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Author(s):

F. F. Liao, X. H. Tang and D. D. Qin

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NEW CONDITIONS ON GROUND STATE SOLUTIONS FOR HAMILTONIAN ELLIPTIC SYSTEMS WITH GRADIENT TERMS

F. F. LIAO, X. H. TANG AND D. D. QIN*

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Abstract. This paper is concerned with the following elliptic system:

$$\begin{cases} -\triangle u + b(x)\nabla u + V(x)u = g(x,v), \\ -\triangle v - b(x)\nabla v + V(x)v = f(x,u), \end{cases}$$

for $x \in \mathbb{R}^N$, where V, b and W are 1-periodic in x, and $f(x,t)$, $g(x,t)$ are Superlinear. In this paper, we give a new technique to show the boundedness of Cerami sequences and establish the existence of ground state solutions with mild assumptions on f and g.

Keywords: Hamiltonian elliptic system, superlinear, ground state solutions, strongly indefinite functionals.
MSC(2010): 35J10; 35J20.

1. Introduction

In this paper, we study the following elliptic system

$$\begin{cases} -\triangle u + b(x)\nabla u + V(x)u = g(x,v), \\ -\triangle v - b(x)\nabla v + V(x)v = f(x,u), \end{cases} \tag{1.1}$$

$x \in \mathbb{R}^N$, where $V \in C(\mathbb{R}^N, \mathbb{R}^N)$, $b = (b_1, \ldots, b_n) \in C(\mathbb{R}^N, \mathbb{R}^N)$ and $f, g \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ are superlinear at infinity. It was called an unbounded Hamiltonian system [3], or an infinite-dimensional Hamiltonian system [4], which can also be obtained from the diffusion system, see [10]. Such a system arises when one is looking for stationary solutions to certain systems of diffusion equtions [15, 21] or systems of optimal control [17]. We make the following assumptions:

(V) $V(x) \in C(\mathbb{R}^N, \mathbb{R}^N)$ is 1-periodic in each of $x_1, x_2, \ldots, x_N$, and $a := \min_{\mathbb{R}^N} V > 0$;

(B) $b(x) \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ is 1-periodic in each of $x_1, x_2, \ldots, x_N$, and $divb = 0$;

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*Corresponding author.

(W1) $f(x,t), g(x,t) \in C(\mathbb{R}^N \times \mathbb{R})$ are 1-periodic in each of $x_1, x_2, \ldots, x_N$, $tf(x,t) \geq 0$, $tg(x,t) \geq 0$, and there exist constants $p \in (2, 2^*)$ and $C > 0$ such that

$$|f(x,t)| \leq C\left(1+|t|^{p-1}\right), |g(x,t)| \leq C\left(1+|t|^{p-1}\right), \quad \left((x,t) \in \mathbb{R}^N \times \mathbb{R}\right).$$

We assume $F(x,t) = \int_0^t f(x,s)ds$ and $G(x,t) = \int_0^t g(x,s)ds$.

(W2) $|f(x,t)| + |g(x,t)| = o(|t|)$, as $|t| \to 0$, uniformly in $x \in \mathbb{R}^N$.

The existence of solutions of Hamiltonian elliptic systems has been a subject of active research in recent years. For the case of a bounded domain the systems like or similar to (1.1) were studied by a number of authors, however they all focused on the case $b \equiv 0$, see [5, 11, 12, 14] and the references therein.

Problems in the whole space $\mathbb{R}^N$ have been considered in various studies, most of which focused on the case $b \equiv 0$, see [1, 2, 5, 9, 13, 19, 20, 26–28, 31–34]. Since the system (1.1) is of Hamiltonian type in $\mathbb{R}^N$, we need to overcome some difficulties. The main difficulty of this problem is the lack of compactness for Sobolev embedding theorem. The usual way to overcome this difficulty is to work on a radically symmetric function space that possesses compact embedding. Using this approach, De Figueiredo and Yang [13] considered this system when $b = 0$ and $V = 1$ and obtained a positive radially symmetric solution that decays exponentially to 0 at infinity. Their results were later generalized by Sirakov [20]. Later, Bartsch and De Figueiredo [5] proved that the system admits infinitely many radial as well as non-radial solutions. By a generalized linking theorem, Li and Yang [19] proved the system has a positive ground state solution for $V = 1$ and an asymptotically quadratic nonlinearity. Another usual way is avoiding the indefinite character of the original functional by using the dual variational method, see for instance Ávila and Yang [1,2] and references therein.

Applying the critical point theory for strongly indefinite functionals developed by Bartsch and Ding, Zhao and coworkers [31, 32] proved the existence and multiplicity of solutions of (1.1) with asymptotically linear and superlinear and periodic assumptions [27, 33, 34]. Using a suitable fractional power of some self-adjoint operator on Sobolev space to define the energy functional, [33] Zhao et al. proved the existence of a ground-state solution with the following monotonous condition:

(Ne) $t \to \frac{f(x,t)}{|t|}$ and $t \to \frac{g(x,t)}{|t|}$ is strictly increasing on $(-\infty, 0) \cup (0, \infty)$.

Zhao and Ding [28] studied the asymptotically quadratic case for (1.1) with periodic or non-periodic potential $V(x)$. They first established a suitable variational framework and obtained the multiplicity of a solution for the non-periodic case. Moreover, without the assumption that the nonlinearity is even in z, they obtained infinitely many geometrically distinct solutions for the periodic case using a reduction method. Yang et al. [26] considered the non-periodic superquadratic case for (1.1) with b is a constant vector and $V = 1$.

Let

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ J_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and let $S = -\triangle + V$ denote the Schrödinger operator. We denote $A_0 := SJ_0$ and

$$A := A_0 + b \cdot \nabla J = \begin{pmatrix} 0 & -\triangle - b \cdot \nabla + V \\ -\triangle + b \cdot \nabla + V & 0 \end{pmatrix}.$$

Then (1.1) can be expressed as

$$Az = H_z(x, z), \quad z = (u, v) \in H^1(\mathbb{R}^N, R^2),$$

where and in the sequel $H(x, z) = F(x, u) + G(x, v)$, $F(x, t) = \int_0^t f(x, s)ds$ and $G(x, t) = \int_0^t g(x, s)ds$.

Recently, Zhang et. al. [30] studied problem (1.1) and obtained the existence of ground state solution (V), (B), (W1), (W2) and the following assumptions:

(H1) $H(x, z)|z|^{-2} \to \infty$ as $|z| \to \infty$ uniformly in x;

(H2) $H(x, z) > 0$, and $\hat{H}(x, z) = \frac{1}{2}(H_z(x, z), z) - H(x, z) > 0$ for all $z \neq 0$, where $(\cdot, \cdot)$ denotes the Euclidean scalar product;

(H3) $(H_z(x, z), w)(z, w) \geq 0$ uniformly in x for all $z, w \in \mathbb{R}^2$;

(H4) $H(x, z) = H(x, w)$ and $(H_z(x, z), w) \leq (H_z(x, z), z)$ uniformly in x if $|z| = |w|$;

(H5) $(H_z(x, z), w) \neq (H_z(x, w), z)$ uniformly in x if $|z| \neq |w|$ and $(z, w) \neq 0$.

Motivated by these researches about Hamiltonian systems, we will continue to study the existence of ground state solutions of problem (1.1). We replace (H2)-(H5) with the following much weaker assumptions:

(W3)

$$\lim_{|(t,s)| \to \infty} \frac{F(x, t) + G(x, s)}{|t|^2 + |s|^2} = \infty, \ a.e. \ x \in \mathbb{R}^N;$$

(W4) there exists a $\eta_0 \in (0, 1)$ such that

$$\frac{1 - \eta^2}{2} tf(x, t) > \int_{\eta t}^{t} f(x, s)ds, \quad \frac{1 - \eta^2}{2} tg(x, t) > \int_{\eta t}^{t} g(x, s)ds, \qquad \forall \ \eta \in [0, \eta_0].$$

Let E be the Hilbert space defined in Section 2. Under assumptions (V), (B), (W1) and (W2), the following functional

$$\Phi(z) := \int_{\mathbb{R}^N} (\frac{1}{2} Az \cdot z - H(x, z))dx, \tag{1.2}$$

is well defined for all $z \in E$, moreover $\Phi \in C^1(E, \mathbb{R})$. Let

$$\mathcal{K} := \{z \in E \setminus \{0\}, \Phi'(z) = 0\}.$$

Now, we are ready to state the main results of this paper:

Theorem 1.1. *Let (V), (B), (W1)-(W4) be satisfied. Then* (1.1) *has a non-trivial* $z_0 \in E$ *such that* $\Phi(z_0) = inf_{\mathcal{K}}\Phi \geq \kappa$, *where* κ *is a positive constant.*

Remark 1.2. Obviously (W3) is fine than (H1), and (W4) implies that $\hat{H}(x, z) \geq 0$ for all $(x, z) \in \mathbb{R}^N \times \mathbb{R}^2$, which is much better than (H2). We point out that condition (W4) first introduced by Tang [24] is weaker than (Ne). In fact, if the weak version of condition (Ne) is satisfied, that is,
(WN) $t \to \frac{f(x,t)}{|t|}$ and $t \to \frac{g(x,t)}{|t|}$ is increasing on $(-\infty, 0) \cup (0, \infty)$.
Then, for any $x \in \mathbb{R}^N$ and $t \neq 0$, set

$$h(\theta) = \frac{1}{2}\theta^2 tf(x, t) - F(x, \theta t).$$

It is easy to check that (WN) implies that $h(1) \geq h(\theta), \forall\ \theta \geq 0$. Then

$$\frac{1-\theta^2}{2}tf(x, t) \geq \int_{\theta t}^{t} f(x, s)ds, \quad \forall\ \theta \geq 0,\ (x, t) \in \mathbb{R}^N \times \mathbb{R}.$$

The same inequality also holds for $g(x, t)$. This shows that (W4) is weaker than (WN).

Remark 1.3. It should be remarked that (W4) is also weaker that the classical Ambrosetti-Rabinowitz condition, (AR) for short. Indeed, if the weaker version of condition (AR) is satisfied, i.e.,
(WAR) there exists a $\mu > 2$ such that

$$0 \leq \mu F(x, t) \leq tf(x, t), \quad 0 \leq \mu G(x, t) \leq tg(x, t), \quad (x, t) \in \mathbb{R}^N \times \mathbb{R}.$$

Then one has

$$\frac{1-\theta^2}{2}tf(x, t) \geq \frac{1}{\mu}tf(x, t) \geq F(x, t) \geq F(x, t) - F(x, \theta t),$$

$$\frac{1-\theta^2}{2}tg(x, t) \geq \frac{1}{\mu}tg(x, t) \geq G(x, t) \geq G(x, t) - G(x, \theta t), \quad \forall\ \theta \in [0, [(\mu-2)/\mu]^{\frac{1}{2}}],$$

which implies that (W4) holds. It is well known that (WN) and (WAR) are complementary. However, the above facts show that they are stronger than (W4). Consequently, (W4) unifies condition (WN), (WAR), and then (Ne) and (AR).

Since the energy functional is strongly indefinite, the classical critical point theory cannot be applied directly, so we will apply a generalized linking theorem for strongly indefinite functionals. In this paper, we give a much more direct and simpler approach to establish the existence of ground state solutions with a new superquadratic condition. The rest of the paper is organized as follows. In section 2, we provide a variational setting. In Section 3, we give the proofs of our theorems.

2. **Variational setting**

We denote by $|\cdot|_p$ the usual L^p-norm and $(\cdot,\cdot)_2$ the L^2 inner product.

Lemma 2.1. *([28, Lemma 2.1, 2.3]) Suppose that (V) and (B) are satisfied. Then the operator A is a self-adjoint operator on $L^2(\mathbb{R}^N,\mathbb{R}^2)$ with domain $\mathcal{D}(A) = H^2(\mathbb{R}^N,\mathbb{R}^2)$, moreover A has only essential spectrum and $\sigma(A) \subset \mathbb{R}\setminus(-a,a)$ is symmetric with respect to the origin.*

Lemma 2.1 implies that the space $L^2 = L^2(\mathbb{R}^N,\mathbb{R}^2)$ has the orthogonal decomposition

$$L^2 = L^- \oplus L^+, \quad z = z^- + z^+, \quad z^\pm \in L^\pm$$

such that A is negative (resp. positive) definite in L^- (resp. L^+). Let $|A|$ denote the absolute of A and $|A|^{\frac{1}{2}}$ be the square root of $|A|$. Let $E = D(|A|^{\frac{1}{2}})$ be the Hilbert space with the inner product

$$\langle z, w\rangle = (|A|^{\frac{1}{2}}z, |A|^{\frac{1}{2}}w)_2$$

and the norm $\|z\| = \langle z,z\rangle^{\frac{1}{2}}$. There is an induced decomposition

$$E = E^- \oplus E^+, \quad E^+ = E\cap L^\pm,$$

that is orthogonal with respect to the inner products $(\cdot,\cdot)_2$ and $\langle\cdot,\cdot\rangle$.

Lemma 2.2. *([28, Lemma 2.4]) $\|\cdot\|$ and $\|\cdot\|_{H^1}$ are equivalent norms. Therefore, E embeds continuously in $L^p(\mathbb{R}^N,\mathbb{R}^2)$ for any $p\in[2,2^*]$ and compactly in $L^p_{loc}(\mathbb{R}^N,\mathbb{R}^2)$ for any $p\in[2,2^*)$, and there exists a constant C_p such that*

$$|z|_p \le C_p\|z\|, \quad \text{for all } z\in E,\ p\in[2,2^*]. \tag{2.1}$$

It follows from Lemma 2.2 that the following functional is well defined for any $z = (u,v)\in E$,

$$\Phi(z) = \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2) - \Psi(z), \tag{2.2}$$

where

$$\Psi(z) = \int_{\mathbb{R}^N} H(x,z)dx = \int_{\mathbb{R}^N}(F(x,u)+G(x,v))\,dx. \tag{2.3}$$

Moreover, $\Phi\in C^1(E,\mathbb{R})$ and for any $z=(u,v), \zeta=(\xi,\eta)\in E$,

$$\begin{aligned}\langle\Phi'(z),\zeta\rangle &= (Az,\zeta)_2 - \int_{\mathbb{R}^N} H_z(x,z)\zeta dx\\ &= (z,\zeta) - \int_{\mathbb{R}^N}(f(x,u)\xi + g(x,v)\eta)dx.\end{aligned} \tag{2.4}$$

Lemma 2.1 also implies that Φ is strongly indefinite and a standard arugment shows that the critical points of Φ are solutions of (1.1)(see [7]).

The following generalized linking theorem plays an important role in proving our main results.

Lemma 2.3. *([7, Lemma 4.5] [18, Lemma 2.1]) Let X be a Hilbert space with $X = X^- \oplus X^+$ and $X^- \perp X^+$, and let $\varphi \in C^1(X, \mathbb{R})$ be of the form*

$$\varphi(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \psi(u), \quad u = u^+ + u^- \in X^- \oplus X^+.$$

Suppose that the following assumptions are satisfied:
(I1) $\psi \in C^1(X, \mathbb{R})$ is bounded from below and weakly sequentially lower semi-continuous;
(I2) ψ' is weakly sequentially continuous;
(I3) there exists $r > \rho > 0$ and $e \in X^+$ with $\|e\| = 1$ such that

$$\kappa := \inf \varphi(S_\rho) > \sup \varphi(\partial Q),$$

where

$$S_\rho = \{u \in X^+ : \|u\| = \rho\}, \quad Q = \{se + v : v \in X^-, s \geq 0, \|se + v\| \leq r\}.$$

Then for some $c \geq k$, there exists a sequence $\{z_n\} \subset E$ satisfying

$$\Phi(z_n) \to c, \quad \|\Phi^{'}(z_n)\|(1 + \|z_n\|) \to 0.$$

Such a sequence is called a Cerami sequence on the level c,or a (C_c) sequence.

Lemma 2.4. *Suppose that (V), (B), (W1) and (W2) are satisfied. Then Ψ is bounded from below, and weakly sequentially lower semicontinuous and Ψ' is weakly sequentially continuous.*

The proof is standard (see [7] and [25]), so we omit it.

Similar to the paper [24] (lemma 2.3), we can get the following Lemma 2.5, whose proof is given in the Appendix.

Lemma 2.5. *Suppose that (V), (B), (W1), (W2) and (W4) are satisfied. Then for $z \in E$, there holds*

$$\begin{aligned} \Phi(z) \quad &\geq \quad \Phi(\theta z^+) + \frac{\theta^2\|z^-\|^2}{2} + \frac{1-\theta^2}{2}\langle \Phi'(z), z\rangle + \theta^2\langle \Phi'(z), z^-\rangle \\ &\qquad -\theta^2 \int_{\theta|z^+|>\eta_0|z|} H_z(x,z)z^+ dx, \qquad \forall\ \theta \geq 0, \end{aligned} \tag{2.5}$$

where η_0 is given in (W4).

3. **Proofs of the main results**

In this section, we give the proofs of our results.

Lemma 3.1. *Suppose that (V), (B), (W1) and (W2) are satisfied. Then there is a $\rho > 0$ such that $k_0 = \inf \Phi(S_{\rho^+}) > 0$, where $S_{\rho^+} = \partial B_\rho \cap E^+$.*

Proof. Given $\varepsilon > 0$, (W1) and (W2) imply the existence of $C_\varepsilon > 0$ such that

$$|f(x,t)| \le \varepsilon|t| + C_\varepsilon|t|^{p-1}, \quad |g(x,t)| \le \varepsilon|t| + C_\varepsilon|t|^{p-1}. \tag{3.1}$$

Therefore

$$|H_z(x,z)| \le \varepsilon|z| + C_\varepsilon|z|^{p-1}, \ \ |H(x,z)| \le \varepsilon|z|^2 + C_\varepsilon|z|^p, \ \forall\ z \in E. \tag{3.2}$$

Then for $z \in E^+$, we have

$$\begin{aligned} \Phi(z) &= \frac{1}{2}\|z\|^2 - \int_{\mathbb{R}^N} H(x,z)dx \\ &\ge \frac{1}{2}\|z\|^2 - (\varepsilon|z|^2 + C_\varepsilon|z|^p) \\ &\ge \frac{1}{2}\|z\|^2 - C_2^2\varepsilon\|z\|^2 - C_p^pC_\varepsilon\|z\|^p, \end{aligned}$$

Choosing an appropiate ε, we see that the desired conclusion holds for some $\rho > 0$. □

Lemma 3.2. *Suppose that (V), (B), (W1) and (W2) are satisfied. Let $e \in E^+$ with $\|e\| = 1$. Then there is a $r_0 > 0$ such that $\sup\Phi(\partial Q) \le 0$, where*

$$Q = \left\{\zeta + se : \zeta \in E^-,\ s \ge 0,\ \|\zeta + se\| \le r_0\right\}. \tag{3.3}$$

Proof. Obviously, $\Phi(z) \le 0$ for $z \in E^-$. Next, it is sufficient to show that $\Phi(z) \to -\infty$ as $z \in E^- \oplus \mathbb{R}e$ and $\|z\| \to \infty$. Suppose that for some sequence $\{z_n\} \subset E^- \oplus \mathbb{R}e$ with $\|z_n\| \to \infty$, there is $M > 0$ such that $\Phi(z_n) \ge -M$ for all $n \in \mathbb{N}$. Set $\xi_n = z_n/\|z_n\| = \xi_n^- + t_ne$, then $\|\xi_n^- + t_ne\| = 1$. Passing to a subsequence, we may assume that $\xi_n \rightharpoonup \xi$ in E, then $\xi_n \to \xi$ a.e. on $\mathbb{R}^N$, $\xi_n^- \rightharpoonup \xi^-$ in E, $t_n \to \bar{t}$, and

$$-\frac{M}{\|z_n\|^2} \le \frac{\Phi(z_n)}{\|z_n\|^2} = \frac{t_n^2}{2} - \frac{1}{2}\|\xi_n^-\|^2 - \int_{\mathbb{R}^N} \frac{H(x,z_n)}{\|z_n\|^2}\mathrm{d}x. \tag{3.4}$$

If $\bar{t} = 0$, then it follows from (3.4) that

$$0 \le \frac{1}{2}\|\xi_n^-\|^2 + \int_{\mathbb{R}^N} \frac{H(x,z_n)}{\|z_n\|^2}\mathrm{d}x \le \frac{t_n^2}{2} + \frac{M}{\|z_n\|^2} \to 0,$$

which yields $\|\xi_n^-\| \to 0$, and so $1 = \|\xi_n\| \to 0$, a contradiction.

If $\bar{t} \neq 0$, then $\xi \neq 0$, it follows from 3.4, (W3) and Fatou's lemma that

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty}\left[\frac{t_n^2}{2} - \frac{1}{2}\|\xi_n^-\|^2 - \int_{\mathbb{R}^N}\frac{H(x,z_n)}{\|z_n\|^2}\mathrm{d}x\right] \\ &= \limsup_{n\to\infty}\left[\frac{t_n^2}{2} - \frac{1}{2}\|\xi_n^-\|^2 - \int_{\mathbb{R}^N}\frac{H(x,z_n)}{|z_n|^2}|\xi_n|^2\mathrm{d}x\right] \\ &\leq \frac{1}{2}\lim_{n\to\infty} t_n^2 - \liminf_{n\to\infty}\int_{\mathbb{R}^N}\frac{H(x,z_n)}{|z_n|^2}|\xi_n|^2\mathrm{d}x \\ &\leq \frac{\bar{t}^2}{2} - \int_{\mathbb{R}^N}\liminf_{n\to\infty}\frac{H(x,z_n)}{|z_n|^2}|\xi_n|^2\mathrm{d}x \\ &= -\infty, \end{aligned}$$

a contradiction. □

Lemma 3.3. *Suppose that (V), (B) and (W1)-(W4) are satisfied. Then there exist a constant* $c_* \in [\kappa_0, \sup\{\Phi(\zeta + se) : \zeta \in E^-,\ s \geq 0\}]$ *and a sequence* $\{z_n\} = \{(u_n, v_n)\} \subset E$ *satisfying*

$$\Phi(z_n) \to c_*, \qquad \|\Phi'(z_n)\|(1 + \|z_n\|) \to 0. \tag{3.5}$$

Proof. Follows directly from Lemmas 2.3, 2.4, 3.1 and 3.2. □

Lemma 3.4. *Suppose that (V), (B) and (W1)-(W4) are satisfied. Then any sequence* $\{z_n\} \subset E$ *satisfying* (3.5) *is bounded in* E.

Proof. To prove the boundedness of $\{z_n\}$, arguing by contradiction, suppose that $\|z_n\| \to \infty$. Let $\xi_n = z_n/\|z_n\|$, then $\|\xi_n\| = 1$. By Lemma 2.2, there exists a constant $C_1 > 0$ such that $\|\xi_n\|_2 \leq C_1$. If

$$\delta := \limsup_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B(y,1)}|\xi_n^+|^2\mathrm{d}x = 0,$$

then by Lion's concentration compactness principle [22] (or [25, Lemma 1.21]), $\xi_n^+ \to 0$ in $L^s(\mathbb{R}^N, \mathbb{R}^2)$ for $2 < s < 2^*$. For any $\varepsilon > 0$, it follows from (3.2) that

$$\begin{aligned} &\lim_{n\to\infty}\frac{R^2}{\|z_n\|}\int_{R|\xi_n^+|>\eta_0|z_n|}H_z(x,z_n)\xi_n^+\mathrm{d}x \\ &\leq \lim_{n\to\infty}\frac{R^2}{\|z_n\|}\int_{R|\xi_n^+|>\eta_0|z_n|}\left(\varepsilon|z_n| + C_\varepsilon|z_n|^{p-1}\right)|\xi_n^+|\mathrm{d}x \\ &\leq \lim_{n\to\infty}\frac{R^2}{\|z_n\|}\int_{R|\xi_n^+|>\eta_0|z_n|}\left(\varepsilon{\eta_0}^{-1}R|\xi_n^+|^2 + C_\varepsilon{\eta_0}^{1-p}R^{p-1}|\xi_n^+|^p\right)\mathrm{d}x \\ &\leq \lim_{n\to\infty}\frac{R^2}{\|z_n\|}\left(\varepsilon{\eta_0}^{-1}R|\xi_n^+|_2^2 + C_\varepsilon{\eta_0}^{1-p}R^{p-1}|\xi_n^+|^p\right) \\ &= 0. \end{aligned} \tag{3.6}$$

Fix $R > [2(1+c_*)]^{1/2}$, for $\varepsilon = 1/4[(RC_1)^2] > 0$, (3.2) implies that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} H(x, R\xi_n^+)\mathrm{d}x \le \varepsilon(RC_1)^2 + R^p C_\varepsilon \lim_{n\to\infty} |\xi_n^+|_p^p = \frac{1}{4}. \tag{3.7}$$

Let $\theta_n = R/\|z_n\|$. Hence, by (3.5)-(3.7) and Lemma 2.5, one has

$$\begin{aligned}
c_* + o(1) &= \Phi(z_n) \\
&\ge \Phi(\theta_n z_n^+) + \frac{\theta_n^2\|z_n^-\|^2}{2} + \frac{1-\theta_n^2}{2}\langle \Phi'(z_n), z_n\rangle + \theta_n^2\langle \Phi'(z_n), z^-\rangle \\
&\quad -\theta^2 \int_{\eta_n|z_n^+|>\theta_0|z_n|} H_z(x, z_n) z_n^+ \mathrm{d}x \\
&= \Phi(R\xi_n^+) + \frac{R^2\|\xi_n^-\|^2}{2} + \left(\frac{1}{2} - \frac{R^2}{2\|z_n\|^2}\right)\langle \Phi'(z_n), z_n\rangle \\
&\quad + \frac{R^2}{\|z_n\|^2}\langle \Phi'(z_n), z_n^-\rangle - \frac{R^2}{\|z_n\|}\int_{R|\xi_n^+|>\eta_0|z_n|} H_z(x, z_n)\xi_n^+ \mathrm{d}x \\
&= \frac{R^2}{2}(\|\xi_n^+\|^2 + \|\xi_n^-\|^2) + \left(\frac{1}{2} - \frac{R^2}{2\|z_n\|^2}\right)\langle \Phi'(z_n), z_n\rangle + \frac{R^2}{\|z_n\|^2}\cdot \\
&\quad \langle \Phi'(z_n), z_n^-\rangle - \frac{R^2}{\|z_n\|}\int_{R|\xi_n^+|>\eta_0|z_n|} H_z(x, z_n)\xi_n^+ \mathrm{d}x - \int_{\mathbb{R}^N} H(x, R\xi_n^+)\mathrm{d}x \\
&= \frac{R^2}{2} - \frac{1}{4} + o(1) > c_* + \frac{3}{4} + o(1),
\end{aligned}$$

which is a contradiction. Thus $\delta > 0$.

Passing to a subsequence, if necessary, we may assume the existence of $k_n \in \mathbb{Z}^N$ such that $\int_{B_{1+\sqrt{N}}(k_n)} |\xi_n^+|^2 dx > \frac{\delta}{2}$. Let $\zeta_n(x) = \xi_n(x + k_n)$. Then

$$\int_{B_{1+\sqrt{N}}(0)} |\zeta_n^+|^2 dx > \frac{\delta}{2}. \tag{3.8}$$

Now we define $\tilde{z}_n(x) = (\tilde{u}_n, \tilde{v}_n) = z_n(x + k_n)$, then $\tilde{z}_n/\|z_n\| = \zeta_n$ and $\|\zeta_n\|^2 = \|\xi_n\|^2$. Passing to a subsequence, we have $\zeta_n \rightharpoonup \zeta$ in E, $\zeta_n \to \zeta$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$, $2 \le s < 2^*$ and $\zeta_n \to \zeta$ a.e. on $\mathbb{R}^N$. Obviously, (3.8) implies that $\zeta^+ \ne 0$. For a.e. $x \in \{y \in \mathbb{R}^N : \zeta^+(y) \ne 0\} := \Omega$, we have $\lim_{n\to\infty} |\tilde{z}_n(x)| = \infty$. Hence, it follows from (2.2), (3.5), (W1), (W3) and the Fatou's lemma that

$$\begin{aligned}
0 &= \lim_{n\to\infty}\frac{c+o(1)}{\|z_n\|^2}=\lim_{n\to\infty}\frac{\Phi(z_n)}{\|z_n\|^2}\\
&= \lim_{n\to\infty}\left[\frac{1}{2}\left(\|\xi_n^+\|^2-\|\xi_n^-\|^2\right)-\int_{\mathbb{R}^N}\frac{H(x,z_n)}{\|z_n\|^2}\mathrm{d}x\right]\\
&= \lim_{n\to\infty}\left[\frac{1}{2}\left(\|\xi_n^+\|^2-\|\xi_n^-\|^2\right)-\int_{\mathbb{R}^N}\frac{H(x,z_n)}{|z_n|^2}|\xi_n|^2\mathrm{d}x\right]\\
&\le \frac{1}{2}-\liminf_{n\to\infty}\int_{\mathbb{R}^N}\frac{H(x,z_n)}{|z_n|^2}|\xi_n|^2\mathrm{d}x\\
&\le \frac{1}{2}-\int_{\Omega}\liminf_{n\to\infty}\frac{H(x,\tilde{z}_n)}{|\tilde{z}_n|^2}|\xi_n|^2\mathrm{d}x=-\infty.
\end{aligned}$$

This contradiction shows that $\{z_n\}$ is bounded. □

Lemma 3.5. *Suppose that (V), (B) and (W1)-(W4) are satisfied. Then $\mathcal{K}\neq\emptyset$, i.e., problem* (1.1) *has a nontrivial solution.*

Proof. Applying Lemmas 3.3 and 3.4, we deduce that there exists a bounded sequence $\{z_n\}\subset E$ satisfying (3.5). Thus there exists a constant $C_2>0$ such that $\|z_n\|_2\le C_2$. If

$$\delta:=\limsup_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|z_n|^2\mathrm{d}x=0, \tag{3.9}$$

then by Lions' concentration compactness principle [22] or [25, Lemma 1.21], $z_n\to 0$ in $L^p(\mathbb{R}^N)$, as $n\to\infty$. Consequently, by (2.1), (2.2), (2.4), (3.2) and (3.5), we have

$$\begin{aligned}
c_*+o(1) &= \Phi(z_n)-\frac{1}{2}\langle\Phi'(z_n),z_n\rangle\\
&= \int_{\mathbb{R}^N}\left(\frac{1}{2}H_z(x,z_n)z_n-H(x,z_n)\right)\mathrm{d}x\\
&\le 2\varepsilon C_2^2\|z_n\|_2^2+2C_\varepsilon|z_n|_p^p=o(1),
\end{aligned}$$

which is a contradiction. Thus $\delta>0$.

Going if necessary to a subsequence, we may assume the existence of $k_n\in\mathbb{Z}^N$ such that $\int_{B_{1+\sqrt{N}}(k_n)}|z_n|^2\mathrm{d}x>\frac{\delta}{2}$. Let us define $\zeta_n(x)=z_n(x+k_n)$ so that

$$\int_{B_{1+\sqrt{N}}(0)}|\zeta_n|^2\mathrm{d}x>\frac{\delta}{2}. \tag{3.10}$$

Since $V(x)$, $f(x,u)$ and $g(x,v)$ are periodic on x, we have $\|\zeta_n\|=\|z_n\|$ and

$$\Phi(\zeta_n)\to c_*,\qquad \|\Phi'(\zeta_n)\|(1+\|\zeta_n\|)\to 0. \tag{3.11}$$

Passing to a subsequence, we have $\zeta_n\rightharpoonup\zeta=(\varphi,\psi)$ in E, $\zeta_n\to\zeta$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and $\zeta_n\to\zeta$ a.e. on $\mathbb{R}^N$. Obviously, (3.10) implies that $\zeta\neq 0$. By a standard argument, we have $\Phi'(\zeta)=0$. Then $\zeta\in\mathcal{K}$, i.e. $\zeta=(\varphi,\psi)$ is a nontrivial solution of (1.1). $\square$

Proof of Theorem 1.1. Lemma 3.5 shows that $\mathcal{K}\neq\emptyset$. Let $\tilde{C}=\inf_{z\in\mathcal{K}}\Phi(z)$. Then, for any $z=(u,v)\in\mathcal{K}$, by (2.2), (2.4) and (W4), we have

$$\begin{aligned}\Phi(z) &= \Phi(z)-\frac{1}{2}\langle\Phi'(z),z\rangle=\int_{\mathbb{R}^N}\left(\frac{1}{2}H_z(x,z)z-H(x,z)\right)\mathrm{d}x\\ &= \int_{\mathbb{R}^N}\left(\frac{1}{2}f(x,u)u-F(x,u)+\frac{1}{2}g(x,v)v-G(x,v)\right)\mathrm{d}x\geq 0.\end{aligned}$$

Therefore, $\tilde{C}\geq 0$. Suppose that $\{z_n\}\subset\mathcal{K}$ such that $\Phi(z_n)\to\tilde{C}$. Then $\langle\Phi'(z_n),\zeta\rangle=0$ for any $\zeta\in E$. According to the proof of Lemma 3.4($c_*>0$ is not necessary), we can certify that $\{z_n\}$ is bounded in E. Denote δ as in (3.9). If $\delta=0$, then Lions' concentration compactness principle implies that $z_n\to 0$ in $L^s(\mathbb{R}^N,\mathbb{R}^2)$ for any $s\in(2,2^*)$. By (3.2) and Lemma 2.2, it is easy to check

$$\int_{\mathbb{R}^N}\left(\frac{1}{2}H_z(x,z_n)(z_n^+-z_n^-)\right)\mathrm{d}x\to 0,\quad n\to\infty.$$

Therefore,

$$\|z_n\|^2=\langle\Phi'(z_n),z_n^+-z_n^-\rangle+\int_{\mathbb{R}^N}\left(\frac{1}{2}H_z(x,z_n)(z_n^+-z_n^-)\right)\mathrm{d}x=o(1).$$

On the other hand,

$$\begin{aligned}\|z_n\| &= \langle\Phi'(z_n),z_n^+-z_n^-\rangle+\int_{\mathbb{R}^N}\left(\frac{1}{2}H_z(x,z_n)(z_n^+-z_n^-)\right)\mathrm{d}x\\ &\leq \varepsilon C_2^2\|z_n\|^2+C_\varepsilon C_p^p\|z_n\|^p,\end{aligned}\tag{3.12}$$

which implies that

$$\|z_n\|^2\geq\left(\frac{1-\varepsilon C_2^2}{C_\varepsilon C_p^p}\right)^{1/(p-2)}>0,\ \text{for some appropriate }\varepsilon.$$

This is a contradiction. Thus $\delta>0$. After a suitable $\mathbb{Z}^N$-translation, a subsequence of $\{z_n\}$ converges weakly to some $z_0\in\mathcal{K}$. Thus $\Phi(z_0)\geq\tilde{C}$. It follows from (W4) and Fatou's lemma that

$$\begin{aligned}\tilde{C} &= \lim_{n\to\infty}\left[\Phi(z_n)-\frac{1}{2}\langle\Phi'(z_n),z_n\rangle\right]=\lim_{n\to\infty}\int_{\mathbb{R}^N}\left[\frac{1}{2}H_z(x,z_n)z_n-H(x,z_n)\right]\\ &\geq \int_{\mathbb{R}^N}\lim_{n\to\infty}\left[\frac{1}{2}H_z(x,z_n)z_n-H(x,z_n)\right]\\ &= \int_{\mathbb{R}^N}\left[\frac{1}{2}H_z(x,z_0)z_0-H(x,z_0)\right]=\Phi(z_0).\end{aligned}$$

This show that $\Phi(z_0) = \tilde{C} = \inf_{z\in\mathcal{K}} \Phi(z)$. Since (3.12) holds also for z_0, let

$$\varepsilon_0 = \frac{\eta_0}{2(2+\eta_0)C_2^2},\ \beta_0 = \left(\frac{1}{\eta_0^{p-1}} + \frac{1}{p}\right) C_{\varepsilon_0} C_p^p. \tag{3.13}$$

Then (3.12) implies that

$$2p\beta_0\|z_0\|^{p-2} > 2C_{\varepsilon_0}C_p^p\|z_0\|^{p-2} \geq 1. \tag{3.14}$$

Let $\theta_0 = \frac{1}{\|z_0\|}(2p\beta_0)^{-1/(p-2)}$. Then (3.14) implies that $0 < \theta_0 < 1$. Since $z_0 \in \mathcal{K}$, it follows from (2.1), (2.5), (3.2) and (3.13) that

$$\begin{aligned}
\Phi(z_0) &\geq \frac{\theta_0^2\|z_0\|^2}{2} - \theta_0^2\int_{\theta_0|z_0^+|>\eta_0|z_0|} H_z(x,z_0)z_0^+\mathrm{d}x - \int_{\mathbb{R}^N} H(x,\theta_0 z_0^+)\mathrm{d}x \\
&\geq \frac{\theta_0^2\|z_0\|^2}{2} - \left(\frac{\theta_0}{\eta_0} + \frac{1}{2}\right)\varepsilon_0\theta_0^2|z_0^+|_2^2 - \left(\frac{\theta_0}{\eta_0^{p-1}} + \frac{1}{p}\right) C_{\varepsilon_0}\theta_0^p|z_0^+|_p^p \\
&\geq \frac{\theta_0^2\|z_0\|^2}{2} - \left(\frac{1}{\eta_0} + \frac{1}{2}\right)\varepsilon_0 C_2^2\theta_0^2\|z_0\|^2 - \left(\frac{1}{\eta_0^{p-1}} + \frac{1}{p}\right) C_{\varepsilon_0}C_p^p\theta_0^p\|z_0\|^p \\
&= \frac{\theta_0^2\|z_0\|^2}{4} - \beta_0\theta_0^p\|z_0\|^p = \left(\frac{1}{2} - \frac{1}{p}\right) 2^{\frac{-p}{p-2}} (\beta_0 p)^{\frac{-2}{p-2}} := \kappa. \qquad (3.15)
\end{aligned}$$

This completes the proof. □

4. Appendices

Proof. Here we give a proof of Lemma 2.5 Fix $x \in \mathbb{R}^N$ and $t, t' \in \mathbb{R}$. Set

$$h(\theta) = \frac{1+\theta^2}{2} f(x,t)t - \theta^2 f(x,t)t' + F(x,\theta t') - F(x,t).$$

If $tt' \leq 0$, then it follows from (W1) that

$$\begin{aligned}
h(\theta) &= \frac{1+\theta^2}{2} f(x,t)t - \theta^2 f(x,t)t' + F(x,\theta t') - F(x,t) \\
&\geq \frac{1+\theta^2}{2} f(x,t)t - F(x,t) \geq 0, \quad \theta \geq 0. \qquad (4.1)
\end{aligned}$$

If $tt^{'} > 0$, set $\eta = \theta t^{'}/t$, then it follows from (W1) and (W4) that

$$
\begin{aligned}
h(\theta) &= \frac{1+\theta^2}{2} f(x,t)t - \theta^2 f(x,t)t^{'} + F(x,\theta t^{'}) - F(x,t) \\
&= \frac{1+\theta^2 - 2\eta\theta}{2} f(x,t)t - \int_{\eta t}^{t} F(x,s)ds \\
&= \frac{(\eta-\theta)^2}{2} f(x,t)t + \frac{1-\eta^2}{2} f(x,t)t - \int_{\eta t}^{t} F(x,s)ds \\
&\geq \frac{1-\eta^2}{2} f(x,t)t - \int_{\eta t}^{t} F(x,s)ds \\
&\geq 0, \quad \forall\ \theta \geq 0, \ \ \theta t^{'}/t \leq \eta_0. \qquad (4.2)
\end{aligned}
$$

Combining the above two cases, for any $\theta \geq 0$, $\theta |t^{'}| \leq \eta_0 |t|$, one has

$$
\frac{1+\theta^2}{2} f(x,t)t - \theta^2 f(x,t)t^{'} + F(x,\theta t^{'}) - F(x,t) \geq 0 \qquad (4.3)
$$

and the same inequality also holds for $g(x,t)$ by a similar argument.

Let $z^+ = (u_1, v_1)$ and $z^- = (u_2, v_2)$. It follows from (W4) and (4.3), for any $\theta \geq 0$ that

$$
\begin{aligned}
& \Phi(z) - \Phi(\theta z^+) \\
= & \frac{1}{2}\left[(Az,z)_2 - (A\theta z, z^+)_2\right] \\
& + \int_{\mathbb{R}^N} [F(x,\theta u_1) + G(x,\theta v_1)] - \int_{\mathbb{R}^N} [F(x,u) + G(x,v)] \\
= & \frac{1}{2}\left[(1-\theta^2)(Az,z)_2 + \theta^2 (Az,z^-)\right] \\
& + \int_{\mathbb{R}^N} [F(x,\theta u_1) + G(x,\theta v_1)] - \int_{\mathbb{R}^N} [F(x,u) + G(x,v)] \\
= & \frac{1-\theta^2}{2}(Az,z)_2 + \frac{\theta^2}{2}\|z^-\|^2 + \theta^2 (Az,z^-)_2 \\
& + \int_{\mathbb{R}^N} [F(x,\theta u_1) + G(x,\theta v_1)] - \int_{\mathbb{R}^N} [F(x,u) + G(x,v)]
\end{aligned}
$$

$$
\begin{aligned}
= \;& \frac{1-\theta^2}{2}\langle \Phi'(z), z\rangle + \frac{\theta^2}{2}\|z^-\|^2 + \theta^2\langle \Phi'(z), z^-\rangle \\
& + \int_{\mathbb{R}^N} \left\{ \frac{1-\theta^2}{2} f(x,u)u + \theta^2 f(x,u)u_2 + F(x,\theta u_1) - F(x,u) \right\} \\
& + \int_{\mathbb{R}^N} \left\{ \frac{1-\theta^2}{2} g(x,v)v + \theta^2 g(x,v)v_2 + G(x,\theta v_1) - G(x,u) \right\} \\
= \;& \frac{1-\theta^2}{2}\langle \Phi'(z), z\rangle + \frac{\theta^2}{2}\|z^-\|^2 + \theta^2\langle \Phi'(z), z^-\rangle \\
& + \int_{\mathbb{R}^N} \left\{ \frac{1+\theta^2}{2} f(x,u)u - \theta^2 f(x,u)u_1 + F(x,\theta u_1) - F(x,u) \right\} \\
& + \int_{\mathbb{R}^N} \left\{ \frac{1+\theta^2}{2} g(x,v)v - \theta^2 g(x,v)v_1 + G(x,\theta v_1) - G(x,u) \right\} \\
\geq \;& \frac{1-\theta^2}{2}\langle \Phi'(z), z\rangle + \frac{\theta^2}{2}\|z^-\|^2 + \theta^2\langle \Phi'(z), z^-\rangle \\
& + \int_{\theta|u_1|>\eta_0|u|} \left\{ \frac{1+\theta^2}{2} f(x,u)u - \theta^2 f(x,u)u_1 + F(x,\theta u_1) - F(x,u) \right\} \\
& + \int_{\theta|v_1|>\eta_0|v|} \left\{ \frac{1+\theta^2}{2} g(x,v)v - \theta^2 g(x,v)v_1 + G(x,\theta v_1) - G(x,u) \right\} \\
\geq \;& \frac{1-\theta^2}{2}\langle \Phi'(z), z\rangle + \frac{\theta^2}{2}\|z^-\|^2 + \theta^2\langle \Phi'(z), z^-\rangle \\
& + \theta^2 \left\{ \int_{\theta|u|>\eta_0|u|} f(x,u)u_1 + \int_{\theta|v|>\eta_0|v|} g(x,v)v_1 \right\} \\
\geq \;& \frac{1-\theta^2}{2}\langle \Phi'(z), z\rangle + \frac{\theta^2}{2}\|z^-\|^2 + \theta^2\langle \Phi'(z), z^-\rangle + \theta^2 \int_{\theta|z^+|>\eta_0|z|} H_z(x,z)z^+.
\end{aligned}
$$

This shows that (2.5) holds. □

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(Fangfang Liao) School of Mathematics and Statistics Central South University Changsha, 410083, Hunan
Department of Mathematics, Xiangnan University, Chenzhou, 423000, Hunan, P.R. China
E-mail address: `liaofangfang1981@126.com`

(X. H. Tang) School of Mathematics and Statistics Central South University Changsha, 410083, Hunan, P.R. China
E-mail address: `tangxh@mail.csu.edu.cn`

(Dong Dong Qin) School of Mathematics and Statistics Central South University Changsha, 410083, Hunan, P.R. China
E-mail address: `qindd132@163.com`