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ZERO SETS IN POINTFREE TOPOLOGY AND STRONGLY z-IDEALS

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Abstract. In this paper a particular case of z-ideals, called *strongly z-ideal*, is defined by introducing zero sets in pointfree topology. We study strongly z-ideals, their relation with z-ideals and the role of spatiality in this relation. For strongly z-ideals, we analyze prime ideals using the concept of zero sets. Moreover, it is proven that the intersection of all zero sets of a prime ideal of $C(L)$, which is ring of real-valued continuous functions for frame L, does not have more than one element. Also, z-filters are introduced in terms of pointfree topology. Then the relationship between z-filters and ideals, particularly maximal ideals, is examined. Finally, it is shown that the family of all zero sets is a base for the collection of closed sets.
Keywords: Frame, ring of real-valued continuous functions, zero set, z-ideal, strongly z-ideal.
MSC(2010): Primary: 06D22; Secondary: 13A15, 13C99.

1. Introduction

To study the ring $C(X)$, X is a topological space, zero sets and z-ideals play important role (for more details see [1–3,12]). Banaschewski and Gilmour study the ring $C(L)$ as the pointfree version of $C(X)$ and took cozero elements as pointfree version of cozero sets (1996, [6]). T. Dube [8–10] applied cozero elements to introduce z-ideals in $C(L)$.

In this paper, by considering prime elements of a given frame L as pointfree points of L, we define the trace of an element α of $C(L)$ on any point p of L that is a real number denoted by $\alpha[p]$, and also, zero set of α by $Z(\alpha) = \{p \in \Sigma L : \alpha[p] = 0\}$ is defined. The real number $\alpha[p]$ is defined by Dedekind cut $(L(p,\alpha), U(p,\alpha))$, where $L(p,\alpha) = \{r \in \mathbb{Q} : \alpha(-,r) \leq p\}$ and $U(p,\alpha) = \{s \in$

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$\mathbb{Q} : \alpha(s,-) \leq p\}$ (Proposition 2.2). Also, the map $\widetilde{p} : C(L) \to \mathbb{R}$ given by $\widetilde{p}(\alpha) = \alpha[p]$ is an f-ring homomorphism (Proposition 2.4).

The necessary background on frames (pointfree topology) is given in Section 2.

In Section 3, a relation between zero sets and cozero elements is given in Lemma 3.2, that is: $p \in Z(\alpha)$ if and only if $coz(\alpha) \leq p$. The basic relations that we expect from zero sets are shown in Proposition 3.3. The property of the family of zero sets which is a base for the closed sets is proved in Proposition 3.8 for completely regular frames. Moreover, if L is a spatial frame and the family $Z[L]$ of all zero sets is a base for the closed sets of ΣL, then L is completely regular.

Some natural relations between ideals of $C(L)$ and z-filters that are filters of the lattice of $Z(L)$ are explained in Section 4. Also, we seek some relations among z-ultrafilters, maximal ideals and families with finite intersection property.

Finally, in the last section, *strongly z-ideals* of $C(L)$ are introduced by using the concept of zero sets. It is proved that for a spatial frame L, every z-ideal is a strongly z-ideal (Proposition 5.7). Prime strongly z-ideals are analyzed in Theorem 5.11. Proposition 5.10 was proved by T. Dube in [8], but we prove it directly by our method. In Proposition 5.14, it is proven that the intersection of all zero sets of a prime ideal in $C(L)$ does not have more than one element for a completely regular frame L.

2. **Preliminaries**

Here, we recall some definitions and results from the literature on frames and the pointfree version of the ring of continuous real valued functions. For further information see [15] on frame-theoretic conceppts and [5] on pointfree function rings.

A *frame* is a complete lattice L in which the distributive law

$$x \wedge \bigvee S = \bigvee \{x \wedge s : s \in S\}$$

holds for all $x \in L$ and $S \subseteq L$. We denote the top element and the bottom element of L by $\top$ and $\bot$, respectively. The frame of open subsets of a topological spase X is denoted by $\mathfrak{O}X$.

A *frame homomorphism (frame map)* is a map between frames which preserves finite meets, including the top element, and arbitrary joins, including the bottom element.

An element a of frame L is said to be *rather below* an element b, written $a \prec b$, in the case when there is an element s, called a *separating element*, such that $a \wedge s = \bot$ and $s \vee b = \top$. On the other hand, a is *completely below* b, written $a \prec\!\prec b$, if there are elements (c_q) indexed by the rational numbers

$\mathbb{Q} \cap [0,1]$ such that $c_0 = a$, $c_1 = b$, and $c_p \prec c_q$ for $p < q$. A frame L is said to be *regular* if $a = \bigvee\{x \in L \mid x \prec a\}$ for each $a \in L$, and *completely regular* if $a = \bigvee\{x \in L \mid x \prec\!\prec a\}$ for each $a \in L$.

An element $p \in M$ is called *prime* if $p < \top$ and $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$. An element $m \in M$ is called *maximal* if $m < \top$ and $m \leq x \leq \top$ implies $m = x$ or $x = \top$. Note that every maximal element is prime.

Recall the contravariant functor Σ from **Frm** to the category **Top** of topological spaces which assigns to each frame L its *spectrum* ΣL of prime elements with $\Sigma_a = \{p \in \Sigma L | a \not\leq p\}$ $(a \in L)$ as its open sets. Also, for a frame map $h : L \to M$, $\Sigma h : \Sigma M \to \Sigma L$ takes $p \in \Sigma M$ to $h_*(p) \in \Sigma L$, where $h_* : M \to L$ is the *right adjoint* of h characterized by the condition $h(a) \leq b$ if and only if $a \leq h_*(b)$ for all $a \in L$ and $b \in M$. Note that h_* preserves primes and arbitrary meets.

Recall [5,6] that the frame $\Re$ of reals is obtained by taking the ordered pairs (p, q) of rational numbers as generators and imposing the following relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$

(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$

(R3) $(p, q) = \bigvee\{(r, s) | p < r < s < q\}$

(R4) $\top = \bigvee\{(p, q) |$ all $p, q\}$.

The set $C(L)$ of all frame homomorphisms from $\Re$ to L has been studied as an f-ring in [4,6].

Corresponding to every continuous operation $\diamond : \mathbb{Q}^2 \to \mathbb{Q}$ (in particular, $+, ., \wedge, \vee$) we have an operation on $C(L)$, denoted by the same symbol $\diamond$, defined by:

$$\alpha \diamond \beta(p, q) = \bigvee\{\alpha(r, s) \wedge \beta(u, w) : (r, s) \diamond (u, w) \leq (p, q)\},$$

where $(r, s) \diamond (u, w) \leq (p, q)$ means that if $r < x < s$ and $u < y < w$ then $p < x \diamond y < q$. For every $r \in \mathbb{R}$, we define the constant frame map $\mathbf{r} \in C(L)$ by $\mathbf{r}(p, q) = \top$, whenever $p < r < q$, and otherwise $\mathbf{r}(p, q) = \bot$.

The *cozero map* is the map $coz : C(L) \to L$, defined by

$$coz(\alpha) = \alpha(-, 0) \vee \alpha(0, -).$$

For $A \subseteq C(L)$, let $Coz(A) = \{coz(\alpha) : \alpha \in A\}$ with the cozero part of a frame L, $Coz(C(L))$, called $CozL$ by previous authors. It is known that L is completely regular if and only if $Coz(C(L))$ generates L. For more details about *cozero map* and its properties which are used in this paper see [5].

For $A \subseteq Coz(L)$, we write $Coz^{\leftarrow}(A)$ to designate the family of frame maps $\{\alpha \in C(L) : coz(\alpha) \in A\}$.

In [9], Dube showed the following proposition. It should be noticed that in [9] he considered the frame as completely regular, but he did not depoly this condition in the proof of the aforementioned proposition.

Proposition 2.1. *The following statements hold:*

(1) *If I is a proper ideal of $C(L)$, then $Coz(I)$ is a proper ideal of $Coz(L)$.*

(2) *If A is a proper ideal of L or $Coz(L)$, then $Coz^{\leftarrow}(A)$ is a proper ideal of $C(L)$.*
(3) *If M is a maximal ideal of $C(L)$, then $coz(M)$ is a maximal ideal of $Coz(L)$ and $Coz^{\leftarrow}(Coz(M)) = M$.*
(4) *If A is a maximal ideal of L or $Coz(L)$, then $Coz^{\leftarrow}(A)$ is a maximal ideal of $C(L)$.*

In this paper, we use the pointfree version of the map $\hat{x} : C(X) \to \mathbb{R}$ given by $\hat{x}(f) = f(x)$. In poinfree version, we have prime elements $p \in L$ to replace points $x \in X$. Here we recall necessary notations, definitions and results form [11]. Let $a \in L$ and $\alpha \in C(L)$. The sets $\{r \in \mathbb{Q} : \alpha(-, r) \leq a\}$ and $\{s \in \mathbb{Q} : \alpha(s, -) \leq a\}$, are denoted by $L(a, \alpha)$ and $U(a, \alpha)$, respectively.

For $a \neq \top$ it is obvious that for each $r \in L(a, \alpha)$ and $s \in U(a, \alpha)$, $r \leq s$. In fact, we have:

Proposition 2.2. [11] Let L be a frame. If $p \in \Sigma L$ and $\alpha \in C(L)$, then $(L(p, \alpha), U(p, \alpha))$ is a Dedekind cut for a real number which is denoted by $\widetilde{p}(\alpha)$.

To learn more about Dedekind cut see [13].

Proposition 2.3. [11] If p is a prime element of a frame L, then there exists a unique map $\widetilde{p} : C(L) \longrightarrow \mathbb{R}$ such that $r \leq \widetilde{p}(\alpha) \leq s$ for all $\alpha \in C(L)$, $r \in L(p, \alpha)$, and $s \in U(p, \alpha)$.

If $L = \mathfrak{O}X$ for a topological space X, then for every $x \in X$, $\hat{x} : C(X) \longrightarrow \mathbb{R}$, given by $\hat{x}(\alpha) = \alpha(x)$, factors through $\widetilde{\overline{\{x\}}'}$; note that in $\mathfrak{O}X$, for every $x \in X$, $\overline{\{x\}}'$ is a prime element. In fact, $\hat{x} = \widetilde{\overline{\{x\}}'} \circ \phi$, where $\phi : C(X) \longrightarrow C(\mathfrak{O}X)$ is the isomorphism given by $\phi(\alpha)(p, q) = \alpha^{-1}(p, q)$. Hence $\widetilde{\overline{\{x\}}'}$ is equal to $\hat{x}$ up to isomorphism. By the following proposition, $\tilde{p}$ is an f-ring homomorphism.

Proposition 2.4. [11] If p is a prime element of a frame L, then $\widetilde{p} : C(L) \longrightarrow \mathbb{R}$ is an onto f-ring homomorphism. Also, $\widetilde{p}$ is a linear map with $\widetilde{p}(\mathbf{1}) = 1$.

Let p be a prime element of L. Throughout this paper for every $\alpha \in C(L)$ we define $\alpha[p] = \widetilde{p}(\alpha)$.

Definition 2.5. *A frame L is called weakly spatial, if $a < \top$, then $\Sigma_a \neq \Sigma_\top$.*

Remark 2.6. *A weakly spatial frame need not be spatial, but a weakly spatial regular frame is spatial.*

3. **Zero set**

We define the pointfree version of the zero set of $f \in C(X)$ given by $Z(f) = \{x \in X : f(x) = 0\}$. In poinfree version, we replace a prime element $p \in L$ with the point $x \in X$.

Definition 3.1. *Let $\alpha \in C(L)$. We define*

$$Z(\alpha) = \{p \in \Sigma L : \alpha[p] = 0\}.$$

Such a set is called a zero-set in L. For $A \subseteq C(L)$, we write $Z[A]$ to designate the family of zero-sets $\{Z(\alpha) : \alpha \in A\}$. The family $Z[C(L)]$ of all zero-sets in L will be denoted, for simplicity, by $Z[L]$.

Lemma 3.2. *Let p be a prime element of a frame L. For $\alpha \in C(L)$, $\alpha[p] = 0$ if and only if $coz(\alpha) \leq p$. Hence $Z(\alpha) = \Sigma L - \Sigma_{coz(\alpha)}$.*

Proof. Suppose that $\alpha[p] \neq 0$, assume that $\alpha[p] > 0$. Hence, there is a rational number r such that $\alpha[p] \geq r > 0$, thus, by [11, Lemma 3.1], $r \in L(p, \alpha)$, so by definition of $L(p, \alpha)$, $\alpha(-, r) \leq p$. Now, if $coz(\alpha) \leq p$, we have $\top = \alpha(0, -) \vee \alpha(-, r) \leq coz(\alpha) \vee p \leq p \vee p = p$ it contradicts p is a prime element. Therefore, $coz(\alpha) \not\leq p$. Similarly to prove in the case $\alpha[p] < 0$.

Conversely, suppose that $\alpha[p] = 0$. So, by [11, Lemma 3.1], for every two rationals $r < 0 < s$, we have $r \in L(\alpha, p)$ and $s \in U(\alpha, p)$, hence, $\alpha(-, r) \vee \alpha(s, -) \leq p$ by definition of L and U. Thus, $coz(\alpha) = \bigvee\{\alpha(-, r) \vee \alpha(s, -) : r < 0 < s\} \leq p$, it contradicts the assumption and the proof is complete. □

The above lemma plays an important role in describing the zero sets. The basic properties of zero sets are gathered in the next proposition..

Proposition 3.3. *For every $\alpha, \beta \in C(L)$, we have*

1. *For every $n \in \mathbb{N}$, $Z(\alpha) = Z(|\alpha|) = Z(\alpha^n)$.*
2. *$Z(\alpha) \cap Z(\beta) = Z(|\alpha| + |\beta|) = Z(\alpha^2 + \beta^2)$.*
3. *$Z(\alpha) \cup Z(\beta) = Z(\alpha\beta)$.*
4. *If α is a unit of $C(L)$, then $Z(\alpha) = \emptyset$.*
5. *$Z[L]$ is closed under countable intersection.*

Proof. By using the properties of the cozero map and the fact that $Z(\alpha) = \Sigma L - \Sigma_{coz(\alpha)}$, it is obvious. □

Corollary 3.4. *Let L be a weakly spatial frame and $\alpha \in C(L)$. If $Z(\alpha) = \emptyset$, then α is a unit of $C(L)$.*

Proof. Since by Lemma 3.2, $\Sigma L = \Sigma L - Z(\alpha) = \Sigma_{coz(\alpha)}$ and L is a weakly spatial frame, we conclude that $coz(\alpha) = \top$. that is, α is a unit of $C(L)$. □

Remark 3.5. *Lemma 3.2 yields zero sets are closed. Let X be a topological space and $f \in C(X)$. We consider the frame map $\mathfrak{O}f : \Re \to \mathfrak{O}(X)$ given by $\mathfrak{O}f(p, q) = \{x \in X : p < f(x) < q\}$. Note that $\mathfrak{O}f$ is the element of $C(L)$ corresponding to f, and*

$$coz(\mathfrak{O}f) = \mathfrak{O}f((-, 0) \vee (0, -)) = \{x \in X : f(x) \neq 0\} = X - Z(f).$$

Let X is a sober space. We know that prime elements of $\mathfrak{O}(X)$ are the form $X - \overline{\{x\}}$ for $x \in X$. So we have

$$\begin{aligned} X - \overline{\{x\}} \in Z(\mathfrak{O}f) &\Leftrightarrow coz(\mathfrak{O}f) \leq X - \overline{\{x\}} \\ &\Leftrightarrow (X - Z(f)) \cap \overline{\{x\}} = \emptyset \\ &\Leftrightarrow x \in Z(f). \end{aligned}$$

Hence, $Z(\mathfrak{O}f) = \{X - \overline{\{x\}} : x \in Z(f)\}$ is the relation of $Z(f)$ and $Z(\mathfrak{O}f)$.

In Proposition 3.7, we shall show that every pointfree zero set $Z(\alpha)$, $\alpha \in C(L)$ is equal to a topological zero set $Z(f)$ for some $f : \Sigma L \to \mathbb{R}$. Before proceedings, we need some necessary tools.

There is a homeomorphism $\tau : \Sigma\Re \to \mathbb{R}$ such that $r < \tau(p) < s$ if and only if $(r, s) \not\leq p$ for all prime elements p of $\Re$ and all $r, s \in \mathbb{Q}$ (see of [6, Proposition 1]).

Lemma 3.6. *Every prime (maximal) element of $\Re$ is of the form $p_x = \bigvee\{(-, r) \vee (s, -) : r, s \in \mathbb{Q}, r \leq x \leq s\}$ for some $x \in \mathbb{R}$, and $\tau(p_x) = x$. In particular, for every $r \in \mathbb{Q}$, $p_r = (-, r) \vee (r, -)$ and $\tau((-, r) \vee (r, -)) = r$.*

Proof. Since $\Re$ is a completely regular frame, the prime elements are precisely the maximal elements, and maximal elements are of the form p_x for some $x \in \mathbb{R}$. □

Proposition 3.7. *Let L be a frame and $\alpha \in C(L)$. Then $Z(\alpha) = Z(\tau \circ \Sigma\alpha)$.*

Proof. Let $p \in \Sigma L$. We have $\tau \circ \Sigma\alpha(p) = 0$ if and only if $\tau \circ \alpha_*(p) = 0$, where α_* is a right adjoint of α. But by Lemma 3.6, $\tau \circ \alpha_*(p) = 0$ if and only if $\alpha_*(p) = (-, 0) \vee (0, -)$ if and only if $\alpha((-, 0) \vee (0, -)) \leq p$, because $\alpha\alpha_* \leq id$. So, $p \in Z(\tau \circ \Sigma\alpha)$ if and only if $coz(\alpha) \leq p$ and hence by Lemma 3.2 the proof is complete. □

Now, we determine when the family $Z[L]$ is a base for the closed sets of ΣL.

Proposition 3.8. *For each frame L, the following statements hold:*

(1) *If L is completely regular, then the family $Z[L]$ of all zero sets is a base for the closed sets of ΣL.*
(2) *If L is a spatial frame and the family $Z[L]$ of all zero sets is a base for the closed sets of ΣL, then L is completely regular.*

Proof. (1) Let F be a closed set of ΣL. Then there is a $a \in L$ such that $\Sigma L \setminus F = \Sigma_a$. Since L is completely regular frame, we conclude that there exists $\{\alpha_i\}_{i \in I} \subseteq C(L)$ such that $a = \bigvee_{i \in I} coz(\alpha_i)$. So that by Lemma 3.2, we have $F = \Sigma L \setminus \Sigma_a = \bigcap_{i \in I}(\Sigma L \setminus \Sigma_{coz(\alpha_i)}) = \bigcap_{i \in I} Z(\alpha_i)$ and the proof is complete.

(2) Let $a \in L$ and suppose that $\{\alpha_i\}_{i \in I} \subseteq C(L)$ such that $\Sigma L \setminus \Sigma_a = \bigcap_{i \in I} Z(\alpha_i)$. By Lemma 3.2, we have $\Sigma_a = \Sigma_{\bigvee_{i \in I} coz(\alpha_i)}$. Since L is spatial frame, we conclude that $a = \bigvee_{i \in I} coz(\alpha_i)$. □

Question 3.9. *Is there a frame L which is neither sapatial nor completely regular such that $Z[L]$ is a basis for the closed sets of ΣL?*

4. Ideals and z-filters

Continuing our study of the relations between algebraic properties of $C(L)$ and lattice properties of L, we now examine the special features of the family of zero-sets of an ideal of functions. Such a family becomes to possess properties analogous to those of a filter; this fact will play a central role in the development.

Definition 4.1. *A nonempty subfamily $\mathcal{F}$ of $Z[L]$ is called a z-filter on L provided that*

(1) $\emptyset \notin \mathcal{F}$,
(2) *if* $Z_1, Z_2 \in \mathcal{F}$, *then* $Z_1 \cap Z_2 \in \mathcal{F}$, *and*
(3) *if* $Z \in \mathcal{F}$, $Z' \in Z[L]$, *and* $Z \subseteq Z'$, *then* $Z' \in \mathcal{F}$.

By (3), ΣL belongs to every z-filter.

Let $\mathcal{A}$ be a nonempty family of sets. $\mathcal{A}$ is said to have the finite intersection property provided that the intersection of any finite number of members of $\mathcal{A}$ is nonempty.

Every family $\mathcal{B}$ of zero-sets of a frame L that has the finite intersection property is contained in a z-filter. Also,

$$\mathcal{F} = \{Z \in Z[L] : \text{ there exists a finite subset } \mathcal{A} \text{ of } \mathcal{B} \text{ such that } \bigcap \mathcal{A} \subseteq Z\}$$

is the smallest z-filter containing $\mathcal{B}$.

Proposition 4.2. *For every frame L, the following statements hold:*

(1) *If L is a weakly spatial frame and I is a proper ideal in $C(L)$, then the family $Z[I] = \{Z(\alpha) : \alpha \in I\}$ is a z-filter on L.*
(2) *If $\mathcal{F}$ is a z-filter on L, then the family $Z^{\leftarrow}[\mathcal{F}] = \{\alpha : Z(\alpha) \in \mathcal{F}\}$ is a proper ideal in $C(L)$.*

Proof. (1) Since I is a proper ideal in $C(L)$, we conclude that I contains no unit, and by Proposition 3.4, $\emptyset \notin Z[I]$. If $\alpha, \beta \in I$, then $\alpha^2 + \beta^2 \in I$ and by Proposition 3.3(2), $Z(\alpha) \cap Z(\beta) = Z(\alpha^2 + \beta^2) \in Z[I]$. If $Z \in Z[I]$, $Z' \in Z(L)$ and $Z \subseteq Z'$, then there exists $\alpha \in I$ and $\beta \in C(L)$ such that $Z = Z(\alpha)$ and $Z' = Z(\beta)$. Since $\alpha\beta \in I$, we can conclude from Proposition 3.3(3) that $Z' = Z(\alpha\beta) \in Z[I]$.

(2) Let $J = Z^{\leftarrow}[\mathcal{F}]$. By Proposition 3.3(4), J contains no unit. Let $\alpha, \beta \in J$ and $\gamma \in C(L)$. Since $Z(\alpha - \beta) \supseteq Z(\alpha) \cap Z(\beta) \in \mathcal{F}$, and $Z(\gamma\alpha) \supseteq Z(\alpha) \in \mathcal{F}$, we conclude that $\gamma\alpha, \alpha - \beta \in J$. □

Remark 4.3. *By a z-ultrafilter on L is meant a maximal z-filter, i.e., one not contained in any other z-filter. Thus, for every L, the following statements hold:*

(1) *A z-ultrafilter is a maximal subfamily of $Z[L]$ with the finite intersection property.*
(2) *Every subfamily of $Z[L]$ with the finite intersection property is contained in some z-ultrafilter.*

Proposition 4.4. *Let L be a weakly spatial frame.*

(1) *If M is a maximal ideal in $C(L)$, then $Z[M]$ is a z-ultrafilter on L.*
(2) *If $\mathcal{F}$ is a z-ultrafilter on L, then $Z^{\leftarrow}[\mathcal{F}]$ is a maximal ideal in $C(L)$.*

Proof. (1) By Proposition 4.2, $Z[M]$ is a z-filter on L. Let $\mathcal{F}$ be a z-filter on L and $Z[M] \subseteq \mathcal{F}$. By Proposition 4.2, $Z^{\leftarrow}[\mathcal{F}]$ is a proper ideal in $C(L)$ and $M \subseteq Z^{\leftarrow}[\mathcal{F}]$, it follows that $M = Z^{\leftarrow}[\mathcal{F}]$. Hence $Z[M] = \mathcal{F}$, that is $Z[M]$ is a z-ultrafilter on L.

(2) By Proposition 4.2, $Z^{\leftarrow}[\mathcal{F}]$ is a proper ideal in $C(L)$. Let I be a proper ideal of $C(L)$ such that $Z^{\leftarrow}[\mathcal{F}] \subseteq I$, it follows that $\mathcal{F} \subseteq Z[I]$. Since $\mathcal{F}$ is a z-ultrafilter on L, we can conclude from Proposition 4.2 that $\mathcal{F} = Z[I]$. Hence $Z^{\leftarrow}[\mathcal{F}] = I$, that is $Z^{\leftarrow}[\mathcal{F}]$ is a maximal ideal in $C(L)$. □

Corollary 4.5. *Let L be a weakly spatial frame.*

(1) *Let M be a maximal ideal in $C(L)$. If $Z(\alpha)$ meets every member of $Z[M]$, then $\alpha \in M$.*
(2) *Let $\mathcal{F}$ be a z-ultrafilter on L. If a zero-set Z meets every member of $\mathcal{F}$, then $Z \in \mathcal{F}$.*

Proof. It follows from Remark 4.3 and Proposition 4.4. □

5. Strongly z-ideals and prime ideals

Let A be a ring and $a \in A$. we define $\mathcal{M}(a)$ by $\{M : a \in M$ and M is a maximal ideal of $A\}$. It is known that an ideal I in $C(X)$ is a z-ideal if $Z(f) \in Z[I]$ implies that $f \in I$; equivalently, if $a \in I$ and $\mathcal{M}(a) \subseteq \mathcal{M}(b)$ implies that $b \in I$ (see [12]). But this does not hold for $C(L)$ (see Remark 5.4). Therefore, we define strongly z-ideal in $C(L)$, as follows:

Definition 5.1. *An ideal I in $C(L)$ is called a strongly z-ideal if $Z(\alpha) \in Z[I]$ implies that $\alpha \in I$, that is $I = Z^{\leftarrow}[Z[I]]$.*

Remark 5.2. *The definition of strongly z-ideal implies that the mapping Z, from the set of all strongly z-ideals onto the set of all z-filters, is one-to-one.*

Example 5.3. *If $\emptyset \neq S \subseteq \Sigma L$, then $M_S = \{\alpha \in C(L) : S \subseteq Z(\alpha)\}$ is a strongly z-ideal of $C(L)$.*

Remark 5.4. *Let L be a frame such that $\Sigma L = \emptyset$. Then, for every proper ideal I in $C(L)$, $Z^{\leftarrow}[Z[I]] = C(L)$, i.e., I is not a strongly z-ideal.*

Proposition 5.5. *Let L be a weakly spatial frame. Every maximal ideal in $C(L)$ is a strongly z-ideal.*

Proof. Suppose that M is a maximal ideal in $C(L)$ and $Z(\alpha) \in Z[M]$. If $\alpha \notin M$, then there exists $\beta \in M$ and $\gamma \in C(L)$ such that $\mathbf{1} = \beta + \gamma\alpha$. So that $\emptyset = Z(\mathbf{1}) = Z(\beta + \gamma\alpha) \supseteq Z(\beta) \cap Z(\alpha\gamma) = (Z(\beta) \cap Z(\alpha)) \cup (Z(\beta) \cap Z(\gamma)) \neq \emptyset$, which is a contradiction. □

Note that the intersection of an arbitrary family of strongly z-ideals is a strongly z-ideal.

An ideal I of $C(L)$ is a z-ideal if, for any $\alpha \in C(L)$ and $\beta \in I$, $coz(\alpha) = coz(\beta)$ implies $\alpha \in I$(for more details, see [8–10]). The following proposition shows strongly z-ideals are z-ideals.

Proposition 5.6. *In the ring $C(L)$, every strongly z-ideal is a z-ideal.*

Proof. Let I be a strongly z-ideal. Let $\alpha \in C(L)$ and $coz(\alpha) \in coz(I)$. Suppose that there exists a $\beta \in I$ such that $coz(\alpha) = coz(\beta)$. It follows that

$$Z(\alpha) = \{p \in \Sigma L : coz(\alpha) \leq p\} = \{p \in \Sigma L : coz(\beta) \leq p\} = Z(\beta) \in Z[I].$$

Hence, $\alpha \in I$, by hypothesis. Therefore, I is a z-ideal. □

By Remark 5.4, the converse of the above proposition is not true. For spatial frames, it is true and we shall prove it in the next proposition. But, it seems that the condition of spatiality is very strong for this, because spatial frames are open sets of some topological spaces. Here, we have a question whether the condition that every z-ideal is a strongly z-ideal implies that L is spatial. In spite of a great deal of effort expended, we have not been able to answer this question.

Proposition 5.7. *Let L be a spatial frame. If I is a z-ideal in $C(L)$, then I is a strongly z-ideal in $C(L)$.*

Proof. Let $\alpha \in C(L)$ and $Z(\alpha) \in Z[I]$. Then there exists a $\beta \in I$ such that $Z(\alpha) = Z(\beta)$. Then we have

$$\begin{aligned} Z(\alpha) = Z(\beta) &\Rightarrow \Sigma L - \Sigma_{coz(\alpha)} = \Sigma L - \Sigma_{coz(\beta)} \\ &\Rightarrow \Sigma_{coz(\alpha)} = \Sigma_{coz(\beta)} \\ &\Rightarrow coz(\alpha) = coz(\beta), \qquad \text{by spatiality.} \end{aligned}$$

Hence $\alpha \in I$ and the proof is complete. □

Remark 5.8. *Note that if $\mathcal{F}$ is a z-filter on L, then $ZZ^{\leftarrow}[\mathcal{F}] = \mathcal{F}$ and if I is an ideal in $C(L)$, then $Z^{\leftarrow}Z[I] \supseteq I$. Let L be a spatial frame. I is a strongly z-ideal in $C(L)$ if and only if $Z^{\leftarrow}Z[I] = I$.*

Now, we seek the relation between prime ideal and strongly z-ideal.

Lemma 5.9. *For every $\alpha, \beta \geq \mathbf{0}$ in $C(L)$, we have*

$$coz(\alpha) \leq coz((\beta-\alpha)^{-}) \vee coz(\beta).$$

Proof. Since $\alpha \geq \mathbf{0}$, $coz(\alpha) = \alpha(0,-)$. We have

$$\begin{aligned} coz(\alpha) &= \alpha(0,-) \\ &= (\alpha-\beta+\beta)(0,-) \\ &\leq (\alpha-\beta)(0,-) \vee \beta(0,-) \\ &\leq (\alpha-\beta)^{+}(0,-) \vee \beta(0,-) \quad \text{(see [5, pae 42])} \\ &= (\beta-\alpha)^{-}(0,-) \vee \beta(0,-) \\ &= coz(\beta-\alpha)^{-} \vee coz(\beta). \end{aligned}$$

□

In the following we give a direct proof for a result of T. Dube [8, Lemma 4.3] which has already proved by using of [10, Lemma 3.8].

Proposition 5.10. *Let I be a proper z-ideal in $C(L)$. The following assertions are equivalent:*

(1) *I is a prime ideal.*
(2) *I contains a prime ideal.*
(3) *For every $\alpha, \beta \in C(L)$, if $\alpha\beta = \mathbf{0}$, then either $\alpha \in I$ or $\beta \in I$.*

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3) If I contains a prime ideal P, and $\beta\gamma = \mathbf{0}$, then $\beta\gamma \in P$, whence either β or γ is in P and hence in I.

(3) $\Rightarrow$ (1): Suppose that $\alpha\beta \in I$. Since I is a z-ideal we can assume that $\alpha, \beta \geq \mathbf{0}$. Since $(\beta-\alpha)^{-}(\beta-\alpha)^{+} = \mathbf{0}$, we conclude that $(\beta-\alpha)^{-} \in I$ or $(\beta-\alpha)^{+} \in I$. Assume that $(\beta-\alpha)^{-} \in I$. By Lemma 5.9, $coz(\alpha) \leq (coz(\beta-\alpha)^{-}) \vee coz(\beta)$. So

$$\begin{aligned} coz(\alpha) &\leq (coz(\beta-\alpha)^{-} \vee coz(\alpha)) \wedge (coz(\beta-\alpha)^{-} \vee coz(\beta)) \\ &= coz(\beta-\alpha)^{-} \vee (coz(\alpha) \wedge coz(\beta)) \\ &= coz((\beta-\alpha)^{-} + \alpha\beta). \end{aligned}$$

Since $(\beta-\alpha)^{-} + \alpha\beta \in I$ and I is a z-ideal, $\alpha \in I$.

If $(\beta-\alpha)^{+} \in I$, consider $(\alpha-\beta)^{-} = (\beta-\alpha)^{+} \in I$, and similar relation by Lemma 5.9, $coz(\beta) \leq (coz(\alpha-\beta)^{-}) \vee coz(\alpha)$, and similarly prove that $\beta \in I$. Therefore, I is prime. □

In the next theorem, we analyze prime ideals by zero sets in the case of strongly z-ideals.

Theorem 5.11. *Let I be a proper strongly z-ideal in $C(L)$. The following assertions are equivalent:*

(1) *I is a prime ideal.*

(2) *I contains a prime ideal.*
(3) *For every $\alpha, \beta \in C(L)$, if $\alpha\beta = \mathbf{0}$, then either $\alpha \in I$ or $\beta \in I$.*
(4) *For every $\alpha \in C(L)$, there is a zero set $Z \in Z[I]$ such that for every $p \in Z$, $\alpha[p] \geq 0$ or for every $p \in Z$, $\alpha[p] \leq 0$.*

Proof. By Proposition 5.10, it suffices to show that (3) implies (4) and (4) implies (1).

(3) $\Rightarrow$ (4) Let $\alpha \in C(L)$. Since $\alpha^+\alpha^- = \mathbf{0}$, we conclude that $\alpha^+ \in I$ or $\alpha^- \in I$. Suppose that $\alpha^+ \in I$. Let $Z = Z(\alpha^+)$. Let $p \in Z$, so, $\alpha^+[p] = 0$ hence, $\tilde{p}(\alpha^+) = 0$. Since $\tilde{p}$ is a f-ring homomorphism, $(\tilde{p}(\alpha))^+ = 0$, thus, $\tilde{p}(\alpha) \leq 0$, and hence, $\alpha[p] \leq 0$. Similarly $\alpha[p] \geq 0$ for all $p \in Z = Z(\alpha^-)$ in the case $\alpha^- \in I$.

(4) $\Rightarrow$ (1) Suppose that $\alpha\beta \in I$. Let $\gamma = |\alpha| - |\beta|$. By hypothesis, there is a zero-set $Z \in Z[I]$ such that for every $p \in Z$, $\gamma[p] \geq 0$ or for every $p \in Z$, $\gamma[p] \leq 0$. Suppose that for every $p \in Z$, $\gamma[p] \geq 0$. Now, we show that $Z \cap Z(\alpha) \subseteq Z(\beta)$. For this let $p \in Z \cap Z(\alpha)$. So, $p \in Z$ and $\alpha[p] = 0$. Hence, $\gamma[p] \geq 0$. Since $\tilde{p}$ is a f-ring homorphism, we have

$$\tilde{p}(|\alpha|) - \tilde{p}(|\beta|) = \tilde{p}(|\alpha| - |\beta|) = \gamma[p] \geq 0.$$

So,

$$|\beta[p]| = |\tilde{p}(\beta)| = \tilde{p}(|\beta|) \leq \tilde{p}(|\alpha|) = |\tilde{p}(\alpha)| = |\alpha[p]| = 0.$$

Thus, $\beta[p] = 0$, $p \in Z(\beta)$, therefore $Z \cap Z(\alpha) \subseteq Z(\beta)$. Hence,

$$Z(\beta) \supseteq Z \cap Z(\beta) = Z \cap (Z(\alpha) \cup Z(\beta)) = Z \cap Z(\alpha\beta) \in Z[I].$$

So that $Z(\beta) \in Z[I]$. Since I is a strongly z-ideal, $\beta \in I$. Similarly if for every $p \in Z$, $\gamma[p] \leq 0$, we conclude that $\alpha \in I$. Therefore, I is a prime ideal. □

As a consequence of Proposition 5.10 we have the next corollary. [10, Lemma 3.8.]).

Corollary 5.12. *Let I be a proper ideal in $Coz(L)$ such that for every $\alpha, \beta \in C(L)$, $coz(\alpha) \wedge coz(\beta) = \perp$ implies that $coz(\alpha) \in I$ or $coz(\beta) \in I$. Then the following statements hold:*

(1) *$Coz^{\leftarrow}(I)$ is a prime z-ideal of $C(L)$.*
(2) *I is a prime ideal of $Coz(L)$.*

Proof. (1) Let $\alpha, \beta \in C(L)$ and $\alpha\beta = \mathbf{0}$. Then $coz(\alpha) \wedge coz(\beta) = \perp$ and by hypothesis, $coz(\alpha) \in I$ or $coz(\beta) \in I$, i.e., $\alpha \in Coz^{\leftarrow}(I)$ or $\beta \in Coz^{\leftarrow}(I)$. Since $Coz^{\leftarrow}(I)$ is a z-ideal of $C(L)$, we can conclude from Proposition 5.10 that $Coz^{\leftarrow}(I)$ is a prime z-ideal of $C(L)$.

(2) Let $\alpha, \beta \in C(L)$ and $coz(\alpha\beta) = coz(\alpha) \wedge coz(\beta) \in I$. Then $\alpha\beta \in Coz^{\leftarrow}(I)$ and by statement (1), $\alpha \in Coz^{\leftarrow}(I)$ or $\beta \in Coz^{\leftarrow}(I)$. Hence, $coz(\alpha) \in I$ or $coz(\beta) \in I$, i.e., I is a prime ideal of $Coz(L)$. □

In proof of Proposition 5.13, we use this fact: *Let J, J' be two ideals. If $J \cap J'$ is prime then either $J \subseteq J'$ or $J' \subseteq J$.*

About the following proposition we must say that it was established by Dube in [9], and we prove it by another way.

Proposition 5.13. *Every prime ideal in $C(L)$ is contained in a unique maximal ideal.*

Proof. We know that every ideal is contained in at least one maximal ideal. Let P be a prime ideal. If M and M' are distinct maximal ideals such that $P \subseteq M \cap M'$. Since their intersection is a z-ideal, we can conclude from Proposition 5.10 that $M \cap M'$ is a prime ideal of $C(L)$, which is a contradiction. □

We regard the Stone-Čech compactification βL of L, as the frame of completely regular ideals of L. We denote the right adjoint of the join map $j_L : \beta L \to L$ by r_L and recall that $r_L(a) = \{x \in L | x \prec\prec a\}$. We define $M^I = \{\alpha \in C(L) | r_L(coz(\alpha)) \subseteq I\}$, for all $1_{\beta L} \neq I \in \beta L$. If $M^I = M^J$, then $I = J$ (see [9]). Besides, M^I is a maximal ideal of $C(L)$ iff I is a prime element in βL (see in [9, Lemma 4.15]).

For each $a \in L$ with $a \leq \top$, we define the subset M_a of $C(L)$ by

$$M_a = \{\alpha \in C(L) | coz(\alpha) \leq a\}.$$

By Lemma 3.2, if p is a prime element of L, then

$$M_p = \{\alpha \in C(L) | p \in Z(\alpha)\}.$$

(for more details, see [7])

Proposition 5.14. *Let L be a completely regular frame. If P is a prime ideal of $C(L)$, then $|\bigcap Z[P]| \leq 1$.*

Proof. If $p, q \in \bigcap Z[P]$, then $P \subseteq M_p$ and $P \subseteq M_q$. Since by Proposition 5.13, $M^{r_L(p)} = M_p = M_q = M^{r_L(q)}$, i.e., $r_L(p) = r_L(q)$, we conclude that $p = q$. □

Lemma 5.15. *If I is an ideal of $C(L)$, then $I_z := Z^{\leftarrow} Z[I]$ is a strongly z-ideal of $C(L)$ and it is the smallest strongly z-ideal containing I.*

Proof. It is clear. □

By a prime z-filter of L, we shall mean a z-filter $\mathcal{F}$ with the following property: whenever the union of two zero-sets belongs to $\mathcal{F}$, then at least one of them belongs to $\mathcal{F}$.

Proposition 5.16. *Let L be a weakly spatial frame.*

(1) *If P is a prime ideal in $C(L)$, then P_z is a prime strongly z-ideal containing the prime ideal P.*
(2) *If P is a prime ideal in $C(L)$, then $Z[P]$ is a prime z-filter of L.*

(3) *If $\mathcal{F}$ is a prime z-filter of L, then $Z^{\leftarrow}[\mathcal{F}]$ is a prime strongly z-ideal in $C(L)$.*

Proof. (1) By Propositions 5.11, 4.2 and Lemma 5.15, P_z is prime strongly z-ideal containing the prime ideal P.

(2) Suppose that $\alpha, \beta \in C(L)$ and $Z(\alpha) \cup Z(\beta) \in Z[P]$. This implies that $Z(\alpha\beta) \in Z[P_z]$, therefore $\alpha\beta$ belongs to the z-ideal P_z. Since P_z is prime, we conclude that $Z(\alpha) \in Z[P_z] = Z[P]$ or $Z(\beta) \in Z[P_z] = Z[P]$.

(3) By Proposition 5.2, $P = Z^{\leftarrow}[\mathcal{F}]$ is a strongly z-ideal. Suppose that $\alpha, \beta \in C(L)$ and $\alpha\beta \in P$. Hence by Proposition 3.3(3), $Z(\alpha\beta) = Z(\alpha) \cup Z(\beta) \in Z[P] = \mathcal{F}$. By hypothesis, $Z(\alpha) \in Z[P]$ or $Z(\beta) \in Z[P]$. Then $\alpha \in P$ or $\beta \in P$. □

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References

[1] A. R. Aliabad, F. Azarpanah and M. Paimann, Relative z-ideals and z^0-ideals in the factor rings of $C(X)$, *Bull. Iranian Math. Soc.* **36** (2010), no. 1, 211–226.
[2] F. Azarpanah and A. Taherifar, Relative z-ideals in $C(X)$, *Topology Appl.* **156** (2009), no. 9, 1711–1717
[3] F. Azarpanah, O. A. S. Karamzadeh and A. Rezai Aliabad, On z^0-ideals in $C(X)$, *Fund. Math.* **160** (1999), no. 1, 15–25
[4] B. Banaschewski, Pointfree topology and the spectra of f-rings, Ordered Algebraic Structures, (Curaçao, 1995), 123–148, Kluwer Acad. Publ., Dordrecht, 1997.
[5] B. Banaschewski, The real numbers in pointfree topology, Textos de Mathemática (Series B), 12, University of Coimbra, Coimbra, 1997.
[6] B. Banaschewski and C. R. A. Gilmour, Pseudocompactness and the cozero part of a frame, *Comment. Math. Univ. Carolin.* **37**(1996), no. 3, 577–587.
[7] T. Dube, Extending and contracting maximal ideals in the function rings of pointfree topology, *Bull. Math. Soc. Sci. Math. Roumanie* **55(103)** (2012), no. 4, 365–374.
[8] T. Dube, Some algebraic characterizations of F-frames, *Algebra Universalis.* **62** (2009), no. 2-3, 273–288.
[9] T. Dube, Some ring-theoretic properties of almost P-frames, *Algebra Universalis* **60** (2009), no. 2, 145–162.
[10] T. Dube and J. Walters-Wayland, Coz-onto frame maps and some applications, *Appl. Categ. Structures* **15** (2007), no. 1-2, 119–133.
[11] M. M. Ebrahimi and A. Karimi Feizabadi, Prime representation of real Riesz maps, *Algebra Universalis* **54** (2005), no. 3, 291–299.
[12] L. Gillman and M. Jerison, Rings of continuous functions, Springer-Verlag, New York-Heidelberg, 1976.
[13] J. Gutiérrez García and J. Picado, How to deal with the ring of (continuous) real-valued functions in terms of scales, Proceedings of the Workshop in Applied Topology WiAT'10, pp. 19–30.
[14] G. Mason, z-ideals and prime ideals, *J. Algebra.* **26** (1973) 280–297.

[15] J. Picado and A. Pultr, Frames and Locales, Topology without Points, Frontiers in Mathematics, Birkhäuser-Springer, Basel, 2012.

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