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FABER POLYNOMIAL COEFFICIENT ESTIMATES FOR BI-UNIVALENT FUNCTIONS DEFINED BY SUBORDINATIONS

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Abstract. A function is said to be bi-univalent on the open unit disk $\mathbb{D}$ if both the function and its inverse are univalent in $\mathbb{D}$. Not much is known about the behavior of the classes of bi-univalent functions let alone about their coefficients. In this paper we use the Faber polynomial expansions to find coefficient estimates for four well-known classes of bi-univalent functions which are defined by subordinations. Both the coefficient bounds and the techniques presented are new and we hope that this paper will inspire future researchers in applying our approach to other related problems.
Keywords: Faber polynomials, bi-univalent, subordinations.
MSC(2010): Primary 30C45; Secondary 30C50.

1. Introduction

Let $\mathcal{A}$ denote the class of functions f which are analytic on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let $\mathcal{S}$ denote the class of functions $f \in \mathcal{A}$ that are univalent in $\mathbb{D}$ and $\mathcal{P}$ be the class of functions $\varphi(z) = 1 + \sum_{n=1}^{\infty} \varphi_n z^n$ that are analytic in $\mathbb{D}$ and satisfy the condition $Re(\varphi(z)) > 0$ in $\mathbb{D}$. By the Caratheodory Lemma (see [13]) we have $|\varphi_n| \leq 2$. A functions $f \in \mathcal{S}$ is said to be starlike in $\mathbb{D}$ if $zf'(z)/f(z) \in \mathcal{P}$ and is said to be convex in $\mathbb{D}$ if $1 + zf''(z)/f'(z) \in \mathcal{P}$ (see [13]).

If $g = f^{-1}$ is the inverse of a function $f \in S$, then g has a Maclaurin series expansion in some disk about the origin [13]. In 1923, Lowner [25] proved that

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the inverse of the Koebe function $f(z) = z/(1-z)^2$ provides the best upper bounds for the coefficients of the inverses of the functions $f \in S$. Sharp bounds for the coefficients of the inverses of univalent functions have been obtained in a surprisingly straightforward way, whereas the case for the subclasses of univalent functions turned out to be a challenge. In 1979, Krzyz, and et. al. [18] obtained sharp upper bounds for the first two coefficients of inverses of classes of starlike functions. In 1982, Libera and Zlotkiewicz [21] found the bounds for the first seven coefficients of the inverse of convex functions. Later in [22] they obtained the bounds for the first six coefficients of the inverse of f provided $f'(z) \in \mathcal{P}$. In [23] they considered the odd functions $f(z) = z + a_3z^3 + a_5z^5 + \cdots$ and showed that if $f'(z) \in \mathcal{P}$ then $[-z + \log((1+z)/(1-z))]^{-1}$ is the extremal function for the inverse of f. In 1986, Juneja and Rajasekaran [17] obtained coefficient estimates for inverses of α-spiral functions. In 1989, Silverman [34] proved that if $f \in S$ is so that $\sum_{n=2}^{\infty} n|a_n| \leq 1$ then the $n-th$ coefficient of the inverse of f is bounded above by $\frac{1}{n}\binom{2n-3}{n-2}\frac{1}{2^{n-2}}$. In 1992, Libera and Zlotkiewicz [24] proved that the $n-th$ coefficients of the inverse of starlike functions are bounded above by $[(2n)!/n!(n+1)!]$. Chou [12] in 1994, proved that if $f \in S$ and $f'(z) \in \mathcal{P}$ then $-z + 2\log(1+z)$ is the extremal function for the inverse of f. Estimates for the first two coefficients of the inverses of subclasses of starlike functions were also obtained in [11] and [36].

Finding coefficient estimates for the inverses of univalent function becomes even more involved when the bi-univalency condition is imposed on these functions. A function $f \in S$ is said to be *bi-univalent* in $\mathbb{D}$ if its inverse map $g = f^{-1}$ is also univalent in $\mathbb{D}$. The class of bi-univalent analytic functions was first introduced and studied by Lewin [20] where it was proved that $|a_2| < 1.51$. Brannan and Clunie [9] improved Lewin's result to $|a_2| \leq \sqrt{2}$ and later Netanyahu [32] proved that $|a_2| \leq 4/3$. Brannan and Taha [10] and Taha [37] also investigated certain subclasses of bi-univalent functions and found estimates for their initial coefficients. Recently, Srivastava, et. al. [35], Frasin and Aouf [14], and Ali, et. al. [7] found estimates for the first two coefficients of certain subclasses of bi-univalent functions. The bi-univalency requirement makes the behavior of the coefficients of the function f and its inverse $g = f^{-1}$ unpredictable. Not much is known about the higher coefficients of bi-univalent functions as Ali, Lee, Ravichandaran and Supramaniam [7] also remarked that finding the bounds for the $n-th$, $(n \geq 4)$ coefficients of classes of bi-univalent functions is an open problem. In this paper, we use the Faber polynomial expansions to find upper bounds for the $n-th$, $(n \geq 3)$ coefficients of four well known classes of analytic functions, namely $\mathcal{R}_b(\varphi)$, $\mathcal{S}^*(\alpha;\varphi)$, $\mathcal{L}(\alpha;\varphi)$ and $\mathcal{M}(\alpha;\varphi)$. An examination of the unexpected behavior of the first two coefficients of each of these four classes are also presented. The use of Faber polynomials for the coefficients of bi-univalent functions and the techniques presented in this paper are new in their kinds. We hope that they inspire future applications of our methods to other related problems.

For $f(z)$ and $F(z)$ analytic in $\mathbb{D}$, we say that $f(z)$ is subordinate to $F(z)$, written $f \prec F$, if there exists a Schwarz function $u(z) = \sum_{n=1}^{\infty} c_n z^n$ with $|u(z)| < 1$ in $\mathbb{D}$, such that $f(z) = F(u(z))$. For the Schwarz function $u(z)$ we note that $|c_n| \leq 1$. (see [13]).

For $z \in \mathbb{D}$, $b \in \mathbb{C}\backslash\{0\}$, $0 \leq \alpha \leq 1$ and for the functions $f \in \mathcal{A}$ and $\varphi \in \mathcal{P}$ we consider the following well-known classes of functions

$$\begin{aligned}
\mathcal{R}_b(\varphi) &:= \left\{1 + \frac{1}{b}(f'(z) - 1) \prec \varphi(z)\right\},\\
\mathcal{S}^*(\alpha;\varphi) &:= \left\{\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)} \prec \varphi(z)\right\},\\
\mathcal{L}(\alpha;\varphi) &:= \left\{\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} \prec \varphi(z)\right\},\\
\mathcal{M}(\alpha;\varphi) &:= \left\{(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z)\right\}.
\end{aligned}$$

The class $\mathcal{R}_1(\varphi)$ consists of bounded turning functions first defined by MacGregor [27]. The functions in the class $\mathcal{S}^*(\alpha;\varphi)$ are starlike (e.g. see Ramesha, Kumar and Padmanabhan [33]). The class $\mathcal{L}(\alpha;\varphi)$ consists of logarithmic $\alpha - convex$ functions (see [19]). The class $\mathcal{M}(\alpha;\varphi)$ consists of $\alpha - convex$ functions first defined by Mocanu [31] (also see Miller [28] and Miller, et. al. [29] and [30]). A unified treatment of various classes of functions consisting of convex and starlike functions for which either or both of the expressions $1+zf''(z)/f'(z)$ and $zf'(z)/f(z)$ are subordinate to certain superordinate functions can also be found in Jahangiri, et. al [15, 16], Ma and Minda [26], and Ali, Lee, Ravichandaran and Supramaniam [6].

2. **Main results**

Prior to stating and proving our theorems, we shall need the following preliminaries. Using the Faber Polynomial expansion for functions $f \in \mathcal{A}$ of the form (1.1), the coefficients of its inverse map $g = f^{-1}$ may be expressed as, (see [3] and [4]),

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \cdots) w^n, \tag{2.1}$$

where

$$
\begin{aligned}
K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!(n-1)!}a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!}a_2^{n-3}a_3 \\
&+ \frac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4 \\
&+ \frac{(-n)!}{(2(-n+2))!(n-5)!}a_2^{n-5}\left[a_5 + (-n+2)a_3^2\right] \\
&+ \frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}\left[a_6 + (-2n+5)a_3a_4\right] + \sum_{j\geq 7} a_2^{n-j}V_j,
\end{aligned}
$$

such that V_j with $7 \leq j \leq n$ is a homogeneous polynomial in the variables $a_3, a_4, \cdots, a_n$. In particular, the first three terms of K_{n-1}^{-n} are $K_1^{-2} = -2a_2$, $K_2^{-3} = +3(2a_2^2 - a_3)$, and $K_3^{-4} = -4(5a_2^3 - 5a_2a_3 + a_4)$.

In general, (see [1,3,38]) for any $p \in \mathbb{R}$, an expansion of K_n^p is given by

$$
K_n^p = pa_n + \frac{p(p-1)}{2}D_n^2 + \frac{p!}{(p-3)!3!}D_n^3 + \cdots + \frac{p!}{(p-n))!n!}D_n^n,
$$

where for $m \leq n$,

$$
D_{n-1}^m\ (a_2, \cdots, a_n) = \sum_{n=2}^{\infty} \frac{m!(a_2)^{\mu_1}\cdots(a_n)^{\mu_{n-1}}}{\mu_1!\cdots\mu_{n-1}!},
$$

and the sum is taken over all nonnegative integers $\mu_1, \cdots, \mu_{n-1}$ satisfying

$$
\begin{cases}
\mu_1 + \mu_2 + \cdots + \mu_{n-1} = m, \\
\mu_1 + 2\mu_2 + \cdots + (n-1)\mu_{n-1} = n-1.
\end{cases}
$$

Evidently: $D_{n-1}^{n-1}(a_2, \cdots, a_n) = a_2^{n-1}$, [2].

We shall also need the following lemma which can be found in [3, Theorem 3.2].

Lemma 2.1. *Consider the Faber polynomial*

$$
(b_1, b_2, \cdots, b_n) \longrightarrow F_n(b_1, b_2, \cdots, b_n).
$$

Therefore

$$
\begin{aligned}
& F_n\left(-F_1(b_1), -F_2(b_1, b_2), \cdots, -F_n(b_1, b_2, \cdots, b_n)\right) \\
= \ & F_n(2b_1, 3b_2, \cdots, (n+1)b_{n+1}) - F_n(b_1, b_2, \cdots, b_n).
\end{aligned}
$$

Our first theorem provides an estimate for the n-th coefficients of the functions in $\mathcal{R}_b(\varphi)$ subject to a gap series condition.

Theorem 2.2. *Let $f \in \mathcal{R}_b(\varphi)$ and $g = f^{-1} \in \mathcal{R}_b(\varphi)$. If $a_k = 0$ for $2 \leq k \leq n-1$ then*

$$
|a_n| \leq \frac{2|b|}{n}; \quad n \geq 3.
$$

Proof. Let f be as given in (1.1). Therefore

$$f'(z)-1=\sum_{n=2}^{\infty} na_n z^{n-1} \tag{2.2}$$

and if by (2.1) we assume that

$$g(w)=f^{-1}(w)=w+\sum_{n=2}^{\infty}\frac{1}{n}K_{n-1}^{-n}(a_2,a_3,\cdots)w^n=w+\sum_{n=2}^{\infty}b_n w^n, \tag{2.3}$$

then

$$g'(w)-1=\sum_{n=2}^{\infty}K_{n-1}^{-n}(a_2,a_3,\cdots)w^{n-1}=\sum_{n=2}^{\infty}nb_n w^{n-1}. \tag{2.4}$$

On the other hand, for $f\in\mathcal{R}_b(\varphi)$ and $\varphi\in\mathcal{P}$ there are two Schwarz functions $u(z)=c_1z+c_2z^2+\cdots$ and $v(w)=d_1w+d_2w^2+\cdots$ such that

$$1+\frac{1}{b}(f'(z)-1)=\varphi(u(z)) \tag{2.5}$$

and

$$1+\frac{1}{b}(g'(w)-1)=\varphi(v(w)) \tag{2.6}$$

where

$$\varphi(u(z))=1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}\varphi_k D_n^k(c_1,c_2,\cdots,c_n)z^n \tag{2.7}$$

and

$$\varphi(v(w))=1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}\varphi_k D_n^k(d_1,d_2,\cdots,d_n)w^n. \tag{2.8}$$

Therefore, from (2.2), (2.5) and (2.7) we may write

$$\frac{1}{b}na_n=\sum_{k=1}^{n-1}\varphi_k D_{n-1}^k(c_1,c_2,\cdots,c_{n-1}), \qquad n\geq 2. \tag{2.9}$$

Similarly, from (2.4),(2.6) and (2.8) we may write

$$\frac{1}{b}nb_n=\sum_{k=1}^{n-1}\varphi_k D_{n-1}^k(d_1,d_2,\cdots,d_{n-1}), \qquad n\geq 2. \tag{2.10}$$

Now, (2.9) and (2.10) for $a_k=0$ $(2\leq k\leq n-1)$, respectively, yield

$$\frac{1}{b}na_n=\varphi_1 c_{n-1}$$

and

$$\frac{1}{b}nb_n=-\frac{1}{b}na_n=\varphi_1 d_{n-1},$$

since by definition of K_n^p we have $b_n = -a_n$.

Upon simplification, we obtain

$$a_n = \frac{b}{n}(\varphi_1 c_{n-1}) \tag{2.11}$$

and

$$a_n = -\frac{b}{n}(\varphi_1 d_{n-1}). \tag{2.12}$$

Taking the absolute values of (2.11) or (2.12) and using the facts that $|\varphi_1| \leq 2$, $|c_{n-1}| \leq 1$ and $|d_{n-1}| \leq 1$ we obtain

$$|a_n| \leq \frac{2|b|}{n}.$$

□

Our next theorem clearly demonstrates the unpredictable behavior of the early coefficients of classes of bi-univalent functions.

Theorem 2.3. *Let $f \in \mathcal{R}_b(\varphi)$ and $g = f^{-1} \in \mathcal{R}_b(\varphi)$. Then*

(i). $|a_2| \leq \begin{cases} |b|, & |b| < \frac{4}{3}; \\ \sqrt[2]{\frac{4|b|}{3}}, & |b| \geq \frac{4}{3}. \end{cases}$

(ii). $|a_3| \leq \begin{cases} \frac{2|b|}{3} + |b|^2, & |b| < \frac{2}{3}; \\ \frac{4|b|}{3}, & |b| \geq \frac{2}{3}. \end{cases}$

(iii). $|a_3 - \mu a_2^2| \leq \dfrac{2\mu|b|}{3}, \qquad \mu = 1, 2.$

Proof. Letting $n = 2$ and $n = 3$ in (2.9) and (2.10), respectively, imply

$$\frac{1}{b}2a_2 = \varphi_1 c_1, \tag{2.13}$$

$$\frac{1}{b}3a_3 = \varphi_1 c_2 + \varphi_2 c_1^2, \tag{2.14}$$

and

$$\frac{1}{b}2b_2 = \varphi_1 d_1, \tag{2.15}$$

$$\frac{1}{b}3b_3 = \varphi_1 d_2 + \varphi_2 d_1^2. \tag{2.16}$$

Comparing the corresponding coefficients of (2.3) with the relations (2.15) and (2.16) we deduce

$$-\frac{1}{b}2a_2 = \varphi_1 d_1 \tag{2.17}$$

and

$$\frac{1}{b}(6a_2^2 - 3a_3) = \varphi_1 d_2 + \varphi_2 d_1^2. \tag{2.18}$$

Obviously $c_1 = -d_1$. Therefore, from either of (2.13) or (2.17) we obtain

$$|a_2| = \frac{|b||\varphi_1 c_1|}{2} \leq |b|.$$

Adding (2.14) to (2.18) we obtain

$$6\frac{1}{b}a_2^2 = \varphi_1(c_2 + d_2) + \varphi_2(c_1^2 + d_1^2)$$

and therefore

$$|a_2| = \sqrt[2]{\frac{|b||\varphi_1(c_2 + d_2) + \varphi_2(c_1^2 + d_1^2)|}{6}} \leq \sqrt[2]{\frac{4|b|}{3}}.$$

Now the bounds for $|a_2|$ are justified since $|b| < \sqrt[2]{\frac{4|b|}{3}}$ for $|b| < \frac{4}{3}$.

From (2.14) we obtain

$$|a_3| = \frac{|b||\varphi_1 c_2 + \varphi_2 c_1^2|}{3} \leq \frac{4|b|}{3}. \tag{2.19}$$

On the other hand subtracting (2.18) from (2.14) implies

$$\frac{6}{b}(a_3 - a_2^2) = \varphi_1(c_2 - d_2) + \varphi_2(c_1^2 - d_1^2) = \varphi_1(c_2 - d_2). \tag{2.20}$$

Solving the above equation for a_3 and taking the absolute values yield

$$|a_3| = \frac{|b|}{6}|\varphi_1(c_2 - d_2)| + |a_2|^2 \leq \frac{2|b|}{3} + |a_2|^2.$$

Applying the estimate $|a_2| \leq |b|$ we obtain

$$|a_3| \leq \frac{2|b|}{3} + |b|^2. \tag{2.21}$$

Now, Theorem 3.2 (ii) follows from (2.19) and (2.21) upon noting that

$$\frac{2|b|}{3} + |b|^2 < \frac{4|b|}{3} \text{ if } |b| < \frac{2}{3}.$$

For the third part of the theorem, we rewrite (2.18) as

$$3(a_3 - 2a_2^2) = -b(\varphi_1 d_2 + \varphi_2 d_1^2).$$

Dividing by 3 and taking the absolute values we get

$$|a_3 - 2a_2^2| \leq \frac{|b||\varphi_1 d_2 + \varphi_2 d_1^2|}{3} \leq \frac{4|b|}{3}.$$

Finally, solving the equation (2.20) for $(a_3 - a_2^2)$ and taking the absolute values we obtain

$$|a_3 - a_2^2| = \frac{|b||\varphi_1(c_2 - d_2)|}{3} \leq \frac{2|b|}{3}.$$

□

Theorem 2.4. *Let $f \in \mathcal{S}^*(\alpha;\varphi)$ and $g = f^{-1} \in \mathcal{S}^*(\alpha;\varphi)$. If $a_k = 0$ for $2 \leq k \leq n-1$ then*

$$|a_n| \leq \frac{2}{(n-1)(\alpha n+1)}; \quad n \geq 3.$$

Proof. Consider the function f given by (1.1). Then for $f \in \mathcal{S}^*(\alpha;\varphi)$ we can write (see [4, equation 1.6])

$$\frac{zf'(z)}{f(z)} = 1 - \sum_{n=2}^{\infty} F_{n-1}(a_2, a_3, \cdots, a_n) z^{n-1}, \tag{2.22}$$

where the first few coefficients of $F_{n-1}(a_2, a_3, \cdots, a_n)$ are as follow:

$$\begin{aligned}
F_1 &= -a_2,\\
F_2 &= a_2^2 - 2a_3,\\
F_3 &= -a_2^3 + 3a_2a_3 - 3a_4,\\
F_4 &= a_2^4 - 4a_2^2a_3 + 4a_2a_4 + 2a_3^2 - 4a_5,\\
F_5 &= -a_2^5 + 5a_2^3a_3 + 5a_2^2a_4 - 5(a_3^2 - a_5)a_2 + 5a_3a_4 - 5a_6,\\
F_6 &= a_2^6 - 6a_2^4a_3 + 6a_2^3a_4 - 6(2a_3a_4 - a_6)a_2 - 2a_3^3 + 9a_2^2a_2^2 + 6a_3a_5\\
&\quad +3a_4^2 - 3a_2^2a_5 - 6a_7.
\end{aligned}$$

In general, [8, Remark 1.1],

$$F_{n-1}(a_2, a_3, \cdots, a_n) = \sum_{i_1+2i_2+\cdots+(n-1)i_{n-1}=n-1} A(i_1, i_2, \cdots, i_{n-1})(a_2^{i_1} a_3^{i_2} \cdots a_n^{i_{n-1}})$$

where

$$A(i_1, i_2, \cdots, i_{n-1}) := (-1)^{(n-1)+2i_1+\cdots+ni_{n-1}} \frac{(i_1 + i_2 + \cdots + i_{n-1} - 1)!(n-1)}{(i_1)!(i_2)!\cdots(i_{n-1})!}.$$

For $f \in \mathcal{S}^*(\alpha;\varphi)$ we also have, by [5, Proposition 3.1 (b)],

$$\frac{z^2 f''(z)}{f(z)} = \tag{2.23}$$

$$\sum_{n}^{\infty} \sum_{m} (-1)^{\sum_{i=1}^{n-1} m_i + 1} \left(\sum_{i=1}^{n-1} m_i - 1\right)! \left(n - 1 + \sum_{i=1}^{n-1} i^2 m_i\right) \frac{(a_2)^{m_1} \cdots (a_n)^{m_{n-1}}}{m_1! m_2! \cdots m_{n-1}!} z^{n-1},$$

where

$$\sum_{m} = \sum_{m_1+2m_2+\cdots+(n-1)m_{n-1}=n-1}, \quad \text{and} \quad n = 2, 3, \cdots.$$

Similarly, for $g=f^{-1}\in\mathcal{S}^*(\alpha;\varphi)$ we have

$$\frac{wg'(w)}{g(w)}=1-\sum_{n=2}^{\infty}F_{n-1}(b_2,b_3,\cdots,b_n)w^{n-1} \tag{2.24}$$

and,

$$\frac{w^2g''(w)}{g(w)}= \tag{2.25}$$

$$\sum_{n}^{\infty}\sum_{m}(-1)^{\sum_{i=1}^{n-1}m_i+1}\left(\sum_{i=1}^{n-1}m_i-1\right)!\left(n-1+\sum_{i=1}^{n-1}i^2m_i\right)\frac{(b_2)^{m_1}\cdots(b_n)^{m_{n-1}}}{m_1!m_2!\cdots m_{n-1}!}w^{n-1}.$$

Now, by definition of subordinations, there exist two Schwarz functions $u(z)=\sum_{n=1}^{\infty}c_nz^n$ and $v(w)=\sum_{n=1}^{\infty}d_nw^n$ so that

$$\frac{zf'(z)}{f(z)}+\alpha\frac{z^2f''(z)}{f(z)}=\varphi(u(z)) \tag{2.26}$$

and

$$\frac{wg'(w)}{g(w)}+\alpha\frac{w^2g''(w)}{g(w)}=\varphi(v(w)), \tag{2.27}$$

where the Faber polynomial expansions of $\varphi(u(z))$ and $\varphi(v(w))$ are given by (2.7) and (2.8).
Under the stated coefficient hypothesis in the theorem, the equations (2.7), (2.22) and (2.23), respectively, yield

$$\varphi(u(z))=1+\varphi_1c_{n-1},$$

$$\frac{zf'(z)}{f(z)}=1+(n-1)a_n,$$

and

$$\frac{z^2f''(z)}{f(z)}=n(n-1)a_n.$$

Therefore, in light of (2.26) we conclude

$$(n-1)(\alpha n+1)a_n=\varphi_1c_{n-1}. \tag{2.28}$$

Similarly, (2.8), (2.24) and (2.25) reduce to

$$\varphi(v(w))=1+\varphi_1d_{n-1},$$

$$\frac{wg'(w)}{g(w)}=1+(n-1)b_n,$$

and

$$\frac{w^2g''(z)}{g(z)}=n(n-1)b_n.$$

These in conjunction with (2.27) imply

$$-(n-1)(\alpha n+1)a_n=\varphi_1d_{n-1}, \tag{2.29}$$

where the relations between a_n and b_n are determined by the equation (2.3). Taking absolute values of both sides of either (2.28) or (2.29) imply

$$(n-1)(\alpha n+1)|a_n| = |\varphi_1||c_{n-1}|.$$

Solving for $|a_n|$ followed by an application of Caratheodory lemma we obtain

$$|a_n| \leq \frac{2}{(n-1)(\alpha n+1)}.$$

□

In the following theorem we present the estimates for a_2 and the coefficient body $(a_3 - a_2^2)$.

Theorem 2.5. *Let $f \in \mathcal{S}^*(\alpha;\varphi)$ and $g = f^{-1} \in \mathcal{S}^*(\alpha;\varphi)$. Then*

(i). $|a_2| \leq \dfrac{2}{1+2\alpha}$,

(ii). $|a_3 - a_2^2| \leq \dfrac{1}{1+3\alpha}$.

Proof. For $n = 2$ and $n = 3$, the equations (2.26) and (2.27), respectively, yield

$$(1+2\alpha)a_2 = \varphi_1 c_1, \tag{2.30}$$

$$-(1+2\alpha)a_2 = \varphi_1 d_1; \tag{2.31}$$

and

$$-(1+2\alpha)a_2^2 + 2(1+3\alpha)a_3 = \varphi_1 c_2 + \varphi_2 c_1^2, \tag{2.32}$$

$$(3+10\alpha)a_2^2 - 2(1+3\alpha)a_3 = \varphi_1 d_2 + \varphi_2 d_1^2. \tag{2.33}$$

Obviously, $c_1 = -d_1$. Taking the absolute values of either (2.30) or (2.31) we obtain

$$|a_2| \leq \frac{2}{1+2\alpha}.$$

Subtracting (2.33) from (2.32) gives

$$4(1+3\alpha)(a_3 - a_2^2) = \varphi_1(c_2 - d_2) + \varphi_2(c_1^2 - d_1^2) = \varphi_1(c_2 - d_2).$$

Dividing by $4(1+3\alpha)$ and taking the absolute values we get

$$|a_3 - a_2^2| \leq \frac{|\varphi_1|(|c_2| + |d_2|)}{4(1+3\alpha)} \leq \frac{1}{1+3\alpha}.$$

□

Theorem 2.6. *Let $f \in \mathcal{L}(\alpha;\varphi)$ and $g = f^{-1} \in \mathcal{L}(\alpha;\varphi)$. If $a_k = 0$ for $2 \leq k \leq n-1$ then*

$$|a_n| \leq \frac{2}{(n-1)[n(1-\alpha)+\alpha]}; \quad n \geq 3.$$

Proof. Suppose that $f \in \mathcal{L}(\alpha, \varphi)$ where f is given by the series expansion (1.1). The terms zf'/f and $1 + zf''/f'$ can be expressed by the Faber polynomial expansions (see [3, page 199])

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha} = 1 + \sum_{n=2}^{\infty} K_{n-1}^{\alpha}\left(-F_1(a_2), -F_2(a_2, a_3), \cdots, -F_{n-1}(a_2, a_3, \cdots, a_n)\right) z^{n-1}$$

and

$$\left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} = 1 + \sum_{n=2}^{\infty} K_{n-1}^{1-\alpha}\left(-F_1(2a_2), -F_2(2a_2, 3a_3), \cdots, -F_{n-1}(2a_2, 3a_3, \cdots, na_n)\right) z^{n-1}.$$

Therefore

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} = \left(\sum_{n=1}^{\infty} K_{n-1}^{\alpha}\left(-F_1(a_2), -F_2(a_2, a_3), \cdots, -F_{n-1}(a_2, a_3, \cdots, a_n)\right) z^{n-1}\right) \times \left(\sum_{n=1}^{\infty} K_{n-1}^{1-\alpha}\left(-F_1(2a_2), -F_2(2a_2, 3a_3), \cdots, -F_{n-1}(2a_2, 3a_3, \cdots, na_n)\right) z^{n-1}\right). \tag{2.34}$$

Similarly

$$\left(\frac{wg'(w)}{g(w)}\right)^{\alpha}\left(1 + \frac{wg''(w)}{g'(w)}\right)^{1-\alpha} = \left(\sum_{n=1}^{\infty} K_{n-1}^{\alpha}\left(-F_1(b_2), -F_2(b_2, b_3), \cdots, -F_{n-1}(b_2, b_3, \cdots, b_n)\right) w^{n-1}\right) \times \left(\sum_{n=1}^{\infty} K_{n-1}^{1-\alpha}\left(-F_1(2b_2), -F_2(2b_2, 3b_3), \cdots, -F_{n-1}(2b_2, 3b_3, \cdots, nb_n)\right) w^{n-1}\right). \tag{2.35}$$

In light of the hypothesis, (2.34) and (2.35), respectively, yield

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} = 1 + (n-1)(n(1-\alpha)+\alpha)a_n z^{n-1} + \alpha n(n-1)^2(1-\alpha)a_n^2 z^{2(n-1)} \tag{2.36}$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^{\alpha}\left(1 + \frac{wg''(w)}{g'(w)}\right)^{1-\alpha} = 1 - (n-1)(n(1-\alpha)+\alpha)a_n w^{n-1} + \alpha n(n-1)^2(1-\alpha)a_n^2 w^{2(n-1)}, \tag{2.37}$$

where the relations between a_n and b_n are determined by the equation (2.3). Since $f \in \mathcal{L}(\alpha;\varphi)$ and $g = f^{-1} \in \mathcal{L}(\alpha;\varphi)$, by definition, there exist two Schwarz functions $u(z) = \sum_{n=1}^{\infty} c_n z^n$ and $v(w) = \sum_{n=1}^{\infty} d_n w^n$ so that

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha} = \varphi(u(z)), \tag{2.38}$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^{\alpha}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\alpha} = \varphi(v(w)). \tag{2.39}$$

A comparison of the corresponding coefficients of (2.36) and (2.38) yields

$$(n-1)(n(1-\alpha)+\alpha)a_n = \varphi_1 c_{n-1}.$$

Similarly, from (2.37) and (2.39) we obtain

$$-(n-1)(n(1-\alpha)+\alpha)a_n = \varphi_1 d_{n-1}.$$

Solving either of the above two equation for a_n and taking the absolute values of both sides coupled by an application of Caratheodory Lemma yield

$$|a_n| \leq \frac{2}{(n-1)[n(1-\alpha)+\alpha]}.$$

□

The estimations for a_2 and the coefficient body $(a_3 - a_2^2)$ are presented in the following result.

Theorem 2.7. *Let $f \in \mathcal{L}(\alpha;\varphi)$ and $g = f^{-1} \in \mathcal{L}(\alpha;\varphi)$. Then*

(i). $|a_2| \leq \dfrac{2}{2-\alpha}$,

(ii). $|a_3 - a_2^2| \leq \dfrac{1}{3-2\alpha}$.

Proof. Letting $n = 2$ in the equations (2.34), (2.35), (2.38) and (2.39) and equating the corresponding coefficients we obtain

$$\begin{cases} (2-\alpha)a_2 = \varphi_1 c_1, \\ -(2-\alpha)a_2 = \varphi_1 d_1; \end{cases}$$

and

$$\begin{cases} 2\alpha(1-\alpha)a_2^2 = \varphi_1 c_2 + \varphi_1 c_1^2, \\ 2\alpha(1-\alpha)a_2^2 = \varphi_1 d_2 + \varphi_1 d_1^2. \end{cases}$$

Solving the first set of the equations for a_2 and taking the absolute values followed by an application of Caratheodory Lemma yield

$$|a_2| \leq \frac{2}{2-\alpha}.$$

By the same token, for $n = 3$, we have

$$\frac{1}{2}(\alpha^2 + 5\alpha - 8)a_2^2 + 2(3 - 2\alpha)a_3 = \varphi_1 c_2 + \varphi_2 c_1^2, \tag{2.40}$$

and

$$\frac{1}{2}(\alpha^2 - 11\alpha + 16)a_2^2 - 2(3 - 2\alpha)a_3 = \varphi_1 d_2 + \varphi_2 d_1^2. \tag{2.41}$$

Subtracting (2.41) from (2.40) yields

$$4(3 - 2\alpha)(a_3 - a_2^2) = \varphi_1(c_2 - d_2) + \varphi_2(c_1^2 - d_1^2) = \varphi_1(c_2 - d_2).$$

Dividing by $4(3-2\alpha)$, taking the absolute values and applying the Caratheodory Lemma yield

$$|a_3 - a_2^2| \leq \frac{|\varphi_1|(|c_2| + |d_2|)}{4|3 - 2\alpha|} \leq \frac{1}{3 - 2\alpha}.$$

□

Theorem 2.8. *Let $f \in \mathcal{M}(\alpha;\varphi)$ and $g = f^{-1} \in \mathcal{M}(\alpha;\varphi)$. If $a_k = 0$ for $2 \leq k \leq n-1$ then*

$$|a_n| \leq \frac{2}{(n-1)(1 + (n-1)\alpha)}; \quad n \geq 3.$$

Proof. Suppose that $f \in \mathcal{M}(\alpha;\varphi)$ and has the series expansion (1.1). Therefore, for the term $1 + zf''/f'$ we have (see [3, page 204])

$$1 + \frac{zf''(z)}{f'(z)} = 1 - \sum_{n=2}^{\infty} F_{n-1}(2a_2, 3a_3, \cdots, na_n)z^{n-1}.$$

Moreover, an application of Lemma 2.1 implies

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 - \sum_{n=2}^{\infty} \left[F_{n-1}(a_2, \cdots, a_n) + \alpha F_{n-1}(-F_1(a_2), -F_2(a_2, a_3), \cdots, -F_{n-1}(a_2, \cdots, a_n))\right] z^{n-1}. \tag{2.42}$$

In a similar manner, for the inverse map $g = f^{-1}$ we obtain

$$(1-\alpha)\frac{wg'(w)}{g(w)} + \alpha\left(1 + \frac{wg''(w)}{g'(w)}\right) = 1 - \sum_{n=2}^{\infty} \left[F_{n-1}(b_2, \cdots, b_n) + \alpha F_{n-1}(-F_1(b_2), -F_2(b_2, b_3), \cdots, -F_{n-1}(b_2, \cdots, b_n))\right] w^{n-1}. \tag{2.43}$$

For f and $g = f^{-1}$ in $\mathcal{M}(\alpha;\varphi)$, by definition, there exist two Schwarz functions $u(z) = \sum_{n=1}^{\infty} c_n z^n$ and $v(w) = \sum_{n=1}^{\infty} d_n w^n$, satisfying the series expansion (2.7) and (2.8) so that

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = \varphi(u(z)) \tag{2.44}$$

and

$$(1-\alpha)\frac{wg'(w)}{g(w)} + \alpha\left(1 + \frac{wg''(w)}{g'(w)}\right) = \varphi(w). \tag{2.45}$$

From (2.7), (2.42) and (2.44) we conclude that

(2.46)
$$\sum_{k=1}^{n-1} \varphi_k D_{n-1}^k(c_1, c_2, \cdots, c_{n-1}) =$$
$$-\left[F_{n-1}(a_2, a_3, \cdots, a_n) + \alpha F_{n-1}(-F_1(a_2), -F_2(a_2, a_3), \cdots, -F_{n-1}(a_2, \cdots, a_n))\right].$$

Similarly, (2.8), (2.43) and (2.45) yield

(2.47)
$$\sum_{k=1}^{n-1} \varphi_k D_{n-1}^k(d_1, d_2, \cdots, d_{n-1}) =$$
$$-\left[F_{n-1}(b_2, b_3, \cdots, b_n) + \alpha F_{n-1}(-F_1(b_2), -F_2(b_2, b_3), \cdots, -F_{n-1}(b_2, \cdots, b_n))\right].$$

Under the assumption $a_k = 0;\ 2 \le k \le n-1$, equations (2.46) and (2.47), respectively, reduce to

$$(n-1)(1+(n-1)\alpha)a_n = \varphi_1 c_{n-1}, \qquad n \ge 2, \tag{2.48}$$

and

$$(n-1)(1+(n-1)\alpha)b_n = \varphi_1 d_{n-1}, \qquad n \ge 2. \tag{2.49}$$

An application of (2.3) reveals that $b_n = -a_n$. So (2.49) can be rewritten as

$$-(n-1)(1+(n-1)\alpha)a_n = \varphi_1 d_{n-1}, \qquad n \ge 2. \tag{2.50}$$

After taking the absolute values of both sides of either (2.48) or (2.50) and applying the Caratheodory lemma we obtain

$$|a_n| \le \frac{2}{(n-1)(1+(n-1)\alpha)}.$$

□

Finally, the bounds for a_2 and the coefficient body $(a_3 - a_2^2)$ are given in the following

Theorem 2.9. *Let $f \in \mathcal{M}(\alpha;\varphi)$ and $g = f^{-1} \in \mathcal{M}(\alpha;\varphi)$. Then*

(*i*). $|a_2| \le \dfrac{2}{1+\alpha}$,

(*ii*). $|a_3 - a_2^2| \le \dfrac{1}{1+2\alpha}$.

Proof. For $n = 2$ and $n = 3$, the equations (2.46) and (2.47), respectively, yield

$$(1+\alpha)a_2 = \varphi_1 c_1, \tag{2.51}$$

$$-(1+\alpha)a_2 = \varphi_1 d_1, \tag{2.52}$$

and

$$2(1+2\alpha)a_3 - (1+3\alpha)a_2^2 = \varphi_1 c_2 + \varphi_2 c_1^2, \tag{2.53}$$

$$(3+5\alpha)a_2^2 - 2(1+2\alpha)a_3 = \varphi_1 d_2 + \varphi_2 d_1^2. \tag{2.54}$$

where we used the identities $b_2 = -a_2$ and $b_3 = 2a_2^2 - a_3$ for the equation (2.54).

Obviously, $c_1 = -d_1$. From either of the equations (2.51) or (2.52) upon an application of Caratheodory Lemma we obtain

$$|a_2| \le \frac{2}{1+\alpha}.$$

Subtracting (2.54) from (2.53) gives

$$4(1+2\alpha)(a_3 - a_2^2) = \varphi_1(c_2 - d_2) + \varphi_2(c_1^2 - d_1^2) = \varphi_1(c_2 - d_2).$$

Dividing by $4(1+2\alpha)$, taking the abosolute values and applying the Caratheodory Lemma we obtain

$$|a_3 - a_2^2| \le \frac{|\varphi_1|(|c_2| + |d_2|)}{4(1+2\alpha)} \le \frac{1}{1+2\alpha}.$$

□

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