Title:
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ENTIRE FUNCTIONS SHARING A SMALL ENTIRE FUNCTION WITH THEIR DIFFERENCE OPERATORS

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Abstract. In this paper, we mainly investigate the uniqueness of the entire function sharing a small entire function with its high difference operators. We obtain a result, which can give a negative answer to an uniqueness question relate to the Brück conjecture dealt by Liu and Yang. Meanwhile, we also establish a difference analogue of the Brück conjecture for entire functions of order less than 2, which improves some results obtained by Liu and Yang.

Keywords: Entire function, difference equation, small function.


1. Introduction

In this paper, a meromorphic function always means it is meromorphic in the whole complex plane $\mathbb{C}$. We assume that the reader is familiar with the standard notations in the Nevanlinna theory of meromorphic functions. We use the following standard notations in value distribution theory (see [7, 10, 15, 16]):

$$T(r, f), m(r, f), N(r, f), \overline{N}(r, f), \cdots.$$ 

We denote by $S(r, f)$ any quantity satisfying

$$S(r, f) = o(T(r, f)), \text{ as } r \to \infty,$$

possibly outside of a set $E$ with finite linear measure, not necessarily the same at each occurrence. A meromorphic function $a(z)$ is said to be a small function with respect to $f(z)$ if $T(r, a) = S(r, f)$. We use $\lambda(f)$ and $\sigma(f)$ to denote the exponent of convergence of zeros of $f$ and the order of $f$, respectively. We use $\sigma_2(f)$ to denote the super order of $f$. We say that two meromorphic functions $f(z)$ and $g(z)$ share the value $a$ IM (ignoring multiplicities) if $f(z) - a$ and $g(z) - a$ have the same zeros. If $f(z) - a$ and $g(z) - a$ have the same
zeros with the same multiplicities, then we say that they share the value a CM (counting multiplicities). A polynomial $Q(f)$ is called a difference polynomial in $f$ if $Q$ is a polynomial in $f$ and its shifts with meromorphic coefficients, say $\{a_\lambda | \lambda \in I\}$, such that $T(r,a_\lambda) = S(r,f)$ for all $\lambda \in I$. We define the difference operators $\Delta f = f(z+1) - f(z)$ and $\Delta^n f = \Delta^{n-1}(\Delta f)$. Moreover, $
abla^n f = \sum_{j=0}^{n} C^n_j (-1)^{n-j} f(z+j)$.

In 1996, Brück [2] studied the uniqueness theory about the entire functions sharing one value with their first derivatives and posed the following interesting conjecture.

**Conjecture 1.** Let $f(z)$ be a non-constant entire function satisfying the super order $\sigma_2(f) < \infty$ which is not a positive integer. If $f(z)$ and $f'(z)$ share one finite value a CM, then

$$f'(z) - a = c(f(z) - a)$$

holds for some constant $c \neq 0$.

He also proved that the conjecture is true provided $a \neq 0$ and $N(r, \frac{1}{f'}) = S(r, f)$ or $a = 0$. But the conjecture is still open by now. Recently the difference analogue of the lemma on the logarithmic derivative and Nevanlinna theory for the difference operator have been founded (see [1, 4–6]), which bring about a number of papers focusing on the uniqueness of meromorphic functions sharing a small function with their difference operators (see [9, 11–14]). For example, Liu and Yang [14] obtained the following result.

**Theorem A.** [14] Let $f$ be a transcendental entire function such that $\sigma(f) < 1$. If $f$ and $\Delta^n f$ share a finite value a CM, then

$$\Delta^n f - a = c(f - a)$$

holds for some nonzero complex number c.

But we find that such probability in the conclusion of Theorem A does not exist. As a matter of fact, we can obtain the following result.

**Theorem 1.1.** Let $f$ be a transcendental entire function such that $\sigma(f) < 1$. Then $f$ and $\Delta^n f$ can not share a finite value a CM.

The hypothesis $\sigma(f) < 1$ plays an important role in the proof of Theorem A. It is natural to ask one question: May the conclusion of Theorem A be still valid when $\sigma(f) \geq 1$? In this direction, we prove the following result.

**Theorem 1.2.** Let $f(z)$ be a transcendental entire function such that $\sigma(f) < 2$, and $\alpha(z) \neq 0$ be an entire function such that $\sigma_2(\alpha) < \sigma(f)$ and $\lambda(f - \alpha) < \sigma(f)$. If $\Delta^n f - \alpha(z)$ and $f(z) - \alpha(z)$ share 0 CM, then $\alpha(z)$ is a polynomial with degree at most $n - 1$ and $f$ must be form of

$$f(z) = \alpha(z) + H(z)e^{dz},$$
where $H(z)$ is a polynomial such that $cH(z) = -\alpha(z)$, and $c, d$ are two nonzero constants such that $e^d = 1$.

From Theorem 1.2 we can obtain the following two results.

**Corollary 1.3.** Let $f(z)$ be a transcendental entire function such that $\sigma(f) < 2$, and $\alpha(z) \neq 0$ be an entire function such that $\sigma(\alpha) < \sigma(f)$ and $\lambda(f - \alpha) < \sigma(f)$. If $\Delta f - \alpha(z)$ and $f(z) - \alpha(z)$ share 0 CM, then $\alpha(z)$ is a constant $\alpha$ and $f$ must be form of

$$f(z) = \alpha + ce^d z,$$

where $c, d$ are two nonzero constants such that $e^d = 1$.

**Corollary 1.4.** Let $f(z)$ be a transcendental entire function such that $\sigma(f) < 2$, and $\alpha$ be a nonzero constant and $\lambda(f - \alpha) < \sigma(f)$. If $\Delta^n f - \alpha$ and $f(z) - \alpha$ share 0 CM, then $f$ must be form of

$$f(z) = \alpha + ce^d z,$$

where $c, d$ are two nonzero constants such that $e^d = 1$.

We are not sure whether the assumption $\alpha(z) \neq 0$ in Theorem 1.2 is necessary or not, so we present the following conjecture.

**Conjecture 2.** Let $f$ be a transcendental entire function such that $\sigma(f) < 2$ and $\lambda(f) < \sigma(f)$. If $f$ and $\Delta^n f$ share 0 CM, then $f$ must be form of

$$f(z) = He^d z,$$

where $H$ and $d$ are two nonzero constants.

2. Some lemmas

To prove our results, we need some lemmas as follows.

**Lemma 2.1.** (see [4]) Let $\eta_1, \eta_2$ be two complex numbers such that $\eta_1 \neq \eta_2$ and let $f(z)$ be a finite order meromorphic function. Let $\sigma$ be the order of $f(z)$, then for each $\varepsilon > 0$, we have

$$m(r, f(z + \eta_1) = O(r^{\sigma - 1 + \varepsilon}).$$

**Lemma 2.2.** (see [15]) Suppose that $f_1(z), f_2(z), \ldots, f_n(z)$ $(n \geq 2)$ are meromorphic functions and $g_1(z), g_2(z), \ldots, g_n(z)$ are entire functions satisfying the following conditions:

(i) $\sum_{j=1}^n f_j(z)e^{g_j(z)} \equiv 0$;

(ii) $g_j(z) - g_k(z)$ are not constants for $1 \leq j < k \leq n$;

(iii) For $1 \leq j \leq n, 1 \leq h < k \leq n$.\ T(r, f_j) = o(T(r, e^{g_j - g_k})), (r \to \infty, r \notin E).

Then $f_j(z) \equiv 0(j = 1, 2, \ldots, n)$.
**Lemma 2.3.** (see [1]) Let $g$ be a transcendental function of order less than 1 and $h$ be a positive constant. Then there exists an $\varepsilon$ set $E$ such that

$$ \frac{g'(z + \eta)}{g(z + \eta)} \to 0, \quad \frac{g(z + \eta)}{g(z)} \to 1, \text{ as } z \to \infty \text{ in } C \setminus E $$

uniformly in $\eta$ for $|\eta| \leq h$. Further, the set $E$ may be chosen so that for large $|z| \notin E$, the function $g$ has no zeroes or poles in $|\zeta - z| \leq h$.

**Remark.** According to Hayman [8], an $\varepsilon$ set is defined to be a countable union of open discs not containing the origin and subtending angles at the origin whose sum is finite. Suppose $E$ is an $\varepsilon$ set, then the set of $r \geq 1$ for which the circle $S(0, r) = \{z : |z| = r\}$ meets $E$ has finite logarithmic measure and for almost all real $\theta$ the intersection of $E$ with the ray $\arg z = \theta$ is bounded.

The following lemma is a trivial well known fact from the theory of finite differences. For example, we can find it in [3] (the case $q(z) \equiv 1$).

**Lemma 2.4.** (see [3]) Let $g$ be a polynomial with degree $n$. Then $\Delta g$ is a polynomial with degree $n - 1$.

### 3. The proofs of theorems

**3.1. Proof of Theorem 1.1.**

**Proof.** On the contrary, we suppose $f$ and $\Delta^n f$ share a finite value $a$ CM. Then by Theorem A, we get

\begin{align}
\Delta^n f - a &= c(f - a) 
\end{align}

holds for some nonzero complex number $c$. Then we rewrite Equation (3.1) as the following form

\begin{align}
\frac{\Delta^n f}{f} - a &= c(1 - \frac{a}{f}).
\end{align}

By Lemma 2.3 and the remark, we get

\begin{align}
\frac{\Delta^n f}{f} = \sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{f(z + j)}{f(z)} \to \sum_{j=0}^{n} C_n^j (-1)^{n-j} = (1 - 1)^n = 0
\end{align}

and

\begin{align}
\frac{a}{f} \to 0 \text{ as } |z| = r \to +\infty, z \in \{z : |f(z)| = M(r, f) \text{ and } z \notin E_\varepsilon\},
\end{align}

where $E_\varepsilon$ is an $\varepsilon$ set. Thus we get $c = 0$, which is a contradiction.

\square
3.2. Proof of Theorem 1.2.

Proof. On the one hand, since \( \Delta^n f - \alpha(z) \) and \( f(z) - \alpha(z) \) share 0 CM, then there exists an entire function \( Q(z) \) such that
\[
(3.3) \quad \Delta^n f - \alpha(z) = [f(z) - \alpha(z)]e^{Q(z)}.
\]
By Lemma 2.1, we see, for any \( \varepsilon > 0 \),
\[
T(r, \Delta^n f) = m(r, \Delta^n f) \leq \sum_{j=0}^{n} m(r, \frac{f(z+j)}{f(z)}) + m(r, f) = O(r^{\sigma-1+\varepsilon}) + T(r, f),
\]
where \( \sigma = \sigma(f) \). Thus from Equation (3.3) and our assumption that \( \alpha(z) \) is a small function with respect to \( f(z) \), we can apparently obtain that
\[
T(r, e^{Q(z)}) \leq 2T(r, f) + O(r^{\sigma-1+\varepsilon}) + S(r, f),
\]
which means \( Q(z) \) is a polynomial and then
\[
\sigma(e^{Q(z)}) = \deg Q(z) \leq \sigma(f) < 2.
\]
That is to say, \( \deg Q(z) = 0 \) or \( \deg Q(z) = 1 \).
On the other hand, since \( \lambda(f - \alpha) < \sigma(f) \), then there exists an entire function \( \tilde{H}(z) \) and a polynomial \( \tilde{Q}(z) \) such that
\[
\tilde{f}(z) - \alpha(z) = \tilde{H}(z)e^{\tilde{Q}(z)},
\]
and \( \sigma(\tilde{H}) = \lambda(\tilde{H}) = \lambda(f - \alpha) < \sigma(f) \). Thus, we can obtain
\[
\sigma(e^{\tilde{Q}(z)}) = \deg \tilde{Q}(z) = \sigma(f) < 2.
\]
Thus, \( \deg \tilde{Q}(z) = 1 \) and then there exists an entire function \( H(z) \) such that
\[
(3.4) \quad f(z) - \alpha(z) = H(z)e^{dz}
\]
and
\[
\sigma(H) = \lambda(H) = \lambda(f - \alpha) < \sigma(f),
\]
where \( d \) is a nonzero constant. By Equation (3.4), we can obtain
\[
\sigma(f) = \sigma(e^{dz}) = 1 \quad \text{and} \quad \sigma(H) < 1.
\]
Set
\[
H_0(z) = H, \quad H_{1}(z) = e^{d}H(z+1) - H(z), \cdots, H_{n}(z) = e^{d}H_{n-1}(z+1) - H_{n-1}(z).
\]
By Lemma 2.1, in a similar way, we see
\[
\sigma(H_i) < 1 \quad \text{for} \quad i = 0, 1, 2, \cdots, n.
\]
We claim that if \( H(z) \) is a transcendental entire function, then \( 2H_j(z) \) are all transcendental entire functions for \( j = 0, 1, \cdots, n \).
In fact, on the contrary, suppose that \( H_{1}(z) \) is a polynomial, then there exists a positive integer \( k \) such that
\[
H_{1}^{(k)}(z) = e^{d}H^{(k)}(z+1) - H^{(k)}(z) = 0.
\]
Let \( g(z) = H^{(k)}(z) \), then \( g(z) \) is a transcendental function of order less than 1 and
\[
e^{-d} = \frac{g(z + 1)}{g(z)}.
\]

By Lemma 2.3, we can obtain there exists an \( \varepsilon \) set \( E \) such that
\[
e^{-d} = \frac{g(z + \eta)}{g(z)} \to 1, \text{ as } z \to \infty \text{ in } C \setminus E.
\]

Thus, \( e^{-d} = 1 \) and \( g(z + 1) = g(z) \). It is easy to see that \( z_k = k \) are some zeros of \( g(z) - g(0) \), which implies
\[
N(r, \frac{1}{g(z) - g(0)}) \geq r(1 + o(1)).
\]

It means \( \sigma(g) \geq 1 \), i.e., \( \sigma(H) \geq 1 \), which is a contradiction. So \( H_1(z) \) is a transcendental entire function. In the same way, we can obtain \( H_j(z) \) are all transcendental entire functions for \( j = 0, 1, \cdots, n \).

Next, we discuss the following two cases separately.

**Case 1.** If \( \deg Q(z) = 1 \), then Equation (3.3) becomes
\[
\Delta^n f - \alpha(z) = [f(z) - \alpha(z)]e^{bz},
\]
where \( b, c \) are two nonzero constants. Moreover, from Equation (3.4), we can obtain
\[
\Delta^n f - \Delta^n \alpha(z) = H_n(z)e^{dz}.
\]

Thus, from Equations (3.4)-(3.6) and the definition of \( H_n \), we can obtain
\[
\Delta^n \alpha(z) + H_n(z)e^{dz} - \alpha(z) = cH(z)e^{(b+d)z}.
\]

Now, we will consider two subcases in this case.

**Subcase 1.1.** If \( H_n \equiv 0 \), then Equation (3.7) becomes
\[
\Delta^n \alpha(z) - \alpha(z) = cH(z)e^{(b+d)z}.
\]

By Equation (3.8) and Lemma 2.2, we can obtain
\[
\Delta^n \alpha(z) - \alpha(z) = cH(z) \text{ and } b + d = 0.
\]

But \( H_n \equiv 0 \), by our claim, we can see \( H(z) \) is a polynomial, and then \( H_j(z) \) are all polynomials for \( j = 1, \cdots, n \). Recalling our definition \( H_n = e^{d}H_{n-1}(z + 1) - H_{n-1}(z) = 0 \). If \( H_{n-1} \neq 0 \), then we can obtain
\[
e^{-d} = \frac{H_{n-1}(z + 1)}{H_{n-1}(z)} \to 1, \text{ as } z \to \infty \text{ in } C \setminus E.
\]

Thus \( e^{-d} = 1 \) and then
\[
H_n = e^{d}H_{n-1}(z + 1) - H_{n-1}(z) = \Delta H_{n-1} = \cdots = \Delta^n H.
\]
Thus $\Delta^n H = 0$. If $H_{n-1} = H_{n-2} = \cdots H_{j+1} \equiv 0$ and $H_j \neq 0$, $0 \leq j \leq n - 2$, then we can obtain $e^d = 1$ and $\Delta^n H = 0$ similarly. Thus, by Lemma 2.4, we can obtain $H$ is a polynomial with degree at most $n - 1$. Recall

\begin{equation}
\Delta^n \alpha(z) - \alpha(z) = cH(z).
\end{equation}

We claim that $\alpha(z)$ is a polynomial. In fact, on the contrary, suppose $\alpha(z)$ is a transcendental entire function with order less than 1, and rewrite Equation (3.9) in a more attractive form as follows

\begin{equation}
\sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{\alpha(z + j)}{\alpha(z)} - 1 = c \frac{H(z)}{\alpha(z)}.
\end{equation}

By Liouville theorem, we can obtain $H(z)/\alpha(z) \to 0$, as $|z| = r \to +\infty$, and $|\alpha(z)| = M(r, \alpha)$.

However, by Lemma 2.3, we can obtain

\begin{equation}
\sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{\alpha(z + j)}{\alpha(z)} \to \sum_{j=0}^{n} C_n^j (-1)^{n-j} = (1 - 1)^n = 0, \quad z \to \infty
\end{equation}

possibly outside a $\varepsilon$ set $E$. Thus we can get a contradiction from Equation (3.10). It follows that then $\alpha(z)$ is a polynomial. By Equation (3.9) and Lemma 2.4, we get

$$\deg \alpha(z) = \deg H(z) \leq n - 1 \text{ and } \alpha(z) + cH(z) = 0.$$ 

**Subcase 1.2.** If $H_n \neq 0$, then by Equation (3.7) and Lemma 2.2, we can obtain $b = 0$, and then Equation (3.7) becomes

\begin{equation}
\Delta^n \alpha(z) - \alpha(z) = [cH(z) - H_n(z)]e^{dz}.
\end{equation}

By Equation (3.11) and Lemma 2.2, we can obtain

$$\Delta^n \alpha(z) - \alpha(z) = cH(z) - H_n(z) = 0.$$

If $\alpha(z)$ is a transcendental entire function with order less than 1, then by Lemma 2.2, we get

$$1 = \frac{\Delta^n \alpha(z)}{\alpha(z)} = \sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{\alpha(z + j)}{\alpha(z)} \to \sum_{j=0}^{n} C_n^j (-1)^{n-j} = (1 - 1)^n = 0$$

as $z \to \infty$ possibly outside a $\varepsilon$ set $E$, which is impossible.

If $\alpha(z)$ is a nonconstant polynomial, then by Lemma 2.4, we get

$$\deg[\Delta^n \alpha(z) - \alpha(z)] = \deg \alpha(z) = 0,$$

which is also impossible. Thus $\alpha(z)$ is a constant, and then

$$\alpha(z) = \Delta^n \alpha(z) \equiv 0,$$
which is a contradiction.

**Case 2.** If \( \deg Q(z) = 0 \), then Equation (3.3) becomes

\[
\Delta^n f - \alpha(z) = c[f(z) - \alpha(z)],
\]

where \( c \) is a nonzero constant. From Equations (3.4), (3.6) and (3.12), we obtain

\[
\Delta^n \alpha(z) - \alpha(z) = [cH(z) - H_n(z)]e^{dz}.
\]

By the above equation and Lemma 2.2, we can obtain

\[
\Delta^n \alpha(z) - \alpha(z) = cH(z) - H_n(z) = 0.
\]

By the same arguments as in subcase 1.2, we get \( \alpha(z) \equiv 0 \), which is a contradiction.

The proof of Theorem 2 is completed. \( \square \)

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