Let

$$
S_k = \begin{pmatrix} A_{11}^k & A_{21}^k & A_{31}^k \\ -A_{12}^k & -A_{22}^k & -A_{32}^k \\ A_{13}^k & A_{23}^k & A_{33}^k \end{pmatrix},\ k = 1, 2, \ldots, n, \tag{2.13}
$$

By substituting (2.12) in (2.11) and using the matrix form we obtain

$$
S_k U_{k-1} + (-S_k - S_{k+1} + Q_k)U_k + S_{k+1}U_{k+1} = \lambda W_k U_k, \tag{2.14}
$$

where $k = 1, \ldots, n-1, U_k = [e_k, c_k, d_k]$ and

$$
W_k = \begin{pmatrix} w_k & w_{1k} & \frac{1}{2}w_{2k} \\ w_{1k} & w_{2k} & \frac{1}{2}w_{3k} \\ \frac{1}{2}w_{2k} & \frac{1}{2}w_{3k} & \frac{1}{4}w_{4k} \end{pmatrix}, Q_k = \begin{pmatrix} q_k & q_{1k} & \frac{1}{2}q_{2k} \\ q_{1k} & q_{2k} & \frac{1}{2}q_{3k} \\ \frac{1}{2}q_{2k} & \frac{1}{2}q_{3k} & \frac{1}{4}q_{4k} \end{pmatrix}. \tag{2.15}
$$

Substituting $k = 0$ in (2.11) we obtain

$$
\begin{aligned}
f_1 - f_0 &= e_0(\lambda w_0 - q_0) + d_k(\lambda w_{10} - q_{10}) + \tfrac{1}{2}c_k(\lambda w_{20} - q_{20}) \\
g_1 - g_0 &= -e_0(\lambda w_{10} - q_{10}) - d_0(\lambda w_{20} - q_{20}) - \tfrac{1}{2}c_0(\lambda w_{30} - q_{30}) \\
h_1 - h_0 &= \tfrac{1}{2}e_0(\lambda w_{20} - q_{20}) + \tfrac{1}{2}d_0(\lambda w_{30} - q_{30}) + \tfrac{1}{4}c_0(\lambda w_{40} - q_{40}).
\end{aligned} \tag{2.16}
$$

For $\alpha, \beta \in (0, \pi)$, applying boundary conditions in (2.16) we have

$$
(-S_1 - C_\alpha + Q_0)U_0 + S_1 U_1 = \lambda W_0 U_0, \tag{2.17}
$$

where

$$C_\alpha = \cot\alpha \begin{pmatrix} 1 & a & \frac{1}{2}a^2 \\ a & a^2-1 & \frac{1}{2}a^3-a \\ \frac{1}{2}a^2 & \frac{1}{2}a^3-a & 1-a^2+\frac{1}{4}a^4 \end{pmatrix}. \tag{2.18}$$

Substituting $k = n$ in (2.11) and using boundary conditions we obtain

$$S_n U_{n-1} + (-S_n + C_\beta + Q_n)U_n = \lambda W_n U_n, \tag{2.19}$$

where

$$C_\beta = \cot\beta \begin{pmatrix} -1 & b & \frac{1}{2}b^2 \\ b & b^2-1 & \frac{1}{2}b^3-b \\ \frac{1}{2}b^2 & \frac{1}{2}b^3-b & 1-b^2+\frac{1}{4}b^4 \end{pmatrix}. \tag{2.20}$$

Equations (2.14),(2.17) and (2.19) may be combined to give matrix eigenvalue problem

$$(P_{\alpha\beta} + Q_{\alpha\beta})U = \lambda W_{\alpha\beta} U,\ \alpha,\beta \in (0,\pi), \tag{2.21}$$

where

$$U = [U_0, U_1, \ldots, U_n],\ U_i = [e_i, d_i, c_i],\ Q_{\alpha\beta} = \mathrm{diag}(Q_0, Q_1, \ldots, Q_n),$$
$$W_{\alpha\beta} = \mathrm{diag}(W_0, W_1, \ldots, W_n),$$
$$P_{\alpha\beta} = \begin{pmatrix} -S_1 - C_\alpha & S_1 & & & \\ S_1 & -S_1 - S_2 & S_2 & & \\ & \ddots & \ddots & \ddots & \\ & & S_{n-1} & -S_{n-1} - S_n & S_n \\ & & & S_n & -S_n + C_\beta \end{pmatrix}.$$

From matrix A_k we see that the matrices S_k and thus matrix $P_{\alpha\beta}$ are symmetric. For any solution $y(x)$ of SLP (1.1),(1.3) and $(u_1, \ldots, u_6)$ of the system (2.2), if we define e_k, d_k and c_k by (2.7) then

$$U = [e_0, d_0, c_0, \ldots, e_n, d_n, c_n],$$

is an eigenvector of (2.21). Conversely, if $U = [e_0, d_0, c_0, \ldots, e_n, d_n, c_n]$ is an eigenvector of (2.21) and we define h_k, g_k, f_k by (2.12) and $u_1, \ldots, u_6$ by (2.7),(2.8) , then $(u_1, \ldots, u_6)$ is a solution of the system (2.7) and by integration of the equations (2.2) we can extend them continuously to the whole interval $J = (a, b)$. In subintervals $[a_k, b_k], u_1, u_2, u_3$ are defined by (2.7), for any $x \in [b_{k-1}, a_k]$ we have

$$u_3(x) - u_3(b_{k-1}) = \int_{b_{k-1}}^{x} r(s)u_4(s)ds = \int_{b_{k-1}}^{x} (h_k + sg_k + \frac{1}{2}s^2 f_k)r(s)ds,$$

$$\begin{aligned}u_2(x)-u_2(b_{k-1}) &= \int_{b_{k-1}}^{x} u_3(s)ds\\ &= \int_{b_{k-1}}^{x} c_{k-1}ds + \int_{b_{k-1}}^{x}\int_{b_{k-1}}^{s}[h_k + ug_k + \frac{1}{2}u^2 f_k]r(u)duds\\ &= c_{k-1}(x-b_{k-1}) + \int_{b_{k-1}}^{x}[h_k + sg_k + \frac{1}{2}s^2 f_k](x-s)r(s)ds,\end{aligned}$$

$$\begin{aligned}u_1(x)-u_1(b_{k-1}) &= \int_{b_{k-1}}^{x} u_2(s)ds = \int_{b_{k-1}}^{x}[u_2(b_{k-1}) + c_{k-1}(s-b_{k-1})]ds\\ &+ \int_{b_{k-1}}^{x}\int_{b_{k-1}}^{s}(h_k + ug_k + u^2 f_k)(s-u)r(u)duds\\ &= u_2(b_{k-1})(x-b_{k-1}) + \frac{1}{2}c_{k-1}(x-b_{k-1})^2\\ &+ \frac{1}{2}\int_{b_{k-1}}^{x}(h_k + sg_k + s^2 f_k)(x-s)^2 r(s)ds.\end{aligned}$$

In subintervals $[b_{k-1}, a_k]$, u_4, u_5, u_6 are defined by (2.8), for any $x \in [a_k, b_k]$ we have

$$u_6(x)-u_6(a_k) = \int_{a_k}^{x}(\lambda w - q)u_1(s)ds = \int_{a_k}^{x}(\lambda w - q)(e_k + sd_k + \frac{1}{2}s^2 c_k)ds,$$

$$\begin{aligned}u_5(x)-u_5(a_k) &= \int_{a_k}^{x} u_6(s)ds\\ &= \int_{a_k}^{x} f_k ds + \int_{a_k}^{x}\int_{a_k}^{s}(\lambda w - q)(e_k + ud_k + \frac{1}{2}u^2 c_k)duds\\ &= f_k(x-a_k) + \int_{a_k}^{x}(x-s)(\lambda w - q)(e_k + sd_k + \frac{1}{2}s^2 c_k)ds,\end{aligned}$$

$$\begin{aligned}u_4(x)-u_4(a_k) &= \int_{a_k}^{x}(\lambda w - q)u_1(s)ds = \int_{a_k}^{x}(u_5(a_k) + f_k(s-a_k))ds\\ &+ \int_{a_k}^{x}\int_{a_k}^{s}(s-u)(\lambda w - q)(e_k + ud_k + \frac{1}{2}u^2 c_k)dsdu\\ &= u_5(a_k)(x-a_k) + \frac{1}{2}f_k(x-a_k)^2\\ &+ \frac{1}{2}\int_{a_k}^{x}(e_k + sd_k + \frac{1}{2}s^2 c_k)(x-s)^2(\lambda w - q)ds.\end{aligned}$$

Summarizing the previous results, we come to the following Theorem:

Theorem 2.4. *Let $P_{\alpha\beta}$, $Q_{\alpha\beta}$,$W_{\alpha\beta}$ be defined as above and assume that for $k = 1, 2, \ldots, n$, $D_k \neq 0$ and let differential equation* (1.1) *be of Atkinson type.*

Then the SLP (1.1),(1.3) *is equivalent to matrix eigenvalue problem* (2.21), *thus such SLP has finite spectrum. Indeed the problem has at most* $3n+3$ *eigenvalues.*

Remark 2.5.

(1) If $\alpha = 0$, $\beta \in (0, \pi)$, then the boundary conditions imply that $e_0 = d_0 = c_0 = 0$. Thus we can eliminate the first row and column of the block tridiagonal matrix $P_{\alpha\beta}$. In this case the SLP is equivalent to

$$(P_\beta + Q_\beta)U_\beta = \lambda W_\beta U_\beta,$$

where

$$U_\beta = [U_1, \dots, U_n], Q_\beta = \operatorname{diag}(Q_1, \dots, Q_n),$$

$$W_\beta = \operatorname{diag}(W_1, \dots, W_n),$$

and P_β obtained by knocking the first row and column of $P_{\alpha\beta}$. Therefore in this case we have at most $3n$ eigenvalues.

(2) If $\beta = 0$, $\alpha \in (0, \pi)$, then the boundary conditions imply that $e_n = d_n = c_n = 0$. Thus we can eliminate the last row and column of the block tridiagonal matrix $P_{\alpha\beta}$. In this case the SLP is equivalent to

$$(P_\alpha + Q_\alpha)U_\alpha = \lambda W_\alpha U_\alpha,$$

where

$$U_\alpha = [U_0, U_1, \dots, U_{n-1}], Q_\alpha = \operatorname{diag}(Q_0, Q_1, \dots, Q_{n-1}),$$

$$W_\alpha = \operatorname{diag}(W_0, W_1, \dots, W_{n-1}),$$

and P_α obtained by knocking the last row and column of $P_{\alpha\beta}$. Therefore the problem has at most $3n$ eigenvalues.

(3) If $\beta = 0, \alpha = 0$, then the boundary conditions imply that $e_n = d_n = c_n = 0$ and $e_0 = d_0 = c_0$. Thus we can eliminate the first and last row and column of the block tridiagonal matrix $P_{\alpha\beta}$. In this case the SLP is equivalent to

$$(P + Q)U = \lambda W U,$$

where

$$U = [U_1, \dots, U_{n-1}], Q = \operatorname{diag}(Q_1, \dots, Q_{n-1}),$$

$$W = \operatorname{diag}(W_1, \dots, W_{n-1}),$$

and P obtained by knocking the first and last row and column of $P_{\alpha\beta}$. Therefore in this case the problem has at most $3n-3$ eigenvalues.

3. SLP with piecewise polynomial coefficients

First, we show that every SLP of Atkinson type is equivalent to a SLP with piecewise polynomial coefficients.

Lemma 3.1. *Let $t_1, t_2 \in \mathbb{R},\ t_1 \neq t_2$. Then for arbitrary real numbers $\{\eta_i\}_0^4$, there exists a unique polynomial function $p(x)$ of order 4 such that*

$$\int_{t_1}^{t_2} x^j p(x)dx = \eta_j,\ j = 0,1,2,3,4. \tag{3.1}$$

Proof. Let $p(x) = c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0$, then conditions (3.1) leads to

$$\begin{cases} \frac{t_2^5-t_1^5}{5}c_4 + \frac{t_2^4-t_1^4}{4}c_3 + \frac{t_2^3-t_1^3}{3}c_2 + \frac{t_2^2-t_1^2}{2}c_1 + (t_2-t_1)c_0 = \eta_0 \\ \frac{t_2^6-t_1^6}{6}c_4 + \frac{t_2^5-t_1^5}{5}c_3 + \frac{t_2^4-t_1^4}{4}c_2 + \frac{t_2^3-t_1^3}{3}c_1 + \frac{t_2^2-t_1^2}{2}c_0 = \eta_1 \\ \frac{t_2^7-t_1^7}{7}c_4 + \frac{t_2^6-t_1^6}{6}c_3 + \frac{t_2^5-t_1^5}{5}c_2 + \frac{t_2^4-t_1^4}{4}c_1 + \frac{t_2^3-t_1^3}{3}c_0 = \eta_2 \\ \frac{t_2^8-t_1^8}{8}c_4 + \frac{t_2^7-t_1^7}{7}c_3 + \frac{t_2^6-t_1^6}{6}c_2 + \frac{t_2^5-t_1^5}{5}c_1 + \frac{t_2^4-t_1^4}{4}c_0 = \eta_3 \\ \frac{t_2^9-t_1^9}{9}c_4 + \frac{t_2^8-t_1^8}{8}c_3 + \frac{t_2^7-t_1^7}{7}c_2 + \frac{t_2^6-t_1^6}{6}c_1 + \frac{t_2^5-t_1^5}{5}c_0 = \eta_4 \end{cases}. \tag{3.2}$$

The determinant of the coefficients matrix of (3.2) is $\frac{(t_2-t_1)^{25}}{266716800000}$. Thus the system (3.2) has a unique solution and there exists a unique polynomial $p(x)$ which satisfies conditions (3.1). □

We denote the polynomial constructed in Lemma 3.1 by $p(t_1, t_2, \eta_0, \eta_1, ..., \eta_4)$, and we define piecewise polynomials $\overline{r}(x), \overline{q}(x)$ and $\overline{w}(x)$ on J as follows

(3.3)

$$\overline{r}(x) = \begin{cases} p(b_{k-1}, a_k, r_k, r_{1k}, r_{2k}, r_{3k}, r_{4k}),\ x \in [b_{k-1}, a_k],\ k = 1, \dots, n \\ 0, x \in [a_k, b_k],\ k = 0, \dots, n \end{cases}$$
$$\overline{q}(x) = \begin{cases} p(a_k, b_k, q_k, q_{1k}, q_{2k}, q_{3k}, q_{4k}),\ x \in [a_k, b_k],\ k = 0, \dots, n \\ 0, x \in [b_k, a_k],\ k = 0, 1, \dots, n-1 \end{cases}$$
$$\overline{w}(x) = \begin{cases} p(a_k, b_k, w_k, w_{1k}, w_{2k}, w_{3k}, w_{4k}),\ x \in [a_k, b_k],\ k = 0, \dots, n \\ 0, x \in [b_k, a_k],\ k = 0, 1, \dots, n-1 \end{cases}$$

Theorem 3.2. *Let Eq.* (1.1) *of Atkinson type be given and r_{ik}, q_{ik}, w_{ik} are defined by* (2.6)*. Let $\overline{r}(x), \overline{q}(x)$ and $\overline{w}(x)$ on J be defined by* (3.3)*. Then the SLP* (1.1),(1.3) *is equivalent to SLP*

$$(\overline{p}(x)y'''(x))''' + \overline{q}(x)y(x) = \lambda\overline{w}(x)y(x),\ x \in (a,b), \tag{3.4}$$

with the same boundary conditions (1.3).

Proof. By Lemma 3.1 and relations (3.3) we find

$$\overline{r}_{ik} = \int_{b_{k-1}}^{a_k} x^i \overline{r}(x)dx = r_{ik},\ \overline{w}_{ik} = \int_{a_k}^{b_k} x^i \overline{w}(x)dx = w_{ik},$$

$$\overline{q}_{ik} = \int_{a_k}^{b_k} x^i \overline{q}(x)dx = q_{ik}.$$

Thus, both SLPs (1.1) and (3.4) have the same r_{ik}, w_{ik}, q_{ik}. Hence by Theorem 2.4, they are equivalent to the same matrix eigenvalue problem of the form (2.21) and they are equivalent. □

4. **SLP representation for matrix eigenvalue problem**

We consider matrix eigenvalue problem

$$HX = \lambda VX, \tag{4.1}$$

where H is a real symmetric block tridiagonal matrix of the form

$$H = \begin{pmatrix} H_{11} & H_{12} & & & \\ H_{12} & H_{22} & H_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & H_{n-2,n-1} & H_{n-1,n-1} & H_{n-1,n} \\ & & & H_{n-1,n} & H_{nn} \end{pmatrix}, \tag{4.2}$$

and $V = \operatorname{diag}(V_1, V_2, \ldots, V_n)$. Here H_{ij} and V_i are 3×3 symmetric matrices. Also we assume that $\det(V_i) \neq 0, V_i(2,2) = 2V_i(1,3)$, and

$$H_{i,i+1} = \begin{pmatrix} h_{11}^i & h_{12}^i & h_{13}^i \\ h_{12}^i & h_{22}^i & h_{23}^i \\ h_{13}^i & h_{23}^i & h_{33}^i \end{pmatrix},$$

where all entries of this matrix except one are known. We suppose that h_{22}^i is unknown. Suppose $h_{22}^i = h$, then determinant of $H_{i,i+1}$ is as follows

$$\det(H_{i,i+1}) = ch + d, \tag{4.3}$$

where c and d are known real numbers. Comparing (4.1) and (2.21) we find

$$H_{i,i+1} = S_i. \tag{4.4}$$

Hence, the determinants of matrices $H_{i,i+1}$ and S_i are equal and we obtain

$$-D_i = \frac{1}{\det(H_{i,i+1})} = \frac{1}{ch+d}. \tag{4.5}$$

If we consider $\frac{1}{ch+d} = q$, by (4.4) and (2.13) we find

$$\begin{cases} (r_{1i})^2 - r_i r_{2i} + h_{11}^i q = 0 \\ -\frac{1}{2} r_i r_{3i} + \frac{1}{2} r_{1i} r_{2i} + h_{12}^i q = 0 \\ -\frac{1}{2} r_{1i} r_{3i} + \frac{1}{2}(r_{2i})^2 + h_{13}^i q = 0 \\ \frac{1}{4} r_i r_{4i} - \frac{1}{4}(r_{2i})^2 - c_1 q + d_1 = 0 \\ \frac{1}{4} r_{1i} r_{4i} - \frac{1}{4} r_{2i} r_{3i} + h_{23}^i q = 0 \\ -\frac{1}{4} r_{2i} r_{4i} + \frac{1}{4}(r_{3i})^2 + h_{33}^i q = 0 \end{cases}. \tag{4.6}$$

We denote the nonlinear system (4.6) by $F(\mathbf{x}) = 0$, and $DF(\mathbf{x})$ denotes the jacobian matrix of $F(\mathbf{x})$. We define matrices $\{A_i\}_1^6$ as follows

$$A_i = DF(e_i),$$

where e_i is the vector whose ith coordinate is one and the others are zero , and $\gamma = \sum_1^6 \|A_i\|$.

Theorem 4.1. *Let $C = \{\mathbf{x} = (x_1, \ldots, x_6) | x_i \neq 0\}$ and $C_0 \subseteq C$ be the convex set in C such that*

(1) $DF(\mathbf{x}_0)^{-1}$ exist and $\|DF(\mathbf{x}_0)^{-1}\| \leq \beta$,
(2) $\|DF(\mathbf{x}_0)^{-1}F(\mathbf{x}_0)\| \leq \alpha$,

for some $\mathbf{x}_0 \in C_0$. Consider the quantities

$$h := \alpha\beta\gamma, \ r_{1,2} = \frac{1 \pm \sqrt{1-2h}}{h}\alpha.$$

If $h \leq \frac{1}{2}$ and $\overline{S_{r_1}(\mathbf{x}_0)} \subset C_0$, then the sequence $\{\mathbf{x}_k\}$ defined by

$$\mathbf{x}_{k+1} := \mathbf{x}_k - DF(\mathbf{x}_k)^{-1}F(\mathbf{x}_k), \ k = 0, 1, \ldots,$$

remains in $S_{r_1}(\mathbf{x}_0)$ and converges to the unique zero of $F(\mathbf{x})$ in $C_0 \cap S_{r_2}(\mathbf{x}_0)$.

Proof. Let $\mathbf{x_1} = [x_1^1, x_2^1, \ldots, x_6^1], \mathbf{x_2} = [x_1^2, x_2^2, \ldots, x_6^2]$. In this case for any $\mathbf{x_1}, \mathbf{x_2} \in C_0$ we have

$$DF(\mathbf{x}_2) - DF(\mathbf{x_1}) = \sum_{i=1}^{6} A_i(x_i^2 - x_i^1),$$

and

$$\|DF(\mathbf{x}_2) - DF(\mathbf{x_1})\| \leq \sum_{i=1}^{6} \|A_i\| \|\mathbf{x_2} - \mathbf{x_1}\|.$$

Thus $\|DF(\mathbf{x}_2) - DF(\mathbf{x_1})\| \leq \gamma \|\mathbf{x}_2 - \mathbf{x_1}\|$, and all conditions of Newton-Kantorovich Theorem [5] are hold. Hence Newton's method converges to the unique solution of system (4.6) in $C_0 \cap S_{r_2}(\mathbf{x_0})$. □

Indeed in Theorem 4.1 we compute $\{r_{ki}\}_{k=0}^4$ and h_{22}^i of matrices $H_{i,i+1}$. By comparing (4.1) and (2.21) we find

$$H_{ii} = -S_{i-1} - S_i, \qquad i = 1, 2, \ldots, n.$$

Thus by definition of S_i (2.13) we obtain

$$\begin{aligned} q_{i-1} &= H_{ii}(1,1) + \tfrac{A_{11}^{i-1}}{D_{i-1}} + \tfrac{A_{11}^{i}}{D_i}, \\ q_{1,i-1} &= H_{ii}(1,2) + \tfrac{A_{21}^{i-1}}{D_{i-1}} + \tfrac{A_{21}^{i}}{D_i}, \\ q_{2,i-1} &= 2(H_{ii}(1,3) + \tfrac{A_{31}^{i-1}}{D_{i-1}} + \tfrac{A_{31}^{i}}{D_i}), \\ q_{3,i-1} &= 2(H_{ii}(2,3) + \tfrac{A_{32}^{i-1}}{D_{i-1}} + \tfrac{A_{32}^{i}}{D_i}), \\ q_{4,i-1} &= 4(H_{ii}(3,3) + \tfrac{A_{33}^{i-1}}{D_{i-1}} + \tfrac{A_{33}^{i}}{D_i}), \qquad i = 2, 3, \ldots, n. \end{aligned} \tag{4.7}$$

By setting $H_{11} = -S_1 - C_\alpha$ we find

$$(4.8)\quad \begin{aligned} q_0 &= H_{11}(1,1) + \tfrac{A^1_{11}}{D_1} + \cot\alpha, \\ q_{1,0} &= H_{11}(1,2) + \tfrac{A^1_{21}}{D_1} + a\cot\alpha, \\ q_{2,0} &= 2(H_{11}(1,3) + \tfrac{A^1_{31}}{D_1} + \tfrac{1}{2}a^2\cot\alpha), \\ q_{3,0} &= 2(H_{11}(2,3) - \tfrac{A^1_{32}}{D_1} - (a - \tfrac{1}{2}a^3)\cot\alpha), \\ q_{4,0} &= 4(H_{11}(3,3) + \tfrac{A^1_{33}}{D_1} + (1 - a^2 + \tfrac{1}{4}a^4)\cot\alpha). \end{aligned}$$

By following $H_{nn} = -S_n + C_\beta$ we find

$$(4.9)\quad \begin{aligned} q_n &= H_{nn}(1,1) + \tfrac{A^n_{11}}{D_n} + \cot\beta, \\ q_{1,n} &= H_{nn}(1,2) + \tfrac{A^n_{21}}{D_n} - b\cot\beta, \\ q_{2,n} &= 2(H_{nn}(1,3) + \tfrac{A^n_{31}}{D_n} - \tfrac{1}{2}b^2\cot\beta), \\ q_{3,n} &= 2(H_{nn}(2,3) - \tfrac{A^n_{32}}{D_n} + (b - \tfrac{1}{2}b^3)\cot\beta), \\ q_{4,n} &= 4(H_{nn}(3,3) + \tfrac{A^n_{33}}{D_n} - (1 - b^2 + \tfrac{1}{4}b^4)\cot\beta). \end{aligned}$$

Let

$$(4.10)\quad \begin{aligned} &w_i = V_i(1,1), \qquad w_{1i} = V_i(1,2), \qquad w_{2i} = 2V_i(1,3), \\ &w_{3i} = 2V_i(2,3), \qquad w_{4i} = 4V_i(3,3). \end{aligned}$$

If we define a partition of (a,b) by (2.3) and construct piecewise polynomial functions $\overline{p}, \overline{q}, \overline{w}$ by (3.3) then, $\overline{p}, \overline{q}, \overline{w}$ satisfy (2.4), (2.5) and equation (3.4) is of Atkinson type. Therefore, by Theorem 3.2, BVP (3.4) and matrix eigenvalue problem (3.1) are equivalent. Summarizing the previous results, we have the following Theorem:

Theorem 4.2. *Let* $n > 2$*,* H *be a real symmetric block tridiagonal matrix of the form* (4.2)*,* $V = \mathrm{diag}(V_1, \ldots, V_n)$ *be a real symmetric block diagonal matrix, where* $\det(V) \neq 0$ *and* $V_i(2,2) = 2V_i(1,3)$*. Then for a given boundary conditions* (1.3)*, the matrix eigenvalue problem* (4.1) *is equivalent to SLP of Atkinson type of the form* (3.4)*.*

5. **Conclusion**

In this paper we considered a typical class of sixth order Sturm-Liouville problems. We proved that the SLP is equivalent to a symmetric matrix eigenvalue problem and thus the SLP has finite spectrum. Also for a given matrix eigenvalue problem $HX = \lambda VX$, where H is a block tridiagonal matrix and V is a block diagonal matrix, we found a sixth order boundary value problem of Atkinson type that is equivalent to matrix eigenvalue problem.

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