ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 41 (2015), No. 4, pp. 1045–1059

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 41 (2015), No. 4, pp. 1045–1059
Online ISSN: 1735-8515

ON THE CONVERGENCE OF SOLUTIONS TO A DIFFERENCE INCLUSION ON HADAMARD MANIFOLDS

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(Communicated by Mohammad Bagher Kashani)

Abstract. The aim of this paper is to study the convergence of solutions of the following second order difference inclusion

$$\begin{cases} \exp_{u_i}^{-1} u_{i+1} + \theta_i \exp_{u_i}^{-1} u_{i-1} \in c_i A(u_i), \quad i \geqslant 1 \\ u_0 = x \in M, \quad \sup_{i \geqslant 0} d(u_i, x) < +\infty, \end{cases}$$

to a singularity of a multi-valued maximal monotone vector field A on a Hadamard manifold M, where $\{c_i\}$ and $\{\theta_i\}$ are sequences of positive real numbers and x is an arbitrary fixed point in M. The results of this paper extend previous results in the literature from Hilbert spaces to Hadamard manifolds for general maximal monotone, strongly monotone multi-valued vector fields and subdifferentials of proper, lower semicontinuous and geodesically convex functions $f : M \to]-\infty, +\infty]$. In the recent case, when $A = \partial f$, we show that the sequence $\{u_i\}$, given by the equation, converges to a point of the solution set of the following constraint minimization problem

$$\underset{x \in M}{Min}\ f(x).$$

Keywords: Maximal monotone operator, multivalued vector field, convergence, subdifferential, minimization problem, Hadamard manifold.
MSC(2010): Primary: 34C40, 37C10; Secondary: 47H05.

1. Introduction

Let H be a real Hilbert space equipped with the inner product $\langle ., . \rangle$ and norm $\| \cdot \|$. Let A be a nonempty subset of $H \times H$, to which we refer as a (nonlinear) possibly multivalued operator in H. A is called monotone if and only if $\langle y_2 - y_1, x_2 - x_1 \rangle \geqslant 0$, for all $[x_i, y_i] \in A$, $i = 1, 2$. A monotone operator

Article electronically published on August 16, 2015.
Received: 26 June 2013, Accepted: 6 July 2014.
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A is said to be maximal monotone if A, regarded as a subset of $H \times H$, is not properly included in any other monotone subset of $H \times H$.

Given any function $f : H \to]-\infty, +\infty]$ with domain $D(f)$, its subdifferential is defined as a multi-valued operator ∂f, where

$$\partial f(x) = \{w \in H \mid f(x) - f(y) \leqslant \langle w, x - y\rangle, \ \forall y \in H\}.$$

The function f is called proper if and only if there exists $x \in H$ such that $f(x) < +\infty$. It is a well-known result that if f is a proper, convex and lower semicontinuous function, then ∂f is a maximal monotone operator. We refer the reader to the book by Morosanu [12] in order to understand monotone operators and subdifferentials of convex functions in Hilbert spaces.

Let A be a maximal monotone operator in H. The second order evolution equation

$$\begin{cases} p(t)u''(t) + r(t)u'(t) \in Au(t), \\ u(0) = u_0, \quad \underset{t \geqslant 0}{sup} \|u(t)\| < +\infty, \end{cases} \tag{1.1}$$

has been studied by many mathematicians for existence, periodicity and asymptotic behavior of solutions (see for instance [5, 10–12] and references therein). Equation (1.1) has applications not only in ordinary and partial differential equations, but also in optimization. In fact, under suitable conditions, the trajectory of the solution of (1.1) converges to a zero of the maximal monotone operator A, which is a minimum point of f, when $A = \partial f$, where f is a convex, proper and lower semicontinuous function (see [5,10,11]). The second author of this paper has showed in [10,11] that equation (1.1), in the special case $p(t) = 0$ and $r(t) = c \geqslant 0$, gives a better rate of convergence to a minimum value of f than the first order evolution equation of monotone type (steepest descent method). In order to extract an algorithm to approximate a singularity of a monotone vector field or to solve unconstraint minimization problem

$$\underset{x \in H}{Min} f(x),$$

where $A = \partial f$, as well as for applying in partial differential equations (see [1]) and in discrete modelings (see [13–15]), we consider the discrete version of (1.1). Discrete analogue of equation (1.1) may be written in the following form:

$$\begin{cases} u_{i+1} - (1 + \theta_i)u_i + \theta_i u_{i-1} \in c_i A u_i, \\ u_0 = x, \quad \underset{i \geqslant 0}{sup} \|u_i\| < +\infty, \end{cases} \tag{1.2}$$

where A is maximal monotone operator and $\{c_i\}$ and $\{\theta_i\}$ are positive real sequences. There are a lot of investigations for existence and asymptotic behavior of solutions to (1.2), which the reader can find them in [9, 12, 13] and in their appeared references. In [9] the author has proved the weak and strong convergence of solutions to (1.2) with the best conditions on parameters $\{c_i\}$ and $\{\theta_i\}$ and he also has given some results for the rate of convergence.

In this paper, we consider an analogue equation of the equation (1.2) on Hadamard manifolds. Then we get convergence results for solutions to this equation, which are similar to the obtained results for solutions to the equation (1.2) in [9].

2. **Some tools of Riemannian geometry**

Here we remind some indispensable backgrounds about Riemannian manifolds from [8, 16].

Let M be a connected m-dimensional Riemannian manifold, with a Riemannian metric $\langle ., . \rangle$ and the corresponding norm denoted by $\| \, . \, \|$. For $p \in M$ the tangent space at p is denoted by T_pM and the tangent bundle of M by $TM = \bigcup_{p \in M} T_pM$. A vector filed A is a mapping from M to TM which maps each point $p \in M$ to a vector $A(p) \in T_pM$. Let p and q be two points in M, and $\gamma : [a, b] \to M$ be a piecewise smooth curve joining p to q. The length of γ is defined as

$$L(\gamma) = \int_a^b \| \, \dot{\gamma}(t) \, \| \, dt,$$

and the Riemannian distance $d(p, q)$ is defined by

$$d(p, q) = inf\{L(\gamma) | \gamma : [a, b] \to M \text{is a piecewise smooth curve with } \gamma(a) = p, \gamma(b) = q\},$$

which induces the original topology on M.

Let ∇ be the Levi-Civita connection on M associated with the Riemannian metric $\langle ., . \rangle$, and γ be a smooth curve in M. A vector filed A is said to be parallel along γ if $\nabla_{\dot{\gamma}} A = 0$. A smooth curve γ is a geodesic, if $\dot{\gamma}$ itself is parallel along γ. If γ is a geodesic, then $\|\dot{\gamma}\|$ is constant. When $\|\dot{\gamma}\| = 1$, γ is said to be normalized. A geodesic joining p to q in M is called minimal if its length is equal to $d(p, q)$.

A Riemannian manifold M is complete if for each $p \in M$ all geodesics emanating from p are defined on whole of $\mathbb{R}$. If M is complete, then by the Hopf-Rinow theorem, any pair of points in M can be joined by a minimal geodesic.

Let M be a connected and complete Riemannian manifold. The exponential map $\exp_p : T_pM \to M$ at p is defined by $\exp_p(v) = \gamma_v(1)$ for each $v \in T_pM$, where $\gamma_v(.)$ is the geodesic with $\gamma_v(0) = p$ and $\dot{\gamma}_v(1) = v$. Then $\exp_p(tv) = \gamma_v(t)$ for each real number t.

Throughout the paper we assume that M is a complete, simply connected Riemannian manifold of non-positive sectional curvature of dimension m, which is called a Hadamard manifold of dimension m.

Proposition 2.1. ([16, p. 221]) *Let $p \in M$. Then $\exp_p : T_pM \to M$ is a diffeomorphism, and for any two points $p, q \in M$ there exists a unique normalized geodesic joining p to q, which is, in fact, a minimal geodesic.*

An immediate consequence of Proposition 2.1 is that $d(p,q) = \| \exp_p^{-1} q \|$, for any two points $p, q \in M$. Proposition 2.1 shows that any m-dimensional Hadamard manifold has the same topology and differential structure as the Euclidean space $\mathbb{R}^m$. In fact, Hadamard manifolds and Euclidean spaces have some similar geometrical properties. One of them is described in the following proposition.

By definition, a geodesic triangle $\Delta(p_1p_2p_3)$ of a Riemannian manifold is a set consisting of three points p_1, p_2 and p_3, and three minimal geodesics joining these points.

Proposition 2.2. ([16, p. 223])(Comparison theorem for triangles) *Let $\Delta(p_1p_2p_3)$ be a geodesic triangle. Denote by $\gamma_i : [0, l_i] \to M$ the geodesic joining p_i to p_{i+1}, and set $l_i := L(\gamma_i)$, $\alpha_i := \angle(\dot{\gamma}_i(0), -\dot{\gamma}_{i-1}(l_{i-1}))$, where $i = 1, 2, 3$ (mod 3). Then*

$$\alpha_1 + \alpha_2 + \alpha_3 \leqslant \pi,$$

$$l_i^2 + l_{i+1}^2 - 2l_il_{i+1}\cos\alpha_{i+1} \leqslant l_{i-1}^2. \tag{2.1}$$

Since

$$\langle \exp_{p_{i+1}}^{-1} p_i, \exp_{p_{i+1}}^{-1} p_{i+2} \rangle = d(p_i, p_{i+1}) d(p_{i+1}, p_{i+2}) \cos\alpha_{i+1},$$

the inequality (2.1) may be rewritten as follows

$$d^2(p_i, p_{i+1}) + d^2(p_{i+1}, p_{i+2}) - 2\langle \exp_{p_{i+1}}^{-1} p_i, \exp_{p_{i+1}}^{-1} p_{i+2} \rangle \leqslant d^2(p_{i+2}, p_i). \tag{2.2}$$

Now we are going to remind some definitions which extends some notions of the monotonicity, from the corresponding notions in Hilbert spaces (see [3,4]), to multi-valued vector fields on Hadamard manifolds. Let $\mathcal{X}(M)$ denote the set of all multi-valued vector fields $A : M \longrightarrow 2^{TM}$ such that $A(x) \subseteq T_xM$ for each $x \in M$ and the domain $\mathcal{D}(A)$ of A is closed and convex, where $\mathcal{D}(A)$ is defined by

$$\mathcal{D}(A) = \{x \in M : \ A(x) \neq \varnothing\}.$$

Definition 2.3. ([3]) Let $A \in \mathcal{X}(M)$. Then A is said to be

(i) *monotone* if the following condition holds for any $x, y \in \mathcal{D}(A)$,

$$\langle u, \exp_x^{-1} y \rangle \leqslant \langle v, -\exp_y^{-1} x \rangle \ , \quad \forall u \in A(x), v \in A(y) : \tag{2.3}$$

(ii) *strongly monotone* if there exists $\rho > 0$ such that, for any $x, y \in \mathcal{D}(A)$, we have

$$\langle u, \exp_x^{-1} y \rangle - \langle v, -\exp_y^{-1} x \rangle \leqslant -\rho d^2(x,y) \ , \quad \forall u \in A(x), v \in A(y) : \tag{2.4}$$

(iii) *maximal monotone* if it is monotone and the following implication holds for any $x \in \mathcal{D}(A)$ and $u \in T_xM$,

(2.5)

$$(\langle u, \exp_x^{-1} y\rangle \leqslant \langle v, -\exp_y^{-1} x\rangle \ , \ \forall y \in \mathcal{D}(A) \quad \text{and} \quad \forall v \in A(y)) \Longrightarrow u \in A(x).$$

Definition 2.4. ([3]) *Let $A \in \mathcal{X}(M)$ and $x_0 \in \mathcal{D}(A)$. Then A is said to be upper Kuratowski semicontinuous at x_0 if, for any sequences $\{x_k\} \subseteq \mathcal{D}(A)$ and $\{u_k\} \subset TM$ with each $u_k \in A(x_k)$, the relations $\lim_{k\to\infty} x_k = x_0$ and $\lim_{k\to\infty} u_k = u_0$ imply that $u_0 \in A(x_0)$. A is said to be upper Kuratowski semicontinuous on M if it is upper Kuratowski semicontinuous at each point $x_0 \in \mathcal{D}(A)$.*

In [3, Proposition 3.5] it is shown that each maximal monotone vector field is upper Kuratowski semicontinuous.

Now we present the equation (1.2) in a Hadamard manifold M as follows.

$$\begin{cases} \exp_{u_i}^{-1} u_{i+1} + \theta_i \exp_{u_i}^{-1} u_{i-1} \in c_i A(u_i), \quad i \geqslant 1 \\ u_0 = x, \quad \sup_{i\geqslant 0} d(u_i, x) < +\infty, \end{cases} \tag{2.6}$$

where $\{c_i\}$ and $\{\theta_i\}$ are positive real sequences and x is an arbitrary point in M.

Throughout the paper, $A(u_i)$ denotes the element $\frac{\exp_{u_i}^{-1} u_{i+1} + \theta_i \exp_{u_i}^{-1} u_{i-1}}{c_i}$ and $a_i = (\theta_1 \cdots \theta_i)^{-1}$.

3. General monotone case

In this section we establish some results on the convergence of solutions to (2.6). We first recall the notion of Fejér convergence and the following related result from [6].

Definition 3.1. *Let X be a complete metric space and $K \subseteq X$ be a nonempty set. A sequence $\{x_n\} \subset X$ is called Fejér convergent to K if*

$$d(x_{n+1}, y) \leqslant d(x_n, y), \quad \forall y \in K \quad \textit{and} \quad n = 0, 1, 2, \dots \, .$$

Lemma 3.2. *Let X be a complete metric space and $K \subseteq X$ be a nonempty set. Let $\{x_n\} \subset X$ be Fejér convergent to K and suppose that any cluster point of $\{x_n\}$ belongs to K. If the set of cluster points of $\{x_n\}$ is nonempty, then $\{x_n\}$ converges to a point of K.*

The following two lemmas, which are from [9], will be used in the sequel.

Lemma 3.3. ([9]) *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive real numbers. If $\{a_n\}$ is nonincreasing and convergent to zero and $\sum_{n=1}^{+\infty} a_n b_n < +\infty$, then $(\sum_{k=1}^{n} b_k) a_n \to 0$ as $n \to +\infty$.*

Lemma 3.4. ([9]) *Let $\{a_n\}$ be a sequence of positive real numbers with $\sum_{n=1}^{+\infty} a_n^{-1} = +\infty$. If $\{b_n\}$ is a bounded sequence, then* $\liminf_{n\to+\infty} a_n(b_{n+1} - b_n) \leqslant 0$.

Lemma 3.5. *Let $\{u_i\}$ be a solution to (2.6). Then $a_{i-1}d(u_i, u_{i-1})$ is either nonincreasing or eventually increasing.*

Proof. From the monotonicity of A and (2.6), we have

$$\frac{1}{c_{i+1}}\langle \exp_{u_{i+1}}^{-1} u_{i+2} + \theta_{i+1} \exp_{u_{i+1}}^{-1} u_i, \exp_{u_{i+1}}^{-1} u_i\rangle \leqslant \frac{1}{c_i}\langle \exp_{u_i}^{-1} u_{i+1} + \theta_i \exp_{u_i}^{-1} u_{i-1}, -\exp_{u_i}^{-1} u_{i+1}\rangle.$$

Since $\|\exp_x^{-1} y\| = \|\exp_y^{-1} x\| = d(x, y)$, for all $x, y \in M$, it follows that

$$\frac{1}{c_i}d(u_{i+1}, u_i) - \frac{\theta_i}{c_i}d(u_i, u_{i-1}) \leqslant \frac{1}{c_{i+1}}d(u_{i+2}, u_{i+1}) - \frac{\theta_{i+1}}{c_{i+1}}d(u_{i+1}, u_i), \tag{3.1}$$

for all $i \geqslant 1$. If $\{a_{i-1}d(u_i, u_{i-1})\}$ is not nonincreasing, then there exists $j \geqslant 1$ such that $a_j d(u_{j+1}, u_j) > a_{j-1}d(u_j, u_{j-1})$ and by (3.1) the sequence $\{a_{i-1}d(u_i, u_{i-1})\}_{i\geqslant j}$ is increasing. □

Lemma 3.6. *Suppose that u_i is a solution to (2.6) and $p \in A^{-1}(0)$. Then $d(u_i, p)$ is nonincreasing or eventually increasing. Moreover, if $\sum_{i=1}^{+\infty} \theta_1 \cdots \theta_i = +\infty$, then $d(u_i, p)$ and $a_{i-1}d(u_i, u_{i-1})$ are nonincreasing and $a_{i-1}d(u_i, u_{i-1})$ converges to zero as $i \to +\infty$.*

Proof. From the monotonicity of A and (2.6), we get

$$\langle \exp_{u_i}^{-1} u_{i+1} + \theta_i \exp_{u_i}^{-1} u_{i-1}, \exp_{u_i}^{-1} p\rangle \leqslant 0. \tag{3.2}$$

Consider the geodesic triangles $\Delta(u_{i+1}u_i p)$ and $\Delta(u_i u_{i-1} p)$. By inequality (2.2) of the comparison theorem for triangles and inequality (3.2), one obtains that

$$d^2(u_{i+1}, p) - d^2(u_i, p) + \theta_i(d^2(u_{i-1}, p) - d^2(u_i, p)) \geqslant d^2(u_{i+1}, u_i) + \theta_i d^2(u_i, u_{i-1}) \geqslant 0. \tag{3.3}$$

If $d(u_i, p)$ is not nonincreasing, there is $j > 0$ such that $d(u_j, p) < d(u_{j+1}, p)$. By (3.3) the sequence $\{d(u_i, p)\}_{i\geqslant j}$ is increasing.
For the second part of the lemma, multiplying (3.3) by a_i, we get

$$a_{i-1}d^2(u_i, u_{i-1}) \leqslant a_i(d^2(u_{i+1}, p) - d^2(u_i, p)) - a_{i-1}(d^2(u_i, p) - d^2(u_{i-1}, p)).$$

Summing from $i = k$ to m, we get

$$\sum_{i=k}^{m} a_{i-1}d^2(u_i, u_{i-1}) \leqslant a_m(d^2(u_{m+1}, p) - d^2(u_m, p)) - a_{k-1}(d^2(u_k, p) - d^2(u_{k-1}, p)).$$

Taking liminf when $m \to +\infty$, by our assumption and Lemma 3.4,

$$\liminf_{m\to+\infty} a_m(d^2(u_{m+1}, p) - d^2(u_m, p)) \leqslant 0,$$

then we get

$$\sum_{i=k}^{+\infty} a_{i-1} d^2(u_i, u_{i-1}) \leqslant a_{k-1}(d^2(u_{k-1}, p) - d^2(u_k, p)). \tag{3.4}$$

Inequality (3.4) implies that $\{d^2(u_k, p)\}$ is nonincreasing and $\sum_{i=1}^{+\infty} a_{i-1}^{-1} a_{i-1}^2 d^2(u_i, u_{i-1}) < +\infty$. The assumption on $\{\theta_i\}$ implies, $\liminf_{i\to+\infty} a_{i-1} d(u_i, u_{i-1}) = 0$. By Lemma 3.5 $a_{i-1} d(u_i, u_{i-1})$ is nonincreasing and therefore $\lim_{i\to+\infty} a_{i-1} d(u_i, u_{i-1}) = 0$. □

Theorem 3.7. *Suppose that u_i is a solution to (2.6) and $A^{-1}(0) \neq \varnothing$. If $\theta_i \geqslant 1$ and $\liminf_{i\to+\infty} i c_i a_i > 0$, then $\{u_i\}$ converges to a singularity of A.*

Proof. We show that the sequence $\{u_i\}$ is Fejér convergent to $A^{-1}(0)$ and any cluster point of $\{u_i\}$ belongs to $A^{-1}(0)$, then one gets the result by Lemma 3.2. Suppose that $p \in A^{-1}(0)$. The assumption on $\{\theta_i\}$ implies that $\sum_{i=1}^{\infty} \theta_1 \cdots \theta_i = +\infty$. Then by Lemma 3.6, $d^2(u_i, p)$ is nonincreasing, i.e. $\{u_i\}$ is Fejér convergent to $A^{-1}(0)$.

To complete the proof, we only need to prove that any cluster point of $\{u_i\}$ belongs to $A^{-1}(0)$. Let q be a cluster point of $\{u_i\}$. Then there exists a subsequence $\{i_k\}$ of $\{i\}$ such that $u_{i_k} \to q$. Since $a_i \leqslant 1$, inequality (3.4) implies that

$$\sum_{i=k}^{\infty} a_{i-1} d^2(u_i, u_{i-1}) \leqslant d^2(u_{k-1}, p) - d^2(u_k, p).$$

Summing from $k = 1$ to m and letting $m \to +\infty$, we get

$$\sum_{k=1}^{\infty} \sum_{i=k}^{\infty} a_{i-1} d^2(u_i, u_{i-1}) \leqslant d^2(u_0, p) - l(p)^2 < +\infty,$$

where $l(p) = \lim_{m\to+\infty} d(u_m, p)$. Then

$$\sum_{k=1}^{\infty} k a_k^{-1} a_k^2 d^2(u_{k+1}, u_k) < +\infty.$$

By Lemma 3.6, $a_k d(u_{k+1}, u_k)$ is nonincreasing and convergent to zero. Then by Lemma 3.3, we get

$$\left(\sum_{k=1}^{n} k a_k^{-1}\right) a_n^2 d^2(u_{n+1}, u_n) \to 0 \Rightarrow \left(\sum_{k=1}^{n} k\right) a_n^2 d^2(u_{n+1}, u_n) \to 0.$$

Therefore $n a_n d(u_{n+1}, u_n) \to 0$. By (2.6), $n c_n a_n \|A(u_n)\| \to 0$ as $n \to +\infty$. The assumption implies that $A(u_n) \to 0$ as $n \to +\infty$. Thus $A(u_{i_k}) \to 0$ as $k \to +\infty$. This shows that $0 \in A(q)$ because A is upper Kuratowski semicontinuous at q, that is, $q \in A^{-1}(0)$. □

Theorem 3.8. *Suppose that u_i is a solution to (2.6) and $A^{-1}(0) \neq \varnothing$. If $0 < \theta_i < 1$, $\sum_{i=1}^{\infty} \theta_1 \cdots \theta_i = +\infty$ and $\liminf_{i \to +\infty} ic_i > 0$, then $\{u_i\}$ converges to a singularity of A.*

Proof. The process of the proof is similar to that of Theorem 3.7. Just, to get $A(u_n) \to 0$ as $n \to +\infty$, we need to show that $nd(u_n, u_{n-1}) \to 0$ as $n \to +\infty$. Inequality (3.4) implies that

$$\sum_{i=k}^{\infty} d^2(u_i, u_{i-1}) \leqslant d^2(u_{k-1}, p) - d^2(u_k, p),$$

because a_i is increasing. Summing from $k = 1$ to $+\infty$, we get

$$\sum_{k=1}^{\infty} kd^2(u_k, u_{k-1}) \leqslant d^2(u_0, p) - l(p)^2 \leqslant d^2(u_0, p) < +\infty,$$

where $l(p) = \lim_{k \to +\infty} d(u_k, p)$. Therefore $\lim_{k \to +\infty} d^2(u_k, u_{k-1}) = 0$. By Lemma 3.6 $a_{k-1} d(u_k, u_{k-1})$ is nonincreasing and a_k is increasing, so $d(u_k, u_{k-1})$ is nonincreasing. Now, by Lemma 3.3, $(\sum_{k=1}^{n} k)\, d^2(u_n, u_{n-1}) \to 0$, then $nd(u_n, u_{n-1}) \to 0$ as $n \to +\infty$. The rest of the proof is similar to that of Theorem 3.7. □

Theorem 3.9. *Let $A \in \mathcal{X}(M)$ be a maximal monotone operator and $A^{-1}(0) \neq \varnothing$. Assume that u_i is a solution to (2.6) and $\sum_{i=1}^{\infty} \theta_1 \cdots \theta_i < \infty$. If c_i satisfies one of the following assumptions*

(1) $\liminf_{i \to +\infty} \sqrt{i} c_i > 0$,

(2) $\limsup_{i \to +\infty} c_i > 0$,

then $u_i \to p$ as $i \to +\infty$, where $p \in A^{-1}(0)$; moreover, $d(u_i, p) = O(\sum_{k=i}^{\infty} \theta_1 \cdots \theta_k)$.

Proof. If $d(u_i, p)$ is nonincrasing, from (3.3), we get

$$d^2(u_i, u_{i-1}) \leqslant d^2(u_{i-1}, p) - d^2(u_i, p). \tag{3.5}$$

Otherwise, by Lemma 3.6, $d(u_i, p)$ is eventually increasing and by (3.3), we obtain

$$d^2(u_{i+1}, u_i) \leqslant d^2(u_{i+1}, p) - d^2(u_i, p) \tag{3.6}$$

for large i. Summing (3.5) and (3.6) from $i = 1$ to $+\infty$, we get

$$\sum_{i=1}^{\infty} d^2(u_{i+1}, u_i) < +\infty. \tag{3.7}$$

On the other hand, multiplying inequalities (3.5) and (3.6) by i and taking liminf, by Lemma 3.4, we get

$$\liminf_{i \to +\infty} id^2(u_{i+1}, u_i) = 0. \tag{3.8}$$

Summing (3.1) from $n = k$ to $n = m - 1$, we obtain

$$\frac{1}{c_k}d(u_{k+1}, u_k) - \frac{1}{c_m}d(u_{m+1}, u_m) \leqslant \frac{\theta_k}{c_k}d(u_k, u_{k-1}) - \frac{\theta_m}{c_m}d(u_m, u_{m-1}). \tag{3.9}$$

Taking *liminf* as $m \to +\infty$, if $\underset{m\to+\infty}{liminf}\ \sqrt{m}c_m > 0$, then by (3.8), we get

$$d(u_{k+1}, u_k) \leqslant \theta_k d(u_k, u_{k-1}). \tag{3.10}$$

If *limsup* $c_m > 0$, then there exists a subsequence $c_{m_j} > c_0 > 0$. Substituting m by m_j and letting $j \to +\infty$, by (3.7), we get again (3.10). For each $n > m$, (3.10) implies

$$d(u_n, u_m) \leqslant \sum_{k=m}^{n-1} d(u_{k+1}, u_k) \leqslant d(u_1, u_0) \sum_{k=m}^{n-1} \theta_1 \cdots \theta_k. \tag{3.11}$$

It follows that u_n is Cauchy, and therefore $u_n \to p \in M$. Now we prove that $p \in A^{-1}(0)$. Suppose that $y \in \mathcal{D}(A)$ and $v \in A(y)$. By the monotonicity of A, (2.6) and the inequality (2.2) of the comparison theorem for geodesic triangles $\Delta(u_{n+1}u_n y)$ and $\Delta(u_n u_{n-1} y)$, we have

$$\begin{aligned}
\langle v, -\exp_y^{-1} u_n\rangle &\geqslant \langle A(u_n), \exp_{u_n}^{-1} y\rangle \\
&= \frac{1}{c_n}\langle c_n A(u_n), \exp_{u_n}^{-1} y\rangle \\
&= \frac{1}{c_n}\langle \exp_{u_n}^{-1} u_{n+1} + \theta_n \exp_{u_n}^{-1} u_{n-1}, \exp_{u_n}^{-1} y\rangle \\
&\geqslant \frac{1}{2c_n}[d^2(u_n, y) - d^2(u_{n+1}, y) - \theta_n(d^2(u_{n-1}, y) - d^2(u_n, y))] \\
&\geqslant \Big(\frac{-\sqrt{n}}{\sqrt{n}c_n}d(u_{n+1}, u_n) - \theta_n\frac{\sqrt{n}}{\sqrt{n}c_n}d(u_n, u_{n-1})\Big)N,
\end{aligned}$$

where $N = \underset{i\geqslant 0}{sup}\ d(u_i, y)$. If $\underset{n\to+\infty}{liminf}\ \sqrt{n}c_n > 0$, from the above inequality and by (3.8), we get $\langle v, -\exp_y^{-1} p\rangle \geqslant 0$. If $\underset{n\to+\infty}{limsup}\ c_n > 0$, there is a subsequence $c_{n_j} \geqslant c_0 > 0$; substituting n_j for n in the above inequality, letting $j \to +\infty$, then using (3.7), we get $\langle v, -\exp_y^{-1} p\rangle \geqslant 0$. Since A is maximal monotone, then $p \in A^{-1}(0)$. Now by letting $n \to +\infty$ in (3.11), we get

$$d(u_n, p) \leqslant d(u_0, u_1) \sum_{n=k}^{+\infty} \theta_1 \cdots \theta_k.$$

It implies $d(u_m, p) = O(\sum_{k=m}^{\infty} \theta_1 \cdots \theta_k)$. $\square$

4. **Strongly monotone case**

In this section we study the convergence of solutions to (2.6), when A is strongly monotone. We study the convergence of u_i with the same assumptions of Theorem 3.7 and 3.8 on $\{\theta_i\}$ and $\{c_i\}$, and the rate of convergence of u_i to the unique element of $A^{-1}(0)$.

Theorem 4.1. *Let $A \in \mathcal{X}(M)$ such that $A^{-1}(0) \neq \varnothing$. Suppose that A is strongly monotone and u_i is a solution to (2.6). If $\theta_i \geqslant 1$ and $\sum_{i=1}^{\infty} ic_ia_i = +\infty$, then $u_n \to p \in A^{-1}(0)$ as $n \to +\infty$; moreover $d^2(u_n, p) = o((\sum_{i=1}^{n} ic_ia_i)^{-1})$.*

Proof. Since strong monotonicity implies strict monotonicity, and $A^{-1}(0) \neq \varnothing$, there exists a unique element p in $A^{-1}(0)$ by [3, Proposition 4.1]. By (2.6) and strong monotonicity of A, we have

$$\frac{1}{c_i}\langle \exp_{u_i}^{-1} u_{i+1} + \theta_i \exp_{u_i}^{-1} u_{i-1}, \exp_{u_i}^{-1} p\rangle \leqslant -\rho d^2(u_i, p), \quad \forall i \geqslant 1.$$

Consider the geodesic triangles $\Delta(u_{i+1}u_ip)$ and $\Delta(u_iu_{i-1}p)$. By inequality (2.2) of the comparison theorem for triangles and inequality (3.2), we have

$$\frac{1}{2}(d^2(u_{i+1}, p) - d^2(u_i, p) + \theta_i(d^2(u_{i-1}, p) - d^2(u_i, p))) \geqslant \rho c_i d^2(u_i, p).$$

Multiplying by a_i, and summing from $i = k$ to m, we get

$$\rho \sum_{i=k}^{m} c_ia_id^2(u_i, p) \leqslant \frac{1}{2}(a_m(d^2(u_{m+1}, p) - d^2(u_m, p)) - a_{k-1}(d^2(u_k, p) - d^2(u_{k-1}, p))).$$

Taking liminf when $m \to +\infty$, by our assumption and Lemma 3.4,

$$\mathop{liminf}_{m\to+\infty}\ a_m(d^2(u_{m+1}, p) - d^2(u_m, p)) \leqslant 0,$$

then we get

$$\rho \sum_{i=k}^{+\infty} c_ia_id^2(u_i, u_{i-1}) \leqslant \frac{1}{2}a_{k-1}(d^2(u_{k-1}, p) - d^2(u_k, p)). \tag{4.1}$$

Summing from $k = 1$ to $+\infty$, since $a_k \leqslant 1$ and $d(u_k, p)$ is nonincreasing, we obtain

$$\sum_{i=1}^{\infty} ic_ia_id^2(u_i, p) < +\infty. \tag{4.2}$$

By the hypothesis, we have $\mathop{liminf}_{i\to+\infty}\ d^2(u_i, p) = 0$. Since by Lemma 3.6, $d(u_i, p)$ is nonincreasing, then $u_i \to p$ as $i \to +\infty$. The rate of convergence is a consequence of (4.2) and Lemma 3.3. □

Theorem 4.2. *Let $A \in \mathcal{X}(M)$ such that $A^{-1}(0) \neq \varnothing$. Suppose that A is strongly monotone and u_i is a solution to (2.6). If $0 < \theta_i < 1$, $\sum_{i=1}^{\infty} \theta_1 \cdots \theta_i =$*

$+\infty$ *and* $\sum_{i=1}^{\infty} ic_i = +\infty$, *then* $u_n \to p \in A^{-1}(0)$ *as* $n \to +\infty$*; moreover* $d^2(u_n, p) = o((\sum_{i=1}^{n} ic_i)^{-1})$.

Proof. Let p be the unique element of $A^{-1}(0)$. By the strong monotonicity of A and our assumption on θ_i, from (4.1), we get

$$\rho \sum_{i=k}^{+\infty} c_i d^2(u_i, u_{i-1}) \leqslant \frac{1}{2}(d^2(u_{k-1}, p) - d^2(u_k, p)).$$

Summing from $k = 1$ to $+\infty$, we get

$$\sum_{i=1}^{\infty} ic_i d^2(u_i, p) < +\infty. \tag{4.3}$$

The rest of the proof is similar to that of Theorem 4.1. □

5. Subdifferential case

Recall that M is a Hadamard manifold. We remind some definitions and facts about functions of M to $]-\infty, +\infty]$ from [17]. A map $f : M \to]-\infty, +\infty]$ is said to be proper if its domain, defined by $D(f) = \{x \in M \mid f(x) < \infty\}$, is nonempty. A subset $C \subseteq M$ is said to be convex if for any two points x and y in C, the geodesic joining x to y is contained in C. The domain of f, is a closed convex subset of M. A proper map f is said to be geodesically convex if for any geodesic γ of M, the composition function $f \circ \gamma : \mathbb{R} \to]-\infty, +\infty]$ is convex; that is,

$$(f \circ \gamma)(ta + (1-t)b) \leqslant tf \circ \gamma(a) + (1-t)f \circ \gamma(b),$$

for any $a, b \in \mathbb{R}$ and $0 \leqslant t \leqslant 1$. The domain of a lower semicontinuous and geodesically convex function f is a closed convex subset of M.

The subdifferential of a map $f : M \to]-\infty, +\infty]$ is the set-valued mapping $\partial f : M \to 2^{TM}$ defined by

$$\partial f(x) = \{u \in T_xM : \langle u, \exp_x^{-1} y\rangle \leqslant f(y) - f(x) \ , \ \forall y \in M\}, \quad \forall x \in M.$$

The subdifferential $\partial f(x)$ at a point $x \in M$ is a closed convex (possibly empty) set. Let $\mathcal{D}(\partial f)$ denote the domain of ∂f defined by

$$\mathcal{D}(\partial f) = \{x \in M : \ \partial f(x) \neq \varnothing\}.$$

If $\mathcal{D}(\partial f) \neq \varnothing$, the subdifferential $\partial f(\cdot)$ is a monotone and upper Kuratowski semicontinuous multi-valued vector field, and if $D(f) = M$, then ∂f is a maximal monotone vector field (see [3, Theorem 5.1]). Consider the following constraint minimization problem

$$\underset{x \in M}{Min} f(x), \tag{5.1}$$

which M is the constraint set. Let S_f denote the solution set of (5.1), that is

$$S_f = \{x \in M : \ f(x) \leqslant f(y) \ , \ \forall y \in M\}.$$

It is easy to check that
$$x \in S_f \Leftrightarrow 0 \in \partial f(x).$$

Let M be an embedded submanifold in the Euclidean space $\mathbb{R}^n$, for some positive integer n. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth map. Considering the minimization problem (5.1) on the constraint set M, the function f may be geodesically convex on M, but not on $\mathbb{R}^n$, as one can see in Example 5.3. Then the following Theorems 5.1 and 5.2 give an iterative method for approximation of a minimum point of f on M.

Theorem 5.1. *Suppose that u_i is a solution to (2.6) with $A = \partial f$, where $f : M \to]-\infty, +\infty]$ is a proper, geodesically convex and lower semicontinuous function with the solution set $S_f \neq \varnothing$. If $\theta_i \geqslant 1$ and $\sum_{i=1}^{+\infty} i c_i a_i = +\infty$, then $\{u_i\}$ converges to a point $p \in S_f$. Moreover, $f(u_n) - f(p) = o((\sum_{i=1}^{n} i c_i a_i)^{-1})$.*

Proof. We first verify that $\{u_i\}$ is Fejér convergent to S_f. Let $p \in S_f$. The assumption on $\{\theta_i\}$ implies that $\sum_{i=1}^{\infty} \theta_1 \cdots \theta_i = +\infty$. Then by Lemma 3.6, $d^2(u_i, p)$ is nonincreasing, i.e. $\{u_i\}$ is Fejér convergent to $A^{-1}(0)$. Thus by Lemma 3.2, to complete the proof, we only need to prove that any cluster point of $\{u_i\}$ belongs to S_f. For this purpose, we first show that $\{f(u_i)\}$ is a nonincreasing sequence, and $f(u_i) \to f(p)$, as $i \to +\infty$.

By the subdifferential inequality, Lemma 3.6 and (2.6), we have

$$\begin{aligned} f(u_i) - f(u_{i-1}) &\leqslant \langle \partial f(u_i), -\exp_{u_i}^{-1} u_{i-1} \rangle \\ &= \frac{1}{c_i} \langle \exp_{u_i}^{-1} u_{i+1} + \theta_i \exp_{u_i}^{-1} u_{i-1}, -\exp_{u_i}^{-1} u_{i-1} \rangle \\ &\leqslant \frac{1}{c_i a_i} (a_i d(u_{i+1}, u_i) - a_{i-1} d(u_i, u_{i-1})) d(u_i, u_{i-1}) \leqslant 0. \end{aligned} \tag{5.2}$$

Let $p \in S_f$. Making use again of the subdifferential inequality and (2.6), we get

$$\begin{aligned} a_i c_i (f(u_i) - f(p)) &\leqslant a_i \langle \exp_{u_i}^{-1} u_{i+1} + \theta_i \exp_{u_i}^{-1} u_{i-1}, -\exp_{u_i}^{-1} p \rangle \\ &\leqslant \frac{1}{2} (a_i (d^2(u_{i+1}, p) - d^2(u_i, p)) - a_{i-1} (d^2(u_i, p) - d^2(u_{i-1}, p))). \end{aligned}$$

Summing from $i = k$ to m, taking *liminf* when $m \to +\infty$ and then again summing from $k = 1$ to $+\infty$, by Lemmas 3.4 and 3.6, since $a_i \leqslant 1$, we obtain

$$\sum_{i=1}^{+\infty} i c_i a_i (f(u_i) - f(p)) \leqslant \frac{1}{2} d^2(u_0, p) < +\infty. \tag{5.3}$$

By the hypothesis, $\mathop{liminf}\limits_{i \to +\infty} (f(u_i) - f(p)) = 0$. Since by (5.2) $\{f(u_i)\}$ is non-increasing, $\mathop{lim}\limits_{i \to +\infty} f(u_i) = f(p)$. Let q be a cluster point of $\{u_i\}$, then there is a subsequence $\{i_j\}$ of $\{i\}$ such that $u_{i_j} \to q$ as $j \to +\infty$. Hence by the lower semicontinuity of f, we have

$$f(q) \leqslant \underset{j\to+\infty}{liminf}\ f(u_{i_j}) = f(p).$$

Since $p \in S_f$, we have $f(q) \leqslant f(x)$, for all $x \in M$, that is, $q \in S_f$. The rate of convergence of $f(u_i)$ to $f(p)$ is a consequence of (5.3) and Lemma 3.3. □

Theorem 5.2. *Suppose that u_i is a solution to (2.6) with $A = \partial f$, where $f : M \to]-\infty, +\infty]$ is a proper, geodesically convex and lower semicontinuous function with the solution set $S_f \neq \varnothing$. If $0 < \theta_i < 1$, $\sum_{i=1}^{+\infty} \theta_1 \cdots \theta_i = +\infty$ and $\sum_{i=1}^{+\infty} ic_i = +\infty$, then $\{u_i\}$ converges to a point $p \in S_f$. Furthermore $f(u_n) - f(p) = o((\sum_{i=1}^{n} ic_i)^{-1})$.*

Proof. By (5.2) and Lemma 3.6, $\{f(u_i)\}$ is nonincreasing. Let $p \in S_f$. Making use again of the subdifferential inequality and (2.6), we get

$$\begin{aligned} a_i c_i(f(u_i) - f(p)) &\leqslant a_i\langle \exp_{u_i}^{-1} u_{i+1} + \theta_i \exp_{u_i}^{-1} u_{i-1}, -\exp_{u_i}^{-1} p\rangle \\ &\leqslant \frac{1}{2}(a_i(d^2(u_{i+1}, p) - d^2(u_i, p)) - a_{i-1}(d^2(u_i, p) - d^2(u_{i-1}, p))). \end{aligned}$$

Summing from $i = k$ to m, and taking $liminf$ when $m \to +\infty$, by Lemma 3.4, we get

$$a_{k-1}\sum_{i=k}^{+\infty} c_i(f(u_i) - f(p)) \leqslant \sum_{i=k}^{+\infty} a_i c_i(f(u_i) - f(p)) \leqslant \frac{1}{2}a_{k-1}(d^2(u_{k-1}, p) - d^2(u_k, p)),$$

(where in the first inequality we used the assumption $0 < \theta_i < 1$ which implies that a_i is increasing). Summing again from $k = 1$ to $+\infty$, we obtain

$$\sum_{i=1}^{+\infty} ic_i(f(u_i) - f(p)) < +\infty. \tag{5.4}$$

The assumption implies that $\underset{i\to+\infty}{liminf}\ (f(u_i) - f(p)) = 0$. Since by (5.2) $\{f(u_i)\}$ is nonincreasing, so $\underset{i\to+\infty}{lim} f(u_i) = f(p)$. If $u_{i_j} \to q$ as $j \to +\infty$, then

$$\underset{j\to+\infty}{liminf}\ f(u_{i_j}) \geqslant f(q).$$

Hence $f(p) = \underset{i\to+\infty}{lim} f(u_i) \geqslant f(q)$. This implies that $q \in S_f$. The rest of the proof is similar to that of Theorem 5.1. □

Here we present an example of a second order difference inclusion on the Minkowski model of the two dimensional hyperbolic space, which is a Hadamard manifold with nonzero sectional curvature.

Example 5.3. Let $\mathbb{E}^{2,1}$ denote the vector space $\mathbb{R}^3$ endowed with the symmetric bilinear form (which is called the Lorentz metric) defined by $\langle x, y\rangle = x_1y_1 + x_2y_2 - x_3y_3$, where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. Let $\mathbb{H}^2 =$

$\{x \in \mathbb{E}^{2,1} : \langle x, x\rangle = -1,\ x_3 > 0\}$, which is the upper sheet of the hyperboloid $\{x \in \mathbb{E}^{2,1} : \langle x, x\rangle = -1\}$. Then $\mathbb{H}^2$, with the induced metric, is a two dimensional Hadamard manifold with sectional curvature $K = -1$ (cf. [2] and [7]). Furthermore, the normalized geodesic γ_v of $\mathbb{H}^2$ starting from x ($\gamma_v(0) = x$) have the equation $\gamma_v(t) = (\cosh t)x + (\sinh t)v$, where $v = \dot{\gamma}_v(0) \in T_x\mathbb{H}^2$ is the tangent unit vector of γ_v in the starting point. Hence $\exp(tv) = (\cosh t)x + (\sinh t)v$, for each unit vector $v \in T_x\mathbb{H}^n$, and

$$\exp_x^{-1} y = \operatorname{arccosh}(-\langle x, y\rangle)\frac{y + \langle x, y\rangle x}{\sqrt{\langle x, y\rangle^2 - 1}}, \tag{5.5}$$

for all $x, y \in \mathbb{H}_n$.
Assume that the map $f : \mathbb{E}^{2,1} \to \mathbb{R}$ is given by the equation

$$(x_1, x_2, x_3) \mapsto \frac{1}{2}(x_1^2 + x_2^2 + \frac{x_3^2}{3} - 1)x_3.$$

Then f is geodesically convex on $\mathbb{H}^2$ (its Hessian is positive definite on $\mathbb{H}^2$). Now, fix an arbitrary point $u_0 = x$ in $\mathbb{H}^2$ and suppose that $\{u_i\}$ is a sequence in $\mathbb{H}^2$ satisfying in the second order difference inclusion (2.6), for $\theta_i \equiv 1$, $c_i \equiv 1$ and $A \equiv \operatorname{grad} f$. By using (5.5), the equation becomes

$$\begin{cases} \operatorname{arccosh}(-\langle u_i, u_{i+1}\rangle)\frac{u_{i+1}+\langle u_i, u_{i+1}\rangle u_i}{\sqrt{\langle u_i, u_{i+1}\rangle^2 - 1}} + \operatorname{arccosh}(-\langle u_i, u_{i-1}\rangle)\frac{u_{i-1}+\langle u_i, u_{i-1}\rangle u_i}{\sqrt{\langle u_i, u_{i-1}\rangle^2 - 1}} \\ = \operatorname{grad} f(u_i), i \geqslant 1 \\ sup\{d(u_i, u_0) | i \geqslant 0\} < +\infty, \end{cases}$$

where $\operatorname{grad} f(u_i) = (u_{i1}u_{i3}, u_{i2}u_{i3}, {u_{i3}}^2 - 1)$, for each $u_i = (u_{i1}, u_{i2}, u_{i3}) \in \mathbb{H}^2$. This is a system of nonlinear equations which its solution set may be estimated by numerical methods. It is easily seen that $S_f = \{(0, 0, 1)\}$. Hence, by Theorem 5.1, $u_i \to (0, 0, 1)$ and $f(u_i) - (-\frac{1}{3}) = o(\frac{1}{i(i+1)})$.

Acknowledgment:

The authors are grateful to the referee for valuable suggestions leading to the improvement of the paper.

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