ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 41 (2015), No. 5, pp. 1061–1070

Title:

The stability of the solution of an inverse spectral problem with a singularity

Author(s):

S. Mosazadeh

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 41 (2015), No. 5, pp. 1061–1070
Online ISSN: 1735-8515

THE STABILITY OF THE SOLUTION OF AN INVERSE SPECTRAL PROBLEM WITH A SINGULARITY

S. MOSAZADEH

(Communicated by Asadollah Aghajani)

Abstract. This paper deals with the singular Sturm-Liouville expressions $\ell y = -y'' + q(x)y = \lambda y$ on a finite interval, where the potential function q is real and has a singularity inside the interval. Using the asymptotic estimates of a spectral fundamental system of solutions of Sturm-Liouville equation, the asymptotic form of the solution of the equation (0.1) and the eigenvalues are obtained, and proves the stability of the solution of the inverse spectral problem.
Keywords: Sturm-Liouville, singularity, stability, boundary value problem, inverse spectral problem.
MSC(2010): Primary: 34B05; Secondary: 34L20.

1. Introduction

In the present paper, we consider the Sturm-Liouville equation of the form

$$-y'' + q(x)y = \lambda y, \qquad x \in [0, T], \tag{1.1}$$

with the initial conditions

$$y(0, \lambda) = r_1, \quad y'(0, \lambda) = r_2, \tag{1.2}$$

where $q(x)$, r_1, r_2 are real, also $q(x)$ has a singular point $x = x_1$, $x_1 \in (0, T)$, and

$$\lambda = \rho^2, \tag{1.3}$$

is the spectral parameter independent of x.

The subject inverse Sturm-Liouville problem of reconstructing the potential function $q(x)$ from the given spectral information can be seen inside the wider context of ordinary differential equations on multistructures in mathematics, physics, engineering and chemistry (for details see [7, 11, 13, 14]). Also, the

Article electronically published on October 17, 2015.
Received: 29 December 2013, Accepted: 8 July 2014.

equation (1.1) with a singularity, has been associated with natural sciences. For example, the system of differential equations

$$\frac{dy_1}{dx} = i\rho R(x) y_2 \quad , \quad \frac{dy_2}{dx} = i\rho \frac{1}{R(x)}) y_1, \quad x \in [0,T], \tag{1.4}$$

appears in many problems such as optics, spectroscopy, in electrodynamic, Radioengineering and acoustic problems, where describes the wave propagation in a stratified medium. The spectral parameter ρ, is the *wave number* in a vacuum, y_1 and y_2 are the *components of the electromagnetic field*, x is the variable in the direction of stratification, and $R(x)$ is the *wave resistance* which describes the refractive properties of the medium (see [7, 15, 16] and references therein). The function $R(x)$ has the form

$$R(x) = |x - x_1|^{p-1} R_0(x),$$

where $0 < x_1 < T$, p is a real number, $R_0(x) > 0$, $R(0) = 1$, $R'(0) = 0$. By the replacement

$$y_1(x,\rho) = \sqrt{R(x)}\, u(x,\rho), \quad y_2(x,\rho) = \frac{1}{\sqrt{R(x)}} v(x,\rho)$$

and after elimination of v, we transform the system (1.4) to the Sturm-Liouville equation (1.1) with a singularity and the potential function $q(x) = h^2(x) - h'(x)$, $h(x) = (2R(x))^{-1} R'(x)$, and $q(x)$ has the form

$$q(x) = \frac{A}{(x - x_1)^2} + q_0(x), \tag{1.5}$$

where $A = \mu^2 - \frac{1}{4}$, $\mu = \frac{1}{4\eta+2}$, $\eta \in N$, $0 < x_1 < T$ and $q_0(x)(x - x_1)^{1-2\mu} \in L(0,T)$ (see [5, 6]). Therefore, in order to study the properties of the wave function $R(x)$, we can study the potential function $q(x)$.

The significance of asymptotic analysis in obtaining information on the solutions, eigenvalues and inverse problem of equation (1.1) with multiple singular points was realized by Freiling and Yurko in [5, 6], Yurko [17] and Harris [2, 3].

Horvath and Kiss in [4] considered the inverse eigenvalue problem $L_1 = L_1(\chi(x), \alpha, \beta)$ for Schrodinger operator

$$\ell_1 Y \equiv -Y'' + \chi(x) Y = \lambda Y, \qquad x \in [0,\pi],$$

$$Y(0)\cos\alpha + Y'(0)\sin\alpha = 0, \qquad Y(\pi)\cos\beta + Y'(\pi)\sin\beta = 0,$$

and shown that if the potential function $\chi(x)$ is in L^p, then the perturbation of the potentials can be estimated by the $l^{p'}$-norm of the sequence of the eigenvalue differences only if $p \geq 2$, where $\frac{1}{p} + \frac{1}{p'} = 1$. For the stability of reconstruction from the spectral function, see also Marchenko and Maslov [8], where the authors obtained an estimate for operators defined on the half-line, also corresponding results obtained by Marletta and Weikard in [9]. In [10], for two second-order inverse spectral problems defined on a bounded interval, it is shown that a change in the form of the differential operator produces a change in the stability result. A differential systems having a singularity and one turning point of even order was studied in [12], and fined the canonical form of the solutions. In the article [18], the author considered a self-adjoint boundary value problem with nonseparated boundary conditions, and obtained the solution of the inverse problem for the boundary value problem and proved stability theorem for it.

In this paper, we consider a boundary value problem consists of (1.1), (1.2) with the condition $y(T,\lambda)=0$, where the potential function $q(x)$ is real, has a singularity inside the interval $(0,T)$ and is the form (1.5). The results which we obtain in the present paper are finding the asymptotic form of the solution of the equation (1.1) with the initial conditions (1.2) (see section 2), obtain a local solution of the inverse problem, and using a different method which allows one to obtain new results in the inverse problem theory for the operators with a singularity, we give necessary and sufficient conditions on spectrum of the boundary value problem, and prove stability of the solution of inverse spectral problem (see section 3). We mention that a set of products of the eigenfunctions of the boundary value problem forms a Riesz basis in $L^2(0,T)$ plays an important role for investigating the stability of the solution of the inverse problem.

2. Asymptotic form of the solution and eigenvalues

We consider the Sturm-Liouville equation (1.1). Since the solutions of equation (1.1) have singularity at $x=x_1$, therefore in general the values of the solutions and their derivatives at $x=x_1$ are not defined.

Remark 2.1. In [17], fundamental system of solutions $\{S_k(x,\lambda)\},\ k=1,2,$ of equation (1.1) were constructed with the following properties:

(1) For each fixed $x\in[0,T]\backslash\{x_1\}$, the functions $S_k^{(\nu)}(x,\lambda),\ \nu=0,1,$ are entire in λ of order $\frac{1}{2}$.

(2) Denote $\mu_k=(-1)^k\mu+\frac{1}{2},\ k=1,2$. Then, from (1.3), for $|\rho(x-x_1)|\le 1$,
$$|S_k(x,\lambda)|\le C_1|(x-x_1)^{\mu_k}|,$$
where C_1 is a positive constant in estimate not depending on x and ρ.

(3) The following relation holds
$$< S_1(x,\lambda),S_2(x,\lambda) >\equiv 1,$$
where $< y_1(x),y_2(x) >:= y_1(x)y_2'(x)-y_1'(x)y_2(x)$ is the Wronskian of y_1 and y_2.

Let $\omega_0=[0,x_1),\ \ \omega_1=(x_1,T]$, from $[1,5]$ for $x\in\omega_0\cup\omega_1$,
$$S_k(x,\lambda)=(x-x_1)^{\mu_k}\sum_{m=0}^{\infty}S_{km}(\rho(x-x_1))^{2m},\ \ k=1,2,$$

where

$$S_{10}S_{20}=(2\mu)^{-1} \quad, S_{km}=(-1)^m S_{k0}(\prod_{s=1}^{m}((2s+\mu_k)(2s+\mu_k-1)-A))^{-1}.$$

Now, we define the functions $\varphi_k(x,\lambda), k=1,2$, by the following formula :

$$\varphi_k(x,\lambda)=(-1)^{k-1}(S_2^{(2-k)}(0,\lambda)S_1(x,\lambda)-S_1^{(2-k)}(0,\lambda)S_2(x,\lambda)), \quad k=1,2.$$

The functions $\varphi_k(x,\lambda)$ are solutions of (1.1) and

$$\varphi_k^{(m-1)}(0,\lambda)=\delta_{k,m}, \quad k,m=1,2, \tag{2.1}$$

($\delta_{k,m}$ is the Kronecker delta). Moreover

$$\langle\varphi_1(x,\lambda),\varphi_2(x,\lambda)\rangle\equiv 1.$$

From [5], we have the following Lemma:

Lemma 2.2. *For* $(\rho,x)\in\Omega:=\{(\rho,x): \ |\rho(x-x_1)|\geq 1\}$, $x\in\omega_s$, $s=0,1$ *:*

$$\varphi_k^{(m-1)}(x,\lambda)=\frac{1}{2}(i\rho)^{m-k}\{\exp(i\rho x)[1]_\gamma+(-1)^{m-k}\exp(-i\rho x)[1]_\gamma \tag{2.2}$$

$$+(-1)^k 2si\cos\pi\mu \ \exp(i\rho(x-2x_1))[1]_\gamma \ \}, \quad |\rho|\to\infty, \quad k,m=1,2,$$

where $[1]_\gamma=1+O((\rho(x-x_1))^{-1})$.

Using the preceding results, from (1.2), (1.3) and (2.1) we have

$$u(x,\rho)=r_1\varphi_1(x,\lambda)+r_2\varphi_2(x,\lambda). \tag{2.3}$$

According to (2.2) and (2.3) we obtain the asymptotic solution of equation (1.1) in the following theorem:

Theorem 2.3. *For* $x\in\omega_s$, $s=0,1$, $(\rho,x)\in\Omega$, $|\rho|\to\infty$, $Im\rho\geq 0$, $m=0,1$*:*

$$u^{(m)}(x,\rho)=\frac{1}{2}(i\rho)^{m-1}(i\rho r_1+r_2)\exp(i\rho x)[1]_\gamma+\frac{1}{2}(-i\rho)^{m-1}(-i\rho r_1+r_2)$$

$$\times\exp(-i\rho x)[1]_\gamma+s(i\rho)^{m-1}(\rho r_1+ir_2)\cos\pi\mu \ \exp(i\rho(x-2x_1))[1]_\gamma.$$

Now, we consider the boundary value problem $L=L(q(x),r_1,r_2)$ for the equation with boundary conditions

$$y(0,\lambda)=r_1, \quad y'(0,\lambda)=r_2, \quad y(T,\lambda)=0. \tag{2.4}$$

The boundary value problem L has a countable set of positive eigenvalues $\{\lambda_n(T)\}_{n=1}^{\infty}$. From the boundary conditions (2.4) and Theorem 2.3, we have the following asymptotic distribution for each $\lambda_n(T)$, $n\geq 1$:

$$\sqrt{\lambda_n(T)}=\frac{n\pi-\frac{\pi}{2}}{T}+O(\frac{1}{n}). \tag{2.5}$$

3. **The stability theorem**

We seek in this section to establish a stability result for potentials q which have been obtained as solutions of an inverse spectral problem. In our method an important role is played by a set of products of the eigenfunctions of the boundary value problem L forms a Riesz basis in $L^2(0,T)$. Theorem 3.4 is the main result of this section.

First, we denote the functions $\alpha_n(x)$ by the formula

$$\alpha_n(x) = \varphi_1^2(x,\lambda_n) - \frac{1}{2}, \qquad n = 1,2,3,..., \tag{3.1}$$

where the function $\varphi_1(x,\lambda_n)$ is defined in the section 2. In order to continue, we need the following lemma that follows from the method of the proof in [18, Lemma 2.1].

Lemma 3.1. *The system of the functions* $\{\alpha_n(x)\}_{n=1}^{\infty}$ *is a Riesz basis in* $L^2(0,T)$.

Let $\mathbf{B}$ be a Banach space and $g \in L^2(0,T)$. Also, we consider the following nonlinear equation in the space $\mathbf{B}$,

$$g + \sum_{i=1}^{\infty} \theta_i(\tau) = \tau. \tag{3.2}$$

Theorem 3.2. *Let for* $\varepsilon_0 \le \frac{1}{4}$,

$$\|\theta_1(\tau)\| \le \varepsilon_0\|\tau\|, \quad \|\theta_1(\tau) - \theta_1(\widetilde{\tau})\| \le \varepsilon_0\|\tau - \widetilde{\tau}\|,$$

and for $i \ge 2$,

$$\|\theta_i(\tau)\| \le (C_0\|\tau\|)^i,$$

$$\|\theta_i(\tau) - \theta_i(\widetilde{\tau})\| \le \|\tau - \widetilde{\tau}\|(C_0 \max(\|\tau\|, \|\widetilde{\tau}\|))^{i-1},$$

where C_0 *is constant, and the sign* $\|\cdot\|$ *denotes the norm in* $L^2(0,T)$ *with respect to* x. *Then there exists a number* $\varepsilon > 0$ *such that equation (3.2) in* $D = \{ \tau \ : \ \|\tau\| \le 2\varepsilon \}$ *has a unique solution* $\tau \in \mathbf{B}$ *for all functions* g *which* $\|g\| \le \varepsilon$, *and* $\|\tau\| \le 2\|g\|$.

Proof. For $C_0 \ge 1$, we define

$$\varepsilon = \frac{1}{8C_1}, \quad \theta(\tau) = \sum_{i=1}^{\infty} \theta_i(\tau),$$

where $C_1 = 2C_0^2$. If $\|\tau\|, \|\widetilde{\tau}\| \le \frac{1}{4C_1}$, then $C_0\|\tau\|\| \le \frac{1}{2}$, and

$$\|\theta(\tau)\| \le \varepsilon_0\|\tau\| + \sum_{i=2}^{\infty}(C_0\|\tau\|)^i.$$

Since $\sum_{i=2}^{\infty}(C_0\|\tau\|)^i$ is a geometric series with a ratio less than $\frac{1}{2}$, thus it is convergent to $\frac{C_0^2\|\tau\|^2}{1-C_0\|\tau\|}$. Therefore,

$$\|\theta(\tau)\| \le \frac{\|\tau\|}{2}. \tag{3.3}$$

Moreover, according to the assumptions of the theorem,

$$\|\theta(\tau)-\theta(\widetilde{\tau})\| \le \|\tau-\widetilde{\tau}\| \ \{\varepsilon_0+\sum_{i=2}^{\infty}(\frac{C_0}{4C_1})^{i-1}\}.$$

Since $\frac{C_0}{4C_1-C_0} \le \frac{1}{4}$, therefore above inequality together (3.3) give

$$\|\theta(\tau)-\theta(\widetilde{\tau})\| \le \frac{1}{2}\|\tau-\widetilde{\tau}\|. \tag{3.4}$$

Let $\|g\| \le \varepsilon$. For $n = 0,1,2,3,...$, we construct $\tau_0 = g$, $\tau_{n+1} = g + \theta(\tau_n)$. Then, by virtue of (3.4), the series $\tau = \tau_0 + \sum_{n=0}^{\infty}(\tau_{n+1}-\tau_n)$ converges to the solution τ of (3.2). This and $\|\theta(\tau)\| \le \frac{\|\tau\|}{2}$ result $\|\tau\| \le 2||g||$. The uniqueness is distinct. Theorem 3.2 is proved. □

Now, we define the biorthogonal basis $\{\beta_n(x)\}_{n=1}^{\infty}$ to the basis $\{\alpha_n(x)\}_{n=1}^{\infty}$, and take $\varepsilon > 0$ and the sequence $\{\widetilde{\lambda}_n\}_{n=1}^{\infty}$ such that

$$\kappa := \{\sum_{n=1}^{\infty}|\lambda_n-\widetilde{\lambda}_n|^2\}^{\frac{1}{2}} < \varepsilon,$$

where $\{\lambda_n\}_{n=1}^{\infty}$ is the sequence of the eigenvalues of the boundary value problem L. Also, we denote some auxiliary functions and prove some preliminary results. We define the Green's function

$$G(x,t,\lambda) = \varphi_1(t,\lambda)\varphi_2(x,\lambda) - \varphi_1(x,\lambda)\varphi_2(t,\lambda)$$

of the Cauchy problem

$$-y'' + q(x)y - \lambda y = g(x), \quad y(0) = y'(0) = 0.$$

Also, we define

$$G_n(x,t) = \begin{cases} 0, & x < t, \\ \\ G(x,t,\widetilde{\lambda}_n), & x \ge t. \end{cases}$$

Take $\varepsilon_1 > 0$ such that for $\kappa < \varepsilon_1$

$$\varphi_1(T,\widetilde{\lambda}_n) \ne 0, \quad \varphi_1'(T,\widetilde{\lambda}_n) \ne 0.$$

We consider the nonlinear integral equation

$$\tau(x) = g(x) + \sum_{i=1}^{\infty}\underbrace{\int_0^T ... \int_0^T}_{i} M_i(x,t_1,...,t_i)\tau(t_1)...\tau(t_i)dt_1...dt_i, \tag{3.5}$$

in $L^2(0,T)$ where

$$g(x)=\sum_{n=1}^{\infty}\beta_n(x)g_n,\quad M_i(x,t_1,...,t_i)=\sum_{n=1}^{\infty}\beta_n(x)Q_{n,i}(t_1,...,t_i), \tag{3.6}$$

and the Fourier coefficients $Q_{n,i}(t_1,...,t_i)$ and g_n are defined by:

(1) $Q_{n,1}(t_1)=\varphi_1(T,\widetilde{\lambda}_n)\{-\frac{1}{2}\varphi_2'(T,\widetilde{\lambda}_n)+\frac{\partial G_n(x,t_1)}{\partial x}|_{x=T}\ \varphi_1(t_1,\widetilde{\lambda}_n)\}$ $+\varphi_1^2(t_1,\lambda_n)-\varphi_1^2(t_1,\widetilde{\lambda}_n)$.

(2) For $i\geq 2$,

$$\begin{aligned}Q_{n,i}(t_1,...,t_i)&=\varphi_1(t_1,\widetilde{\lambda}_n)\ (\ \frac{1}{2}\delta_{i2}-G_n(t_1,t_2)...G_n(t_{i-2},t_{i-1})\\&\times\{G_n(t_{i-1},t_i)\varphi_1(t_i,\widetilde{\lambda}_n)-\frac{1}{2}\varphi_2(t_{i-1},\widetilde{\lambda}_n)(1-\delta_{i2})\}\ )\\&+\varphi_1(T,\widetilde{\lambda}_n)\frac{\partial G_n(x,t_1)}{\partial x}|_{x=T}\ G_n(t_1,t_2)...G_n(t_{i-2},t_{i-1})\\&\times\{G_n(t_{i-1},t_i)\varphi_1(t_i,\widetilde{\lambda}_n)-\frac{1}{2}\varphi_2(t_{i-1},\widetilde{\lambda}_n)\},\end{aligned}$$

where δ_{ik} is the Kronecker delta.

(3) $g_n=\varphi_1(T,\widetilde{\lambda}_n)\varphi_1'(T,\widetilde{\lambda}_n)$.

Clearly,

$$|\varphi_1(x,\widetilde{\lambda}_n)|<C,\quad |\varphi_1(T,\widetilde{\lambda}_n)|<\frac{C}{n+1}|\lambda_n-\widetilde{\lambda}_n|,\quad |\varphi_1'(T,\widetilde{\lambda}_n)|<\frac{C}{n+1},$$

$$|\varphi_1^2(t_1,\lambda_n)-\varphi_1^2(t_1,\widetilde{\lambda}_n)|<\frac{C}{n+1}|\lambda_n-\widetilde{\lambda}_n|,\quad |\varphi_2(x,\widetilde{\lambda}_n)|<\frac{C}{n+1},$$

also

$$|G_n(x,t)|<\frac{C}{n+1},\quad |\frac{\partial G_n(x,t)}{\partial x}|<C,$$

and consequently, from (3.6),

$$\|M_1(x,t_1)\|<C\kappa,\ \|M_i(x,t_1,...,t_i)\|<C^i,\ i\geq 2,\ \|g(x)\|<C\kappa, \tag{3.7}$$

where here and in the sequel, we shall denote by C different constants depending on L. Therefore, we obtain the following corollary:

Corollary 3.3. *According to (3.7) and by using the Holder's and Schwarz's inequalities, the conditions of Theorem 3.2 for (3.5) are valid. Thus, there exists a number $\varepsilon>0$ such that if $\kappa<\varepsilon$, then there is a solution of (3.5) $\tau(x)\in L^2(0,T)$, $\|\tau(x)\|<C\kappa$.*

Theorem 3.4. (*Stability theorem*) *There exists a number* $\varepsilon > 0$ *(depending on L) such that if numbers* $\{\widetilde{\lambda}_n\}_{n=1}^{\infty}$ *satisfy the condition*

$$\kappa = \{\sum_{n=1}^{\infty} |\lambda_n - \widetilde{\lambda}_n|^2\}^{\frac{1}{2}} < \varepsilon,$$

where $\{\lambda_n\}_{n=1}^{\infty}$, *the sequence of the eigenvalues of the boundary value problem L, defined in (2.5). Then, there is a real valued function* $\widetilde{q}(x) \in L^2(0,T)$ *such that the numbers* $\{\widetilde{\lambda}_n\}_{n=1}^{\infty}$ *are the eigenvalues of the boundary value problem* $\widetilde{L} = L(\widetilde{q}(x), r_1, r_2)$, *and*

$$\|q(x) - \widetilde{q}(x)\| < \kappa C,$$

where C *is a constant depending on* L.

Proof. Applying Corollary 3.3, there exists a number $\varepsilon > 0$ such that if $\kappa < \varepsilon$, then there is a solution $\tau(x) \in L^2(0,T)$ of (3.5) such that $\|\tau(x)\| < C\kappa$. Now, we denote

$$\widetilde{q}(x) = \tau(x) + q(x). \tag{3.8}$$

Therefore,

$$\|q(x) - \widetilde{q}(x)\| < C\kappa.$$

Now, we consider $\widetilde{\varphi}_1(x, \widetilde{\lambda}_n)$, which is the solution of the following equation

$$-\widetilde{\varphi}_1''(x, \widetilde{\lambda}_n) + \widetilde{q}(x)\widetilde{\varphi}_1(x, \widetilde{\lambda}_n) = \widetilde{\lambda}_n \widetilde{\varphi}_1(x, \widetilde{\lambda}_n), \tag{3.9}$$

with the initial conditions

$$\widetilde{\varphi}_1(0, \widetilde{\lambda}_n) = 1, \quad \widetilde{\varphi}_1'(0, \widetilde{\lambda}_n) = 0. \tag{3.10}$$

Thus, it follows that

$$\widetilde{\varphi}_1(x, \widetilde{\lambda}_n) = \varphi_1(x, \widetilde{\lambda}_n) + \eta\varphi_2(x, \widetilde{\lambda}_n) + \int_0^T G_n(x,t)\tau(t)\widetilde{\varphi}_1(t, \widetilde{\lambda}_n)dt, \tag{3.11}$$

where $\eta = -\frac{1}{2}\int_0^T \tau(x)dx$. Now, we have to show that $\widetilde{\varphi}_1(T, \widetilde{\lambda}_n) = 0$. From (3.8)-(3.10) we can write

$$\begin{aligned}\int_0^T \tau(x)\varphi_1(x, \widetilde{\lambda}_n)\widetilde{\varphi}_1(x, \widetilde{\lambda}_n)dx &= \int_0^T (\widetilde{q}(x) - q(x))\varphi_1(x, \widetilde{\lambda}_n)\widetilde{\varphi}_1(x, \widetilde{\lambda}_n)dx \\ &= \{(\widetilde{\varphi}_1'\varphi_1 - \varphi_1'\widetilde{\varphi}_1)(x, \widetilde{\lambda}_n)\}\rfloor_0^T,\end{aligned}$$

therefore,

$$\int_0^T \tau(x)\varphi_1(x, \widetilde{\lambda}_n)\widetilde{\varphi}_1(x, \widetilde{\lambda}_n)dx = \tag{3.12}$$

$$\varphi_1(T, \widetilde{\lambda}_n)\widetilde{\varphi}_1'(T, \widetilde{\lambda}_n) - \widetilde{\varphi}_1(T, \widetilde{\lambda}_n)\varphi_1'(T, \widetilde{\lambda}_n) - \eta.$$

Solving the integral equation (3.11) by using the successive approximation method, we get

$$(3.13)\quad \widetilde{\varphi}_1(x,\widetilde{\lambda}_n)=\varphi_1(x,\widetilde{\lambda}_n)+\eta\varphi_2(x,\widetilde{\lambda}_n)+\sum_{i=1}^{\infty}\underbrace{\int_0^T\ldots\int_0^T}_{i}\{G_n(x,t_1)$$

$$\times G_n(t_1,t_2)\ldots G_n(t_{i-1},t_i)\tau(t_1)\ldots\tau(t_i)(\varphi_1(t_i,\widetilde{\lambda}_n)+\eta\varphi_2(t_i,\widetilde{\lambda}_n))dt_1\ldots dt_i\}.$$

Multiply (3.5) by $\alpha_n(x)$ and integrate it from 0 to T with respect to x, then

$$(3.14)\quad \int_0^T\tau(x)\alpha_n(x)dx=\int_0^T g(x)\alpha_n(x)dx$$

$$+\int_0^T\{\ \sum_{i=1}^{\infty}\underbrace{\int_0^T\ldots\int_0^T}_{i}\alpha_n(x)Q_{n,i}(t_1,\ldots,t_i)\tau(t_1)\ldots\tau(t_i)dt_1\ldots dt_i\ \}dx.$$

According to (3.1), (3.13), and by substituting $Q_{n,1}(t_1)$, $Q_{n,i}(t_1,\ldots,t_i)$ into (3.14) we derive

$$(3.15)\quad \int_0^T\tau(x)\varphi_1^2(x,\lambda_n)dx=g_n+\int_0^T\tau(x)(\varphi_1^2(x,\lambda_n)-\varphi_1^2(x,\widetilde{\lambda}_n))dx$$

$$-\int_0^T\tau(x)\varphi_1(x,\widetilde{\lambda}_n)\zeta_n(x)dx-\eta\{\ 1+\int_0^T\tau(x)\varphi_1(x,\widetilde{\lambda}_n)\varphi_2(x,\widetilde{\lambda}_n)dx\ \}$$

$$+\varphi_1(T,\widetilde{\lambda}_n)\{\ \eta\varphi_2'(T,\widetilde{\lambda}_n)+\zeta_n'(T)\ \},$$

where

$$\zeta_n(x)=\sum_{i=1}^{\infty}\underbrace{\int_0^T\ldots\int_0^T}_{i}G_n(x,t_1)G_n(t_1,t_2)\ldots G_n(t_{i-1},t_i)\tau(t_1)\ldots\tau(t_i)$$

$$\times(\varphi_1(t_i,\widetilde{\lambda}_n)+\eta\varphi_2(t_i,\widetilde{\lambda}_n))dt_1\ldots dt_i.$$

Therefore, from (3.1), (3.13) and (3.15), we calculate

$$(3.16)\quad \int_0^T\tau(x)\varphi_1(x,\widetilde{\lambda}_n)\widetilde{\varphi}_1(x,\widetilde{\lambda}_n)dx=\varphi_1(T,\widetilde{\lambda}_n)\widetilde{\varphi}_1'(T,\widetilde{\lambda}_n)-\eta.$$

By using (3.12) and (3.16), we have

$$\widetilde{\varphi}_1(T,\widetilde{\lambda}_n)=0.$$

Therefore, the numbers $\{\widetilde{\lambda}_n\}_{n=1}^{\infty}$ are the eigenvalues of $\widetilde{L}$, and $\widetilde{\varphi}_1(x,\widetilde{\lambda}_n)$ are their eigenfunctions. This completes the proof. □

References

[1] J. B. Conway, Functions of One Complex Variable, Springer-Verlag, New York, 1995.

[2] B. J. Harris, A note on a paper of Atkinson concerning the asymptotics of an eigenvalue problem with interior singularity, *Proc. Roy. Soc. Edinburgh Sect. A* **110** (1988), no. 1-2, 63–71.

[3] B. J. Harris and D. Race, Asymptotics of eigenvalues for Sturm-Liouville problems with an interior singularity, *J. Differential Equations* **116** (1995), no. 1, 88–118.

[4] M. Horvath and M. Kiss, Stability of Direct and inverse eigenvalue problems for Schrödinger operators on finite intervals, *Int. Math. Res. Not. IMRN* **2010** (2009), no. 11, 2022–2063.

[5] G. Freiling and V. Yurko, On the determination of differential equations with singularities and turning points, *Results Math* **41** (2002), no. 3-4, 275–290.

[6] G. Freiling and V. Yurko, On constructing differential equations with singularities from incomplete spectral information, *Inverse problems* **14** (1998), no. 5, 1131–1150.

[7] O. N. Litvinenko and V. I. Soshnikov, The theory of Heterogenous Lines and their Applications in Radio Engineering, Moscow Radio, Moscow, 1964.

[8] V. A. Marchenko and K. V. Maslov, Stability of the reconstruction of Sturm-Liouville operators by the spectral function, *Matematicheskii Sbornik.* **81** (1970), no. 4, 525-551.

[9] M. Marletta and R. Weikard, Weak stability for an inverse Sturm-Liouville problem with finite spectral data and complex potential, *Inverse Problems* **21** (2005), no. 4, 1275–1290.

[10] J. R. Mclaughlin, Stability theorems for two inverse spectral problems, *Inverse Problems* **4** (1988), no. 2, 529–540.

[11] L. Motl and A. Neitzke, Asymptotic black hole quasinormal frequencies, *Adv. Theor. Math. Phys.* **7** (2003), no. 2, 307–330.

[12] A. Neamaty and S. Mosazadeh, On the canonical solution of the Sturm-Liouville problem with singularity and turning point of even order, *Canad. Math. Bull.* **54** (2011), no. 3, 506–518.

[13] E. Schrödinger, An undulatory theory of the mechanics of atoms and molecules, *Phys. Rev.* **28** (1926), no. 6, 1049–1070.

[14] J. K. Shaw, A. P. Baronavski, H. D. Ladouceur and W. O. Amrein, Applications of the Walker method, Spectral Theory and Computational Methods of Sturm-Liouville problems, 377–395, Lecture Notes in Pure and Appl. Math., 191,Dekker, New York, 1997.

[15] A. G. Sveshnikov and A. S. Il'inskii, Design problems in electrodynamics, *Dokl. Akad. Nauk SSSR.* **204** (1972), no. 5, 1077–1080.

[16] A. V. Tikhonravov, Theoretically attainable exactness of the solution of a synthesis problem, *Zh. Vychisl. Mat. Mat. Fiz.* **22** (1982), no. 6, 1421–1433.

[17] V. A. Yurko, An inverse problem for differential equations with a singularity, *Differential Equations* **28** (1992), no. 8, 1100–1107.

[18] V. A. Yurko, The inverse spectral problem for differential operators with nonseparated boundary conditions, *J. Math. Anal. Appl.* **250** (2000), no. 1, 266–289.

(Seyfollah Mosazadeh) Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Kashan, Kashan, Iran

E-mail address: `s.mosazadeh@kashanu.ac.ir`