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ENTIRE FUNCTIONS SHARING A SMALL ENTIRE FUNCTION WITH THEIR DIFFERENCE OPERATORS

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Abstract. In this paper, we mainly investigate the uniqueness of the entire function sharing a small entire function with its high difference operators. We obtain a result, which can give a negative answer to an uniqueness question relate to the Brück conjecture dealt by Liu and Yang. Meanwhile, we also establish a difference analogue of the Brück conjecture for entire functions of order less than 2, which improves some results obtained by Liu and Yang.
Keywords: Entire function, difference equation, small function.
MSC(2010): Primary: 30D35; Secondary: 34M10.

1. Introduction

In this paper, a meromorphic function always means it is meromorphic in the whole complex plane $\mathbb{C}$. We assume that the reader is familiar with the standard notations in the Nevanlinna theory of meromorphic functions. We use the following standard notations in value distribution theory (see [7,10,15,16]):

$$T(r,f), m(r,f), N(r,f), \overline{N}(r,f), \cdots .$$

We denote by $S(r,f)$ any quantity satisfying

$$S(r,f) = o\{T(r,f)\}, \quad \text{as } r \to \infty,$$

possibly outside of a set E with finite linear measure, not necessarily the same at each occurrence. A meromorphic function $a(z)$ is said to be a small function with respect to $f(z)$ if $T(r,a) = S(r,f)$. We use $\lambda(f)$ and $\sigma(f)$ to denote the exponent of convergence of zeros of f and the order of f, respectively. We use $\sigma_2(f)$ to denote the super order of f. We say that two meromorphic functions $f(z)$ and $g(z)$ share the value a IM (ignoring multiplicities) if $f(z) - a$ and $g(z) - a$ have the same zeros. If $f(z) - a$ and $g(z) - a$ have the same

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zeros with the same multiplicities, then we say that they share the value a CM (counting multiplicities). A polynomial $Q(f)$ is called a difference polynomial in f if Q is a polynomial in f and its shifts with meromorphic coefficients, say $\{a_\lambda | \lambda \in I\}$, such that $T(r, a_\lambda) = S(r, f)$ for all $\lambda \in I$. We define the difference operators $\Delta f = f(z+1) - f(z)$ and $\Delta^n f = \Delta^{n-1}(\Delta f)$. Moreover, $\Delta^n f = \sum_{j=0}^{n} C_n^j (-1)^{n-j} f(z+j)$.

In 1996, Brück [2] studied the uniqueness theory about the entire functions sharing one value with their first derivatives and posed the following interesting conjecture.

Conjecture 1. Let $f(z)$ be non-constant entire function satisfying the super order $\sigma_2(f) < \infty$ which is not a positive integer. If $f(z)$ and $f'(z)$ share one finite value a CM, then

$$f'(z) - a = c(f(z) - a)$$

holds for some constant $c \neq 0$.

He also proved that the conjecture is true provided $a \neq 0$ and $N(r, \frac{1}{f'}) = S(r, f)$ or $a = 0$. But the conjecture is still open by now. Recently the difference analogue of the lemma on the logarithmic derivative and Nevanlinna theory for the difference operator have been founded (see [1, 4–6]), which bring about a number of papers focusing on the uniqueness of meromorphic functions sharing a small function with their difference operators (see [9, 11–14]). For example, Liu and Yang [14] obtained the following result.

Theorem A. *[14] Let f be a transcendental entire function such that $\sigma(f) < 1$. If f and $\Delta^n f$ share a finite value a CM, then*

$$\Delta^n f - a = c(f - a)$$

holds for some nonzero complex number c.

But we find that such probability in the conclusion of Theorem A does not exist. As a matter of fact, we can obtain the following result.

Theorem 1.1. *Let f be a transcendental entire function such that $\sigma(f) < 1$. Then f and $\Delta^n f$ can not share a finite value a CM.*

The hypothesis $\sigma(f) < 1$ plays an important role in the proof of Theorem A. It is natural to ask one question: May the conclusion of Theorem A be still valid when $\sigma(f) \geq 1$? In this direction, we prove the following result.

Theorem 1.2. *Let $f(z)$ be a transcendental entire function such that $\sigma(f) < 2$, and $\alpha(z) \not\equiv 0$ be an entire function such that $\sigma(\alpha) < \sigma(f)$ and $\lambda(f - \alpha) < \sigma(f)$. If $\Delta^n f - \alpha(z)$ and $f(z) - \alpha(z)$ share 0 CM, then $\alpha(z)$ is a polynomial with degree at most $n-1$ and f must be form of*

$$f(z) = \alpha(z) + H(z)e^{dz},$$

where $H(z)$ is a polynomial such that $cH(z) = -\alpha(z)$, and c, d are two nonzero constants such that $e^d = 1$.

From Theorem 1.2 we can obtain the following two results.

Corollary 1.3. *Let $f(z)$ be a transcendental entire function such that $\sigma(f) < 2$, and $\alpha(z) \not\equiv 0$ be an entire function such that $\sigma(\alpha) < \sigma(f)$ and $\lambda(f-\alpha) < \sigma(f)$. If $\Delta f - \alpha(z)$ and $f(z) - \alpha(z)$ share 0 CM, then $\alpha(z)$ is a constant α and f must be form of*

$$f(z) = \alpha + ce^{dz},$$

where c, d are two nonzero constants such that $e^d = 1$.

Corollary 1.4. *Let $f(z)$ be a transcendental entire function such that $\sigma(f) < 2$, and α be a nonzero constant and $\lambda(f-\alpha) < \sigma(f)$. If $\Delta^n f - \alpha$ and $f(z) - \alpha$ share 0 CM, then f must be form of*

$$f(z) = \alpha + ce^{dz},$$

where c, d are two nonzero constants such that $e^d = 1$.

We are not sure whether the assumption $\alpha(z) \not\equiv 0$ in Theorem 1.2 is necessary or not, so we present the following conjecture.

Conjecture 2. Let f be a transcendental entire function such that $\sigma(f) < 2$ and $\lambda(f) < \sigma(f)$. If f and $\Delta^n f$ share 0 CM, then f must be form of

$$f(z) = He^{dz},$$

where H and d are two nonzero constants.

2. **Some lemmas**

To prove our results, we need some lemmas as follows.

Lemma 2.1. *(see [4]) Let η_1, η_2 be two complex numbers such that $\eta_1 \neq \eta_2$ and let $f(z)$ be a finite order meromorphic function. Let σ be the order of $f(z)$, then for each $\varepsilon > 0$, we have*

$$m(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}) = O(r^{\sigma-1+\varepsilon}).$$

Lemma 2.2. *(see [15]) Suppose that $f_1(z), f_2(z), \cdots, f_n(z)$ $(n \geq 2)$ are meromorphic functions and $g_1(z), g_2(z), \cdots, g_n(z)$ are entire functions satisfying the following conditions:*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;
(ii) $g_j(z) - g_k(z)$ *are not constants for* $1 \leq j < k \leq n$;
(iii) For $1 \leq j \leq n, 1 \leq h < k \leq n$. $T(r, f_j) = o\{T(r, e^{g_h - g_k})\}, (r \to \infty, r \notin E)$.

Then $f_j(z) \equiv 0 (j = 1, 2, \cdots, n)$.

Lemma 2.3. *(see [1]) Let g be a transcendental function of order less than 1 and h be a positive constant. Then there exists an ε set E such that*

$$\frac{g'(z+\eta)}{g(z+\eta)} \to 0, \ \frac{g(z+\eta)}{g(z)} \to 1, \textit{as } z \to \infty \textit{ in } C \backslash E$$

uniformly in η for $|\eta| \leq h$. Further, the set E may be chosen so that for large $|z| \notin E$, the function g has no zeroes or poles in $|\zeta - z| \leq h$.

Remark. According to Hayman [8], an ε set is defined to be a countable union of open discs not containing the origin and subtending angles at the origin whose sum is finite. Suppose E is an ε set, then the set of $r \geq 1$ for which the circle $S(0,r) = \{z : |z| = r\}$ meets E has finite logarithmic measure and for almost all real θ the intersection of E with the ray $\arg z = \theta$ is bounded.

The following lemma is a trivial well known fact from the theory of finite differences. For example, we can find it in [3] (the case $q(z) \equiv 1$).

Lemma 2.4. *(see [3]) Let g be a polynomial with degree n. Then Δg is a polynomial with degree $n-1$.*

3. The proofs of theorems

3.1. Proof of Theorem 1.1.

Proof. On the contrary, we suppose f and $\Delta^n f$ share a finite value a CM. Then by Theorem A, we get

$$\Delta^n f - a = c(f - a) \tag{3.1}$$

holds for some nonzero complex number c. Then we rewrite Equation (3.1) as the following form

$$\frac{\Delta^n f}{f} - \frac{a}{f} = c(1 - \frac{a}{f}). \tag{3.2}$$

By Lemma 2.3 and the remark, we get

$$\frac{\Delta^n f}{f} = \sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{f(z+j)}{f(z)} \to \sum_{j=0}^{n} C_n^j (-1)^{n-j} = (1-1)^n = 0$$

and

$$\frac{a}{f} \to 0 \ \text{ as } \ |z| = r \to +\infty, z \in \{z : |f(z)| = M(r,f) \text{ and } z \notin E_\varepsilon\},$$

where E_ε is an ε set. Thus we get $c = 0$, which is a contradiction.

□

3.2. **Proof of Theorem 1.2.**

Proof. On the one hand, since $\Delta^n f - \alpha(z)$ and $f(z) - \alpha(z)$ share 0 CM, then there exists an entire function $Q(z)$ such that

$$\Delta^n f - \alpha(z) = [f(z) - \alpha(z)]e^{Q(z)}. \tag{3.3}$$

By Lemma 2.1, we see, for any $\varepsilon > 0$,

$$T(r, \Delta^n f) = m(r, \Delta^n f) \leq \sum_{j=0}^{n} m(r, \frac{f(z+j)}{f(z)}) + m(r, f) = O(r^{\sigma-1+\varepsilon}) + T(r, f),$$

where $\sigma = \sigma(f)$. Thus from Equation (3.3) and our assumption that $\alpha(z)$ is a small function with respect to $f(z)$, we can apparently obtain that

$$T(r, e^{Q(z)}) \leq 2T(r, f) + O(r^{\sigma-1+\varepsilon}) + S(r, f),$$

which means $Q(z)$ is a polynomial and then

$$\sigma(e^{Q(z)}) = \deg Q(z) \leq \sigma(f) < 2.$$

That is to say, $\deg Q(z) = 0$ or $\deg Q(z) = 1$;
On the other hand, since $\lambda(f - \alpha) < \sigma(f)$, then there exists an entire function $\tilde{H}(z)$ and a polynomial $\tilde{Q}(z)$ such that

$$f(z) - \alpha(z) = \tilde{H}(z)e^{\tilde{Q}(z)},$$

and $\sigma(\tilde{H}) = \lambda(\tilde{H}) = \lambda(f - \alpha) < \sigma(f)$. Thus, we can obtain

$$\sigma(e^{\tilde{Q}(z)}) = \deg \tilde{Q}(z) = \sigma(f) < 2.$$

Thus, $\deg \tilde{Q}(z) = 1$ and then there exists an entire function $H(z)$ such that

$$f(z) - \alpha(z) = H(z)e^{dz} \tag{3.4}$$

and

$$\sigma(H) = \lambda(H) = \lambda(f - \alpha) < \sigma(f),$$

where d is a nonzero constant. By Equation (3.4), we can obtain

$$\sigma(f) = \sigma(e^{dz}) = 1 \text{ and } \sigma(H) < 1.$$

Set

$$H_0(z) = H,\ H_1(z) = e^d H(z+1) - H(z), \cdots, H_n(z) = e^d H_{n-1}(z+1) - H_{n-1}(z).$$

By Lemma 2.1, in a similar way, we see

$$\sigma(H_i) < 1 \text{ for } i = 0, 1, 2, \cdots, n.$$

We claim that if $H(z)$ is a transcendental entire function, then $2H_j(z)$ are all transcendental entire functions for $j = 0, 1, \cdots, n$.
In fact, on the contrary, suppose that $H_1(z)$ is a polynomial, then there exists a positive integer k such that

$$H_1^{(k)}(z) = e^d H^{(k)}(z+1) - H^{(k)}(z) = 0.$$

Let $g(z) = H^{(k)}(z)$, then $g(z)$ is a transcendental function of order less than 1 and

$$e^{-d} = \frac{g(z+1)}{g(z)}.$$

By Lemma 2.3, we can obtain there exists an ε set E such that

$$e^{-d} = \frac{g(z+\eta)}{g(z)} \to 1, \text{ as } z \to \infty \text{ in } C \backslash E.$$

Thus, $e^{-d} = 1$ and $g(z+1) = g(z)$. It is easy to see that $z_k = k$ are some zeros of $g(z) - g(0)$, which implies

$$\overline{N}(r, \frac{1}{g(z) - g(0)}) \geq r(1 + o(1)).$$

It means $\sigma(g) \geq 1$, i.e., $\sigma(H) \geq 1$, which is a contradiction. So $H_1(z)$ is a transcendental entire function. In the same way, we can obtain $H_j(z)$ are all transcendental entire functions for $j = 0, 1, \cdots, n$.
Next, we discuss the following two cases separately.

Case 1. If $\deg Q(z) = 1$, then Equation (3.3) becomes

$$\Delta^n f - \alpha(z) = [f(z) - \alpha(z)]ce^{bz}, \tag{3.5}$$

where b, c are two nonzero constants. Moreover, from Equation (3.4), we can obtain

$$\Delta^n f - \Delta^n \alpha(z) = H_n(z)e^{dz}. \tag{3.6}$$

Thus, from Equations (3.4)-(3.6) and the definition of H_n, we can obtain

$$\Delta^n \alpha(z) + H_n(z)e^{dz} - \alpha(z) = cH(z)e^{(b+d)z}. \tag{3.7}$$

Now, we will consider two subcases in this case.
Subcase 1.1. If $H_n \equiv 0$, then Equation (3.7) becomes

$$\Delta^n \alpha(z) - \alpha(z) = cH(z)e^{(b+d)z}. \tag{3.8}$$

By Equation (3.8) and Lemma 2.2, we can obtain

$$\Delta^n \alpha(z) - \alpha(z) = cH(z) \text{ and } b + d = 0.$$

But $H_n \equiv 0$, by our claim, we can see $H(z)$ is a polynomial, and then $H_j(z)$ are all polynomials for $j = 1, \cdots, n$. Recalling our definition $H_n = e^d H_{n-1}(z+1) - H_{n-1}(z) = 0$. If $H_{n-1} \not\equiv 0$, then we can obtain

$$e^{-d} = \frac{H_{n-1}(z+1)}{H_{n-1}(z)} \to 1, \text{ as } z \to \infty \text{ in } C \backslash E.$$

Thus $e^d = 1$ and then

$$H_n = e^d H_{n-1}(z+1) - H_{n-1}(z) = \Delta H_{n-1} = \cdots = \Delta^n H.$$

Thus $\Delta^n H = 0$. If $H_{n-1} = H_{n-2} = \cdots H_{j+1} \equiv 0$ and $H_j \not\equiv 0$, $0 \le j \le n-2$, then we can obtain $e^d = 1$ and $\Delta^n H = 0$ similarly.
Thus, by Lemma 2.4, we can obtain H is a polynomial with degree at most $n-1$. Recall

$$\Delta^n \alpha(z) - \alpha(z) = cH(z). \tag{3.9}$$

We claim that $\alpha(z)$ is a polynomial. In fact, on the contrary, suppose $\alpha(z)$ is a transcendental entire function with order less than 1, and rewrite Equation (3.9) in a more attractive form as follows

$$\sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{\alpha(z+j)}{\alpha(z)} - 1 = c\frac{H(z)}{\alpha(z)}. \tag{3.10}$$

By Liouvill theorem, we can obtain

$$\frac{H(z)}{\alpha(z)} \to 0, \text{ as } |z| = r \to +\infty, \text{ and } |\alpha(z)| = M(r,\alpha).$$

However, by Lemma 2.3, we can obtain

$$\sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{\alpha(z+j)}{\alpha(z)} \to \sum_{j=0}^{n} C_n^j (-1)^{n-j} = (1-1)^n = 0, \text{ as } z \to \infty$$

possibly outside a ε set E. Thus we can get a contradiction from Equation (3.10). It follows that then $\alpha(z)$ is a polynomial. By Equation (3.9) and Lemma 2.4, we get

$$\deg \alpha(z) = \deg H(z) \le n-1 \text{ and } \alpha(z) + cH(z) = 0.$$

Subcase 1.2. If $H_n \not\equiv 0$, then by Equation (3.7) and Lemma 2.2, we can obtain $b = 0$, and then Equation (3.7)) becomes

$$\Delta^n \alpha(z) - \alpha(z) = [cH(z) - H_n(z)]e^{dz}. \tag{3.11}$$

By Equation (3.11) and Lemma 2.2, we can obtain

$$\Delta^n \alpha(z) - \alpha(z) = cH(z) - H_n(z) = 0.$$

If $\alpha(z)$ is a transcendental entire function with order less than 1, then by Lemma 2.2, we get

$$1 = \frac{\Delta^n \alpha(z)}{\alpha(z)} = \sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{\alpha(z+j)}{\alpha(z)} \to \sum_{j=0}^{n} C_n^j (-1)^{n-j} = (1-1)^n = 0$$

as $z \to \infty$ possibly outside a ε set E, which is impossible.
If $\alpha(z)$ is a nonconstant polynomial, then by Lemma 2.4, we get

$$\deg[\Delta^n \alpha(z) - \alpha(z)] = \deg \alpha(z) = 0,$$

which is also impossible. Thus $\alpha(z)$ is a constant, and then

$$\alpha(z) = \Delta^n \alpha(z) \equiv 0,$$

which is a contradiction.

Case 2. If $\deg Q(z) = 0$, then Equation (3.3) becomes

$$\Delta^n f - \alpha(z) = c[f(z) - \alpha(z)], \tag{3.12}$$

where c is a nonzero constant. From Equations (3.4), (3.6) and (3.12), we obtain

$$\Delta^n \alpha(z) - \alpha(z) = [cH(z) - H_n(z)]e^{dz}.$$

By the above equation and Lemma 2.2, we can obtain

$$\Delta^n \alpha(z) - \alpha(z) = cH(z) - H_n(z) = 0.$$

By the same arguments as in subcase 1.2, we get $\alpha(z) \equiv 0$, which is a contradiction.
The proof of Theorem 2 is completed. □

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