ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 41 (2015), No. 5, pp. 1093–1102

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 41 (2015), No. 5, pp. 1093–1102
Online ISSN: 1735-8515

ON DIMENSIONS OF DERIVED ALGEBRA AND CENTRAL FACTOR OF A LIE ALGEBRA

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(Communicated by Saeid Azam)

Abstract. Some Lie algebra analogues of Schur's theorem and its converse are presented. As a result, it is shown that for a capable Lie algebra L we always have $\dim L/Z(L) \leq 2(\dim(L^2))^2$. ¡We also give¿ some examples supporting our results.
Keywords: Capable Lie algebra, minimal generator, derived algebra, central factor.
MSC(2010): Primary: 17B99; Secondary: 16W25.

1. Introduction

Relations between the center and derived subgroups of a group G back to 1904 when Schur proved that if G is a group such that the order of $G/Z(G)$ is finite, then the order of G' is finite too. But the converse of Schur's theorem is not true in general. There are various papers concerning Schur's theorem and its converse¡; for example see [1, 3, 7] and references therein.¿

Let L be a Lie algebra over a fixed field Λ and ¡let¿ $Z(L)$, $Z_2(L)$, L^2 and $\Phi(L)$ be the center, the second center, the derivative algebra and the Frattini subalgebra of L, respectively. Similar to Schur's theorem, Maneyhun [2] in 1993 proved that if $\dim L/Z(L) = n$, then $\dim(L^2) \leq \frac{1}{2}n(n-1)$. The natural question that arises is whether the finiteness of dimension of L^2 implies the finiteness of dimension of $L/Z(L)$? As for groups, the infinite dimensional Heisenberg algebra gives a counterexample to this question. ¡The aim of this paper is to give some partial positive answer to the aforementioned question in the realm of Lie algebras.¿

Definition 1.1. Let L be a Lie algebra. The linear map $D : L \to L$ is said to be a *derivation* if

$$D([x, y]) = [D(x), y] + [x, D(y)]$$

Article electronically published on October 15, 2015.
Received: 5 November 2013, Accepted: 8 July 2014.
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for all $x, y \in L$. The set of all derivations of L is denoted by $\mathrm{Der}(L)$.

It is well-known that $\mathrm{Der}(L)$ is a Lie algebra through the following bracket

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1,$$

for all $D_1, D_2 \in \mathrm{Der}(L)$.

The map $\mathrm{ad}_x : L \to L$ giving by $y \mapsto [x, y]$ is a derivation called *adjoint map* for all $x \in L$. The set of all adjoint derivations of L is denoted by $\mathrm{IDer}(L)$¡. It is easy to verify that $\mathrm{IDer}(L)$ is an ideal of $\mathrm{Der}(L)$.¿

Definition 1.2. A derivation of a Lie algebra L is called an *ID-derivation* if its image is contained in the derived algebra. The set of all ID-derivations is denoted by $\mathrm{ID}(L)$. ¡T¿he set of all ID-derivations which map central elements to 0 is denoted by $\mathrm{ID}^*(L)$. We always have

$$\mathrm{IDer}(L) \leq \mathrm{ID}^*(L) \leq \mathrm{ID}(L) \leq \mathrm{Der}(L).$$

The aim of this paper is to prove the foll¡o¿iwng results.

Theorem A. *Let L be a Lie algebra such that L^2 is finite dimensional and $L/Z(L)$ is generated by d elements. Then*

$$\dim \mathrm{ID}^*(L) \leq d \cdot \dim L^2.$$

Theorem B. *Let L be a Lie algebra such that L^2 is finite dimensional. Then $L/Z(L)$ is finite dimensional if and only if $Z_2(L)/Z(Z_2(L))$ is finite dimensional, where $Z_2(L)$ is the second term in the upper central series of L.*

Theorem C. *Let L be a Lie algebra that $\dim L^2/L^2 \cap Z(L) = n$. Then $\dim L/Z_2(L) \leq 2n^2$.*

2. Proof of Theorem A

Proof of Theorem A. Let $\{x_1 + Z(L), \ldots, x_d + Z(L)\}$ be a minimal generating set of $L/Z(L)$. ¡It is easy to see that¿

$$\begin{array}{rcl} \psi : \mathrm{ID}^*(L) & \longrightarrow & L^2 \oplus \cdots \oplus L^2 \\ \alpha & \longmapsto & (\alpha(x_1), \ldots, \alpha(x_d)). \end{array}$$

ψ is an injective linear map. Hence the result follows. □

Example 2.1. Let $L = \mathcal{G}en x_1, \ldots, x_m, y_{ij} : [x_i, x_j] = y_{ij}, 1 \leq i < j \leq m$ be a nilpotent Lie algebra of class 2 and dimension $\frac{1}{2}m(m+1)$. Then $Z(L) = L^2 = \mathcal{G}en y_{ij} : 1 \leq i < j \leq m$, $\dim(L^2) = \frac{1}{2}m(m-1)$ and $\dim(L/Z(L)) = d(L/Z(L)) = m$. Since a typical element of $\mathrm{ID}^*(L)$ has the following matrix form

$$\begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \alpha_{1,m+1} & \cdots & \alpha_{m,m+1} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{1,\frac{m(m+1)}{2}} & \cdots & \alpha_{m,\frac{m(m+1)}{2}} & 0 & \cdots & 0 \end{bmatrix}, \qquad \alpha_{i,j} \in \Lambda$$

it follows that $\dim \mathrm{ID}^*(L) = \frac{1}{2}m^2(m-1)$.

Example 2.2. Let H_n be a Lie algebra defined by $H_n = \mathcal{G}enx_1, \ldots, x_n : [x_1, x_i] = x_{i+1},\ 2 \leq i \leq n-1$. Then $L^2 = \mathcal{G}enx_3, \ldots, x_n$, $\dim(L^2) = n-2$, $H_n/Z(H_n) \cong H_{n-1}$ and $d(H_n/Z(H_n)) = 2$. A simple verification shows that a typical element of $\mathrm{ID}^*(L)$ has the following matrix form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \alpha_{1,3} & \alpha_{2,3} & 0 & 0 & \cdots & 0 \\ \alpha_{1,4} & \alpha_{2,4} & \alpha_{2,3} & 0 & \cdots & 0 \\ \vdots & \vdots & \alpha_{2,4} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{1,n} & \alpha_{2,n} & \alpha_{2,n-1} & 0 & \cdots & 0 \end{bmatrix}.$$

Hence, $\dim \mathrm{ID}^*(L) = 2(n-2)$.

The above examples indicate that the upper bound given in Theorem A is sharp.

Example 2.3. Let $L = H(k) = \mathcal{G}en\ x_1, \ldots, x_{2k+1} : [x_{2i-1}, x_{2i}] = x_{2k+1}$, $1 \leq i \leq k$ be Heisenberg Lie algebra of dimension $2k+1$. Then $L^2 = Z(L) = \mathcal{G}enx_{2k+1}$ and $d(L/Z(L)) = \dim(L/Z(L)) = 2k$. An element of $\mathrm{ID}^*(L)$ in the matrix ¡form¿ is represented by

$$\begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ \alpha_{1,2k+1} & \cdots & \alpha_{2k,2k+1} & 0 \end{bmatrix},$$

which implies that $\dim \mathrm{ID}^*(L) = 2k$. Since $H(k) \oplus A^{n-2k-1}$, where $\dim A = 1$ satisfies the same property, all nilpotent Lie algebras L with $\dim L^2 = 1$ have the mentioned properties.

Theorem A has the following interesting consequences.

Corollary 2.4. *Let L be a Lie algebra. Then $\mathrm{IDer}(L)$ is finite dimensional if and only if $\mathrm{ID}^*(L)$ is finite dimensional.*

Proof. Since $L/Z(L) \cong \mathrm{IDer}(L)$, the finiteness of dimension of $\mathrm{IDer}(L)$ implies the finiteness of dimension of L^2 and hence, by Theorem A, the finiteness of dimension of $\mathrm{ID}^*(L)$. On the other hand, $\mathrm{IDer}(L) \leq \mathrm{ID}^*(L)$, which implies the converse is also true. □

Corollary 2.5. *Invoking the hypothesis of Theorem A, we have*

$$\dim \frac{L}{Z(L)} \leq d \cdot \dim(L^2).$$

Let A, B be two Lie algebra¡s¿ over a field Λ and $T(A, B)$ be the set of all linear transformations from A to B. Then $T(A, B)$ equipped with ¡the¿ Lie bracket $[f, g](x) = [f(x), g(x)]$ for all $x \in A$ and $f, g \in T(A, B)$ is a Lie algebra.

Let L be a Lie algebra and

$$C^* = \{\alpha \in \mathrm{Der}(L) : \alpha(x) \in Z(L)\ \forall x \in L \text{ and } \alpha(x) = 0\ \forall x \in Z(L)\}.$$

For any $\alpha \in C^*$, the map $\psi_\alpha : L/(L^2 + Z(L)) \to Z(L)$ defined by $\psi_\alpha(x + L^2 + Z(L)) = \alpha(x)$, for all $x \in L$ is a linear transformation. It is easy to see that the map $\psi : C^* \to T(\frac{L}{L^2+Z(L)}, Z(L))$ defined by $\psi(\alpha) = \psi_\alpha$ is a Lie isomorphism. Therefore $C^* \cong T(L/(L^2 + Z(L)), Z(L))$.

The descending central series of a Lie algebra L is defined by $\{L^i\}$, where $L^1 = L$ and $L^i = [L^{i-1}, L]$ for all $i > 1$. A Lie algebra L is *nilpotent* if there exists a non-negative integer k such that $L^k = 0$. The smallest integer k for which $L^{k+1} = 0$ is called the *nilindex* of L. A Lie algebra L with $\dim(L) = n$ is called *filiform* (or 1-filiform) if it satisfies $\dim(L^i) = n - i$ for all $2 \leq i \leq n$. These algebras have the maximal nilindex $n - 1$. The Lie algebras with a nilindex $n - 2$ are called *quasifiliform* (or 2-filiform) and those with nilindex 1 are called *abelian*.

Corollary 2.6. *Let L be an n-dimensional Lie algebra which attain the upper bound in the previous corollary. Then*

(1) *if L is filiform, then $n = 3$, and*
(2) *if L is quasifiliform, then $n = 4$ or 5.*

Proof. (1) We have $\dim Z(L) = 1$ and $\dim(L^2) = n - 2$. Therefore $d(L) = 2$ and by Corollary 2.5, $n - 1 = 2(n - 2)$, which implies that $n = 3$.

(2) Clearly $\mathrm{ID}^*(L) = \mathrm{IDer}(L)$. Also, since L is a quasifiliform, we have $1 \leq \dim(Z(L)) \leq 2$ and $n - 3 \leq \dim(L^2) \leq n - 2$. If $Z(L) \leq L^2$, then $C^* \leq \mathrm{ID}^*(L) = \mathrm{IDer}(L)$, hence $C^* = Z(\mathrm{IDer}(L))$. On the other hand, $d = d(L)\mathcal{G}eq2$. Thus $Z(\mathrm{IDer}(L))$ is not 1-dimensional. First suppose that $\dim(Z(L)) = 1$. Then $\dim(Z_2(L)) = 3$ and $\dim(C^*) = \dim(Z(\mathrm{IDer}(L))) = 2$. So $d(L) = d(L/Z(L)) = 2$. If $\dim(L^2) = n - 2$, then since $\dim(L/Z(L)) = d \cdot \dim(L^2)$, it follows that $n = 3$, which is a contradiction. If $\dim(L^2) = n - 3$, then by Corollary 2.5, $n - 1 = 2(n - 3)$, which implies that $n = 5$. Second suppose that

$\dim(Z(L)) = 2$. Then $\dim(Z_2(L)) = 3$ and hence $Z(\mathrm{IDer}(L))$ is 1-dimensional, which is a contradiction.

Now, assume that $Z(L) \nleq L^2$. Then $\dim(Z(L)) = 2$. If $\dim(L^2) = n-2$, then $L^2 = \Phi(L)$ and hence $d(L/Z(L)) = 1$, which contradicts the assumption that L is non-abelian. If $\dim(L^2) = n-3$, then $d(L/Z(L)) = d(L/(L^2 + Z(L))) = 2$ and by Corollary 2.5, it follows that $n = 4$. □

3. Proof of Theorems B and C

In what follows we state some lemmas that will be used in the proof of theorems.

Lemma 3.1. *Let H be a subalgebra of a Lie algebra L generated by $x_1, x_2, \cdots, x_d$ and $Z(L)$. If $\dim \mathrm{Imad}_{x_i} = m_i$ for $1 \leq i \leq d$, then $\dim L/C_L(H) \leq \sum_{i=1}^{d} m_i$, where $C_L(H)$ is the centralizer of H in L.*

Proof. The same as in the proof of Theorem A, we just need to consider the following mapping.

$$\begin{array}{rcl} \psi : \frac{L}{C_L(H)} & \longrightarrow & \mathrm{Imad}_{x_1} \oplus \cdots \oplus \mathrm{Imad}_{x_d} \\ x + C_L(H) & \longmapsto & ([x_1, x], \ldots, [x_d, x]). \end{array}$$

Clearly ψ is an injective linear map, from which the result follows. □

Corollary 3.2. *Let L be a Lie algebra such that $L/Z(L)$ is generated by $x_1 + Z(L), \ldots, x_d + Z(L)$. If $\dim \mathrm{Imad}_{x_i} = m_i$ for $1 \leq i \leq d$, then $\dim L/Z(L) \leq \sum_{i=1}^{d} m_i$. Moreover, L^2 is finite dimensional and $\dim L/Z(L) \leq d \cdot \dim L^2$.*

Proof. Put $H = L$ in Lemma 3.1. □

The Heisenberg algebra $H(k)$ of dimension $2k+1$ is an example in which the equality holds in the above corollary. In the sequel, we state some lemma which will be used in the proof of Theorem C.

Lemma 3.3. *Let L be a Lie algebra and $H < L$ be a proper subalgebra of L. Then $L^2 = [L \setminus H, L]$.*

Proof. It is enough to show $[H, L] \subseteq [L \setminus H, L]$. Suppose y is an arbitrary element of $L \setminus H$. Then $[x, t] = [x + y, t] - [y, t]$ for all $x \in H$ and $t \in L$, which implies that $[x, t] \in [L \setminus H, L]$. □

Lemma 3.4. *Let $Z = L \cap Z(L)$ and U, V be subalgebras of L such that $Z \leq U$ and $V \leq L^2$. Then there exist elements x, y of L satisfying the following properties:*

1. *if $Z < U$, then $U \cap C_L(x) < U$, and*
2. *if $V < L^2$, then $V < \mathcal{G}en V, [x, y]$.*

Proof. Let $C = C_L(U)$. If $Z < U$ then $C < L$. Thus $U \cap C_L(x) < U$ for all $x \in L \setminus C$. By Lemma 3.3, $L^2 = [L \setminus C, L]$. If $V < L^2$ we can choose $x \in L \setminus C$ and $y \in L$ such that $V < \mathcal{G}enV, [x, y]$. If $Z = U$ and $V < L^2$ we can find x, y such that $[x, y] \notin V$. Thus $V < \mathcal{G}enV, [x, y]$. □

Lemma 3.5. *Let* $Z = L^2 \cap Z(L)$ *and suppose* $\dim L^2/Z = n$. *Let* T *be a subalgebra of* L *with* $L^2 \leq T \leq L$ *having the following properties:*

(1) $L^2 = T^2 + Z$,
(2) $L^2 \cap Z(T) = Z$, *and*
(3) T/Z *can be generated by* k *elements.*

Then there exists $M \leq L$ *such that* $[M, L, L] = 0$ *and* $\dim L/M \leq nk$.

Proof. Let $M/Z = C_{L/Z}(T/Z)$. By lemma 3.1, $\dim L/M \leq nk$. Now $[T, M, L] = 0$. In particular, $[T, M, T] = 0$. However, according to the Jacobi identity we can see $[T, T, M] = 0$ and using Jacobi identity once more we have $[M, L, T] = 0$. Thus $[M, L] \leq Z(T) \cap L^2 = Z$. Therefore $[M, L, L] = 0$. □

Lemma 3.6. *Let* L *be a finite dimensional Lie algebra and* $\dim L^2/Z = n$, *in which* $Z = L^2 \cap Z(L)$. *Then there exists a subalgebra* T *of* L *satisfying the conditions of Lemma 3.5 such that* $k \leq 2n$.

Proof. By lemma 3.5, we have elements x_{i+1}, y_{i+1} $(0 \leq i \leq l - 1)$ such that $V_i = \mathcal{G}enZ, [x_1, y_1], \ldots, [x_i, y_i]$ and $U_i = C_{L^2}(V_i)$. Now, we have

$$Z = V_0 \leq V_1 \leq V_2 \leq \cdots \leq V_l = L^2$$

and

$$L^2 = U_0 \mathcal{G}eq U_1 \mathcal{G}eq U_2 \mathcal{G}eq \cdots \mathcal{G}eq U_l = Z$$

where l is the smallest integer such that $V_l = L^2$ and $U_l = Z$.
Let $T = \mathcal{G}enZ, x_1, y_1, \ldots, x_l, y_l$. We can see that $n \mathcal{G}eq l$. Hence $k \leq 2n$ and it can be easily verified that the subalgebra T of L satisfies the conditions of Lemma 3.5. □

Now, we are in a position to prove Theorem C.

Proof of Theorem C. By Lemma 3.6, there exists a subalgebra T of L such that $k \leq 2n$ and T satisfies the conditions of Lemma 3.5. Thus there exists a subalgebra M of L such that $[M, L, L] = 0$ and $\dim L/M \leq nk$. Hence $[M, L] \subseteq Z(L)$ and $M/Z(L) \leq Z_2(L)/Z(L)$. Therefore $\dim L/Z_2(L) \leq \dim L/M \leq nk \leq 2n^2$. □

Example 3.7. Let $L = \mathcal{G}en\ x_1, \ldots, x_{2n} : [x_1, x_2] = x_3, [x_1, x_i] = x_{i+n-1}$, $3 \leq i \leq n + 1$. Clearly, $L^2 = \mathcal{G}enx_2, x_{n+2}, \ldots, x_{2n}$, $Z(L) = \mathcal{G}enx_{n+2}, \ldots, x_{2n}$ and $Z_2(L) = \mathcal{G}enx_3, x_4, \ldots, x_{2n}$. Hence $\dim(L/Z_2(L)) = 2$ and $\dim(L^2/L^2 \cap Z(L)) = 1$, which imply that the upper bound introduced in Theorem C is sharp.

The following corollaries are direct consequences of Theorem C.

Corollary 3.8. *Let L be a Lie algebra such that L^2 is finite dimensional. Then $L/Z_2(L)$ is also finite dimensional.*

A Lie algebra H is said to be *capable* if there exists a Lie algebra L such that $H \cong L/Z(L)$.

Corollary 3.9. *Let H be a capable Lie algebra such that $\dim H^2 = n$. Then $\dim H/Z(H) \leq 2n^2$.*

A group theoretical analogue of the above corollary is prove by Podoski and Szegedy in [5].

Another consequence of Corollaries 3.2 and 3.7 is

Corollary 3.10. *Let L be a Lie algebra such that L^2 is finite dimensional. Then $L/Z(L)$ is finite dimensional if and only if $Z_2(L)/Z(L)$ is finitely generated.*

Theorem B gives a better result.

Lemma 3.11. *Let L be a Lie algebra with an abelian ideal A such that $L/C_L(A)$ is finite dimensional and L/A is generated by elements $x_1+A, \ldots, x_d+A$, where $\dim \mathrm{Imad}_{x_i} < \infty$ for $1 \leq i \leq d$. Then $L/Z(L)$ is finite dimensional.*

Proof. Let $X = \{x_1, \cdots, x_d\}$ and Y be a generating set for A. Then $L = \mathcal{G}enX, Y$ and $Z(L) = C_L(X) \cap C_L(Y)$. Since $L/C_L(A)$ is finite dimensional, $L/C_L(Y)$ is finite dimensional too. On the other hand, since Imad_{x_i} is finite dimensional for $1 \leq i \leq d$, by Corollary 3.2, $L/C_L(X)$ has finite dimension. Therefore $L/Z(L)$ is finite dimensional, as required. □

Proof of Theorem B. Suppose that $Z_2(L)/Z(Z_2(L))$ is finitely generated. If $L/Z(L)$ has infinite dimension, then by Corollary 3.7, $Z_2(L)/Z(L)$ has infinite dimension. On the other hand, since $L/Z_2(L)$ is finite dimensional, $L/Z(Z_2(L))$ is finitely generated. Now, since $Z_2(L) \leq C_L(Z(Z_2(L)))$, by Lemma 3.10, $L/Z(L)$ is finite dimensional, which is a contradiction. Therefore $L/Z(L)$ is finite dimensional. The converse is obvious. □

Lemma 3.12. *Let L be a Lie algebra and A be an abelian subalgebra of L such that $\dim L/A = m$ and $dim L^2 = n$. Then*

$$\dim \frac{L}{Z(L)} \leq m(n+1).$$

Proof. We can chose a subspace X of L such that $L = \mathcal{G}enA, X$ and $\dim X = r \leq m$. Similarly, as in the proof of Lemma 3.1, we can show that $\dim L/C_L(X) \leq rn$. Since A is abelian, we can see that $A \cap C_L(X) \subseteq Z(L)$. Thus $\dim L/Z(L) \leq m + mn = m(n+1)$. □

By Theorem B and Lemma 3.12, we can prove a partial of converse Schur theorem in Lie algebra.

Corollary 3.13. *If L is a Lie algebra such that $\dim L^2 = n$ and $Z_2(L)$ is abelian, then*

$$\dim \frac{L}{Z(L)} \leq 2(n^3 + n^2).$$

4. Lie algebras with the property $\mathcal{G}$

Let $\mathcal{G}$ be the family of all Lie algebras L whose derived subalgebras are finite dimensional and $2(\dim L^2)^2$ is greater that or equal to the dimension of their central factor. As it is illustrated in Corollary 3.8, capable Lie algebras belong to this family. hence, a natural question to ask is whether there no-capable Lie algebras belonging to this family? For example, An abelian Lie algebra of dimension 1 and Heisenberg Lie algebras $H(k)$ of dimension $2k + 1$ for $k > 1$ are non-capable Lie algebras which belong to this family (see [4] for details).

In this section we shall determine some families of non-capable Lie algebras which belong to this family. The first example is given as in the following theorem.

Theorem 4.1. *Let L be a Lie algebra with trivial Frattini subalgebra. If L^2 is finite dimensional, then L belongs to $\mathcal{G}$.*

Proof. Since the Frattini subalgebras of L is trivial, $L^2 \cap Z(L) = 0$. Hence $Z_2(L) = Z(L)$ and by Theorem C, the result follows. □

The above theorem gives us non-capable Lie algebras which belong to $\mathcal{G}$. Indeed, the abelian Lie algebra of dimension 1 is a non-capable Lie algebra satisfying the conditions of the above theorem.

To introduce the other family, we need to give a definition. The following important equivalence relation was defined by salemkar [6].

Definition 4.2. Let L and H be two Lie algebras. Then L and H are called *n-isoclinic* and denoted by $L \underset{n}{\sim} H$ if there exists a pair of isomorphisms $\alpha : L/Z_n(L) \to H/Z_n(H)$ and $\beta : L^{n+1} \to H^{n+1}$ such that the following diagram is commutative.

$$\begin{array}{ccc} \frac{L}{Z_n(L)} \oplus \cdots \oplus \frac{L}{Z_n(L)} & \longrightarrow & L^{n+1} \\ \Big\downarrow \alpha^{n+1} & & \Big\downarrow \beta \\ \frac{H}{Z_n(H)} \oplus \cdots \oplus \frac{H}{Z_n(H)} & \longrightarrow & H^{n+1}. \end{array}$$

in which the horizontal maps are defined by $(\overline{x}_1, \ldots, \overline{x}_{n+1}) \mapsto [x_1, \ldots, x_{n+1}]$. If $n = 1$, then L and H are called *isoclinic* and denoted by $L \sim H$.

Clearly, if L and H are isoclinic Lie algebras and $H \in \mathcal{G}$, then so is L. Indeed, if L and H are Lie algebras whose central factors are isomorphic and $\dim H^2 \leq \dim L^2$, then the condition $H \in \mathcal{G}$ implies that $L \in \mathcal{G}$ too.

Now, we enjoy to know under which conditions two Lie algebras with isomorphic central factors one of them belonging to $\mathcal{G}$ implies that the other is also belongs to $\mathcal{G}$.

Proposition 4.3. *Let L be a finite dimensional Lie algebra and S be a subalgebra of L such that the central factors of L and S are isomorphic. If L belongs to the family $\mathcal{G}$, then so is S.*

Proof. It is not difficult to show that $L = S + Z(L)$. Hence $L^2 = S^2$ and the result follows. □

Definition 4.4. Let L be a Lie algebra. Then L is said to be an *n-stem* Lie algebra if $Z(L) \subseteq L^{n+1}$.

Salemkar [6] showed that each n-isoclinism class of Lie algebras contains at least a n-stem Lie algebra.

Theorem 4.5. *[6] Let H be a Lie algebra with $\dim H^n$ finite. Then $Z(H) \cap H^n$ is a subalgebra of H^{n+1} if and only if for each Lie algebra L n-isoclinic to H, $\dim H^n \leq \dim L^n$.*

Utilizing the above theorem, we have

Theorem 4.6. *Let L and H be Lie algebras such that $L \underset{n}{\sim} H$ and $L/Z(L) \cong H/Z(H)$. If H is n-stem and $H \in \mathcal{G}$, then L belongs to $\mathcal{G}$ too.*

Proof. Since the central factors of L and H are isomorphic, we have $L^n/L^n \cap Z(L) \cong H^n/H^n \cap Z(H)$. On the other hand, H is n-stem, and by the previous theorem, $\dim H^n \leq \dim L^n$. Hence $\dim H^n \cap Z(H) \leq \dim L^n \cap Z(L)$ so that $\dim Z(H) \leq \dim L^2 \cap Z(L)$. Therefore, $\dim H^2 \leq \dim L^2$ and the result follows. □

The following example illustrates how the above theorem can be used to find non-capable Lie algebras belonging to $\mathcal{G}$.

Example 4.7. Let $L_n = \mathcal{G}en x_1, \ldots, x_n : [x_i, x_j] = x_{i+j},\ 1 \leq i < j \leq n-i$ and $H_n = \mathcal{G}en x_1, \ldots, x_n : [x_1, x_i] = x_{i+1},\ 2 \leq i \leq n-1$. Then L_n and H_n are capable nilpotent Lie algebras of dimension n for $L_n/Z(L_n) \cong L_{n-1}$ and $H_n/Z(H_n) \cong H_{n-1}$. On the other hand, $L_5/Z(L_5) \cong H_5/Z(H_5)$, L_5 is 3-stem and $L_5 \underset{3}{\sim} H_5$. Now, assume that $L = L_5$ and $H = H_5 \oplus A$. Then L and H satisfy the conditions of the previous theorem while H is not capable.

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