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SOME CONCAVITY PROPERTIES FOR GENERAL INTEGRAL OPERATORS

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ABSTRACT. Let $C_0(\alpha)$ denote the class of concave univalent functions defined in the open unit disk \mathbb{D} . Each function $f \in C_0(\alpha)$ maps the unit disk \mathbb{D} onto the complement of an unbounded convex set. In this paper, we study the mapping properties of this class under integral operators.

Keywords: Unit disk, univalent function, concave function, integral operator.

MSC(2010): Primary: 30C45.

1. Introduction

Let \mathcal{A} denote the class of functions $f(z)$ of the form

$$(1.1) \quad f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\},$$

and satisfy the following usual normalization condition

$$f(0) = f'(0) - 1 = 0.$$

Further, by \mathcal{S} we shall denote the class of all functions in \mathcal{A} which are univalent in \mathbb{D} . Also, let φ be the class of all functions which are analytic in \mathbb{D} and satisfy $p(0) = 1$, $Re p(z) > 0$.

In 1915, Alexander [1] introduced the first integral operator defined by

$$A(z) = \int_0^z \frac{f(t)}{t} dt,$$

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while the Libera operator [24] is defined by

$$L(z) = \frac{2}{z} \int_0^z f(t) dt,$$

and the Bernardi operator [5], which generalizes the Libera operator, is defined for $\gamma = 1, 2, 3, \dots$,

$$L_\gamma(z) = \frac{1+\gamma}{z^\gamma} \int_0^z f(t) t^{\gamma-1} dt.$$

Many authors have studied special integral operators by employing different methods (see [12, 20, 25, 27]). All of these special integral operators are the particular cases of the other general operator of the form

$$(1.2) \quad I[f](z) = \left[\frac{\beta + \gamma}{z^\gamma \phi(z)} \int_0^z f^\alpha(t) t^{\delta-1} \varphi(t) dt \right]^{\frac{1}{\beta}},$$

where $\varphi, \phi \in \mathbb{D}$, $\gamma, \beta, \delta, \alpha$ are complex numbers. Note that, the operator I given in (1.2) was defined by Miller, Mocanu and Reade [26] in 1978. In 2002, Breaz and Breaz [8] defined the following operator:

$$(1.3) \quad F_n[f](z) = \int_0^z \left(\frac{f_1(t)}{t} \right)^{\gamma_1} \dots \left(\frac{f_n(t)}{t} \right)^{\gamma_n} dt,$$

for $f_i(z) \in \mathcal{A}$ and $\gamma_i > 0$, for all $i \in 1, 2, 3, \dots, n$.

By using the concept of convolution, many authors had generalized the operator by Breaz and Breaz (1.3) in several directions (see [2, 9, 11, 28]). Frasin [21] (see also, [7]) introduced and studied a generalized operator $I(f_i, g_i)$ as follows

$$(1.4) \quad I(f_i, g_i)(z) = \int_0^z \left(\frac{(f_i * g_i)(t)}{t} \right)^{\alpha_i} dt, \quad i = 1, \dots, m.$$

The operator $I(f_i, g_i)$ reduces to many well-known integral operators by varying the parameters α_i and by choosing suitable functions instead of $g_i(z)$. For example if we take $m = 1, g_1(z) = \frac{z}{1-z}, \alpha_1 = \alpha \in [0, 1]$ in (1.4), we obtain the integral operator studied in [26] given as

$$(1.5) \quad \int_0^z \left(\frac{f(t)}{t} \right)^\alpha dt.$$

Other types of integral operators are also studied by researchers in this area. For example, integral operators of the types $J(z), H(z), F(z), G(z), F_{\alpha\beta, n}$ and $G_{\alpha, n}(z)$ which were also proved by Breaz et al., [10]. In addition, Ibrahim and Darus [19] defined and studied integral operator as follows:

$$J_{\lambda, \delta}^{k, \alpha, \beta}$$

$$= \left\{ \underbrace{\phi(z) * \dots * \phi(z)}_{k\text{-times}} * \underbrace{\frac{z^\beta}{(1-z)^2} * \dots * \frac{z^\beta}{(1-z)^2}}_{\alpha k\text{-times}} * \frac{z}{(1-z)^{(\delta+1)}} \right\}^{-1} * f(z)$$

$$= z + \sum_{n=2}^{\infty} \frac{a_n}{[(\beta_n)^\alpha + (n-1)(\beta_n)^\alpha \lambda]^k C(\delta, n)} z^n,$$

where $k, \alpha \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\beta \geq 1, \lambda \geq 0, \delta \geq 0, z \in \mathbb{D}$.

Many results were obtained by the authors of [14]- [31] related to this integral operator.

In 2013, Frasin [23] obtained the general integral operator $B_\beta(z)$ defined by

$$(1.6) \quad B_\beta(z) = \left[\beta \int_0^z t^{\beta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t} \right)^{\gamma_i} \rho_i^{\zeta_i}(t) dt \right]^{\frac{1}{\beta}},$$

where $f_i \in \mathcal{A}, \rho_i \in \wp, \beta \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $\gamma_i, \zeta_i \in \mathbb{C}$ for all $i = 1, 2, \dots, n$.

Note that the integral operator B_β generalized special operators introduced and studied by several authors including [8] and [22].

The aim of this work is to investigate some mapping properties of the integral operators (1.5) and

$$(1.7) \quad F(z) = \frac{2}{z^{\lambda+1}} \int_0^z t^\lambda f(t) dt, \lambda > 0,$$

for the class of concave univalent functions.

Remark 1.1. Note that when $\lambda = 0$ in (1.7) we get Libera operator.

We need the following definition in the sequel.

Definition 1.2. A function $f : \mathbb{D} \rightarrow \mathbb{C}$ is said to belong to the family $C_0(\alpha)$ if f satisfies the following conditions:

(i) f is analytic in \mathbb{D} with the standard normalization $f(0) = f'(0) - 1 = 0$. In addition it satisfies $f(1) = \infty$.

(ii) f maps \mathbb{D} conformally onto a set whose complement with respect to \mathbb{C} is convex.

(iii) The opening angle of $f(\mathbb{D})$ at ∞ is less than or equal to $\pi\alpha, \alpha \in (1, 2]$.

The class $C_0(\alpha)$ is referred to as concave univalent functions and for a detailed discussion about concave functions, we may refer to [3, 4, 13] and the references therein.

We recall the analytic characterization for the functions in $C_0(\alpha)$, $\alpha \in (1, 2]$: $f \in C_0(\alpha)$ if and only if $\Re(P_{f(z)}(z)) > 0$, $z \in \mathbb{D}$, where

$$P_f(z) = \frac{2}{\alpha - 1} \left[\frac{(\alpha + 1)}{2} \frac{1 + z}{1 - z} - 1 - z \frac{f''(z)}{f'(z)} \right].$$

In [6], the authors used this characterization for their work.

2. Main results

One of our main results is contained in the following theorem:

Theorem 2.1. *If $\alpha \in (1, 2]$ and $f \in \mathcal{A}$ satisfy the inequality*

$$(2.1) \quad \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1, \quad z \in \mathbb{D},$$

then for all $\lambda \in \mathbb{R}$

$$(2.2) \quad F(z) = \int_0^z \left(\frac{f(t)}{t} \right)^\lambda dt \in C_0(\alpha).$$

Proof. Our aim is to prove that for any complex number w we have (see [10])

$$(2.3) \quad |w - 1| < 1 \Leftrightarrow \Re \frac{1}{w} > \frac{1}{2}.$$

If we let

$$\frac{1}{w} = \frac{2}{\alpha - 1} \left[\frac{\alpha + 1}{2} \frac{1 + z}{1 - z} - 1 - z \frac{F''(z)}{F'(z)} \right],$$

then

$$(2.4) \quad |w - 1| = \left| \frac{2z((1 - z)F''(z) - (1 + \alpha)F'(z))}{(\alpha + \alpha z + 3z - 1)F'(z) - 2z(1 - z)F''(z)} \right|.$$

From (2.2), we obtain

$$F'(z) = \left(\frac{f(z)}{z} \right)^\lambda,$$

and

$$F''(z) = \lambda \left(\frac{f(z)}{z} \right)^{\lambda-1} \left(\frac{zf'(z) - f(z)}{z^2} \right).$$

We substitute $F'(z)$ and $F''(z)$ in (2.4), and make a simple computation to get

$$\begin{aligned} & \left| \frac{2\lambda(1-z) \left(\frac{zf'(z)}{f(z)} - 1 \right) - (1+\alpha)}{(\alpha + \alpha z + 3z - 1) - 2\lambda(1-z) \left(\frac{zf'(z)}{f(z)} - 1 \right)} \right| \\ & \leq \frac{2|\lambda||1-z| \left| \frac{zf'(z)}{f(z)} - 1 \right| + (1+\alpha)}{|\alpha + \alpha z + 3z - 1| - 2|\lambda||1-z| \left| \frac{zf'(z)}{f(z)} - 1 \right|} \\ & \leq \frac{2|\lambda||1-z| + (1+\alpha)}{|\alpha + \alpha z + 3z - 1| - 2|\lambda||1-z|}, \end{aligned}$$

where

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1.$$

Taking $z \rightarrow 1^+$ (by the maximum principle theorem), yields

$$\left| \frac{2\lambda(1-z) \left(\frac{zf'(z)}{f(z)} - 1 \right) - (1+\alpha)}{(\alpha + \alpha z + 3z - 1) - 2\lambda(1-z) \left(\frac{zf'(z)}{f(z)} - 1 \right)} \right| \leq \frac{1}{2} < 1.$$

In view of conclusion (2.3), we have $F(z) \in C_0(\alpha)$. This completes the proof. \square

When $\lambda = 1$, we obtain the next result.

Corollary 2.2. *If $\lambda = 1$ in Theorem 2.1, then the Alexander operator is a concave univalent function.*

Theorem 2.3. *Let $f \in \mathcal{A}$ and $\alpha \in (1, 2]$. If $f \in C_0(\alpha)$, then the integral operator*

$$(2.5) \quad F(z) = \frac{2}{z^{\lambda+1}} \int_0^z t^\lambda f(t) dt, \quad \lambda > 0,$$

satisfies the inequality

$$\Re \left\{ \frac{(z^2 F''(z))'}{F'(z)} \right\} \leq \frac{(\alpha - 1)(\lambda + 2)}{2},$$

whenever

$$(2.6) \quad \Re \left\{ \frac{zF''(z)}{F'(z)} \right\} > 0.$$

Proof. Since

$$F(z) = \frac{2}{z^{\lambda+1}} \int_0^z t^\lambda f(t) dt,$$

we have

$$\frac{z^{\lambda+1}}{2} F(z) = \int_0^z t^\lambda f(t) dt.$$

By taking the derivatives, we obtain

$$\frac{\lambda + 1}{2} F(z) + \frac{z}{2} F'(z) = f(z),$$

which implies that

$$(2.7) \quad \frac{\lambda + 2}{2} F'(z) + \frac{z}{2} F''(z) = f'(z).$$

We deduce

$$(2.8) \quad \frac{\lambda + 3}{2} F''(z) + \frac{z}{2} F'''(z) = f''(z).$$

Since

$$\Re \left(\frac{2}{\alpha - 1} \left(\frac{\alpha + 1}{2} \frac{1 + z}{1 - z} - 1 - z \frac{f''(z)}{f'(z)} \right) \right) > 0,$$

this is equivalent to

$$\Re \left(\frac{\alpha + 1}{2} \frac{1 + z}{1 - z} - 1 - z \frac{f''(z)}{f'(z)} \right) > 0.$$

Since

$$\Re \left(\frac{\alpha + 1}{2} \frac{1 + z}{1 - z} - 1 \right) \leq \left| \frac{\alpha + 1}{2} \frac{1 + z}{1 - z} - 1 \right|,$$

by letting $z \rightarrow 0$, we conclude that

$$\Re \left(\frac{\alpha + 1}{2} \frac{1 + z}{1 - z} - 1 \right) \leq \frac{\alpha - 1}{2}.$$

Hence we obtain

$$0 < \Re \left(\frac{\alpha + 1}{2} \frac{1 + z}{1 - z} - 1 - z \frac{f''(z)}{f'(z)} \right) \leq \Re \left(\frac{\alpha - 1}{2} - z \frac{f''(z)}{f'(z)} \right),$$

which yields the inequality

$$(2.9) \quad \Re \left(\frac{z f''(z)}{f'(z)} \right) \leq \frac{\alpha - 1}{2}.$$

Substituting (2.7) and (2.8) in (2.9), we have

$$\Re \{ (7 - \alpha + 2\lambda) F''(z) + 2z F'''(z) \} \leq (\alpha - 1)(\lambda + 2) \Re \left\{ \frac{F'(z)}{z} \right\}.$$

Consequently, we have

$$\Re \left\{ \frac{(7 - \alpha + 2\lambda) z F''(z)}{2 F'(z)} + \frac{z^2 F'''(z)}{F'(z)} \right\} \leq \frac{(\alpha - 1)(\lambda + 2)}{2}.$$

Since for $\alpha \in (1, 2]$, $\lambda > 0$ and $3 - \alpha + 2\lambda > 0$, we have

$$\frac{7 - \alpha + 2\lambda}{2} > 2,$$

then, by the assumption (2.6), we conclude that

$$\begin{aligned} \Re \left\{ 2 \frac{z F''(z)}{F'(z)} + \frac{z^2 F'''(z)}{F'(z)} \right\} &\leq \Re \left\{ \frac{(7 - \alpha + 2\lambda) z F''(z)}{2 F'(z)} + \frac{z^2 F'''(z)}{F'(z)} \right\} \\ &\leq \frac{(\alpha - 1)(\lambda + 2)}{2}. \end{aligned}$$

This completes the proof. \square

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