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DERIVATIONS INTO N-TH DUALS OF IDEALS OF BANACH ALGEBRAS

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Abstract. We introduce two notions of amenability for a Banach algebra $\mathcal{A}$. Let $n \in \mathbb{N}$ and I be a closed two-sided ideal in $\mathcal{A}$. $\mathcal{A}$ is $n - I-$weakly amenable if the first cohomology group of $\mathcal{A}$ with coefficients in the n-th dual space $I^{(n)}$ is zero; i.e., $H^1(\mathcal{A}, I^{(n)}) = \{0\}$. Further, $\mathcal{A}$ is n-ideally amenable if $\mathcal{A}$ is $n-I-$weakly amenable for every closed two-sided ideal I in $\mathcal{A}$. We find some relationships of $n - I-$ weak and $m - J-$ weak amenabilities for some different m and n or for different closed ideals I and J of $\mathcal{A}$.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}$-module; that is, X is a Banach space and an $\mathcal{A}$-module such that the module operations $(a, x) \longmapsto ax$ and $(a, x) \longmapsto xa$ from $\mathcal{A}\times X$ into X are jointly continuous. The dual space X^* of X is also a Banach $\mathcal{A}$-module by the follwing module actions:

$$\langle x, ax^*\rangle = \langle xa, x^*\rangle,$$
$$\langle x, x^*a\rangle = \langle ax, x^*\rangle, \qquad (a \in \mathcal{A},\ x \in X,\ x^* \in X^*).$$

In particular, for every $n \in \mathbb{N}$, the n-th dual $X^{(n)}$ of X is a Banach $\mathcal{A}$-module, and so for every closed ideal I of $\mathcal{A}$, I is a Banach $\mathcal{A}$-module

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and $I^{(n)}$ is a dual $\mathcal{A}$-module for every $n \in \mathbb{N}$.

Let X be a Banach $\mathcal{A}$-module. Then a continuous linear map $D : \mathcal{A} \longrightarrow \mathcal{X}$ is called a derivation if

$$D(ab) = a \cdot D(b) + D(a) \cdot b \qquad (a, b \in \mathcal{A}). \tag{1.1}$$

For $x \in X$, we define $\delta_x : \mathcal{A} \longrightarrow \mathcal{X}$ as follows:

$$\delta_x(a) = a \cdot x - x \cdot a \qquad (a \in \mathcal{A}).$$

It is easy to show that δ_x is a derivation. Such derivations are called inner derivations. We denote the set of continuous derivations from $\mathcal{A}$ into X by $Z^1(\mathcal{A}, \mathcal{X})$ and the set of inner derivations by $B^1(\mathcal{A}, \mathcal{X})$. We denote space by $H^1(\mathcal{A}, \mathcal{X})$ and the quotient space by $Z^1(\mathcal{A}, \mathcal{X})/\mathcal{B}^\infty(\mathcal{A}, \mathcal{X})$. The space $H^1(\mathcal{A}, \mathcal{X})$ is called the first cohomology group of $\mathcal{A}$ with coefficients in X. $\mathcal{A}$ is called amenable if every derivation from $\mathcal{A}$ into every dual $\mathcal{A}$-module is inner; this definition was introduced by B. E. Johnson in [18] (see [22] and [17]). $\mathcal{A}$ is called weakly amenable if, $H^1(\mathcal{A}, \mathcal{A}^*) = \{0\}$ (see [20], [5], [12], [13] and [14]). Bade, Curtis and Dales [2] have introduced the concept of weak amenability for commutative Banach algebras. Let $n \in \mathbb{N}$. A Banach algebra $\mathcal{A}$ is called n-weakly amenable if, $H^1(\mathcal{A}, \mathcal{A}^{(n)}) = \{0\}$. Dales, Ghahramani and Gronbaek started the concept of n-weak amenability of Banach algebras in [3]. A Banach algebra $\mathcal{A}$ is called ideally amenable if $H^1(\mathcal{A}, I^*) = \{0\}$, for every closed ideal I of $\mathcal{A}$ (see [8]). Here, we shall study $H^1(\mathcal{A}, I^{(n)})$ for a closed ideal I of $\mathcal{A}$. The following definition describes the main new property in our work.

Definition 1.1. Let $\mathcal{A}$ be a Banach algebra, $n \in \mathbb{N}$ and I be a closed two-sided ideal in $\mathcal{A}$. Then $\mathcal{A}$ is $n-I-$weakly amenable if $H^1(\mathcal{A}, I^{(n)}) = \{0\}$, $\mathcal{A}$ is n-ideally amenable if $\mathcal{A}$ is $n - I-$weakly amenable for every closed two-sided ideal I in $\mathcal{A}$ and $\mathcal{A}$ is permanently ideally amenable if $\mathcal{A}$ is $n-I-$weakly amenable for every closed two-sided ideal I in $\mathcal{A}$ and for each $n \in \mathbb{N}$.

Example 1.2. Let $\mathcal{A} = \ell^1(\mathbb{N})$. We define the product on $\mathcal{A}$ by $f \cdot g = f(1)g$ $(f, g \in \mathcal{A})$. $\mathcal{A}$ is a Banach algebra with this product and norm $\| \cdot \|_1$. Let I be a closed two-sided ideal of $\mathcal{A}$. It is easy to see that if $I \neq \mathcal{A}$, then $I \subseteq \{f \in \mathcal{A};\ f(1) = 0\}$. Then, the right module action of $\mathcal{A}$ on I is trivial, and therefore the right module action of $\mathcal{A}$ on $I^{(2k)}$ is trivial for every $k \in \mathbb{N}$. On the other hand, $\mathcal{A}$ having left identity, by proposition 1.5 of [18], we have $H^1(\mathcal{A}, I^{(2k+1)}) = \{0\}(k \geq 0)$. If $I = \mathcal{A}$ then by Assertion 2 of [23], $H^1(\mathcal{A}, I^{(2k+1)}) = \{0\}, (k \geq 0)$. Thus, for every $k \geq 0$, $\mathcal{A}$ is $2k+1$ ideally amenable. It is well known that $\mathcal{A}$ is not 2-weakly amenable (see [23]). Thus, $\mathcal{A}$ is not 2-ideally amenable.

The second dual space $\mathcal{A}^{**}$ of a Banach algebra $\mathcal{A}$ admits a Banach algebra product known as first (left) Arens product. We briefly recall the definition of this product. For $m, n \in \mathcal{A}^{**}$, their first (left) Arens product indicated by mn is given by

$$\langle mn, f \rangle = \langle m, nf \rangle \quad (f \in \mathcal{A}^*),$$

where $nf \in \mathcal{A}^*$ is defined by

$$\langle nf, a \rangle = \langle n, fa \rangle \quad (a \in \mathcal{A}) \quad [\mathcal{A}].$$

Let X be a Banach $\mathcal{A}-$module. We can extend the actions of $\mathcal{A}$ on X to actions of $\mathcal{A}^{**}$ on X^{**} via

$$a''.x'' = w^* - \lim_i \lim_j a_i\, x_j$$

and

$$x''.a'' = w^* - \lim_j \lim_i x_j\, a_i,$$

such that (a_i) and (x_j) are nets in $\mathcal{A}$ and X, respectively, and that $a'' = w^* - \lim_i a_i,\ \ x'' = w^* - \lim_j x_j$.

Definition 1.3. Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}-$module. We define the topological center of the right module action of $\mathcal{A}$ on X as follows:

$$Z_{\mathcal{A}}(X^{**}) := \{x'' \in X^{**} \ : \text{ the mapping } a'' \mapsto x''.a'' : \\ \mathcal{A}^{**} \to X^{**} \text{ is } weak^* - weak^* \text{ continuous } \}.$$

The right module action of $\mathcal{A}$ on X is Arens regular if and only if $Z_{\mathcal{A}}(X^{**}) = X^{**}$ (see [1] and [6]). For a Banach algebra $\mathcal{A}$, the set $Z_{\mathcal{A}}(\mathcal{A}^{**})$ is the topological center of $\mathcal{A}^{**}$ with the first Arens product.
Let $\mathcal{A}$ be a Banach algebra and let X be a Banach $\mathcal{A}-$module. Set $P : X^{****} \to X^{**}$ the adjoint of the inclusion map $i : X^* \to X^{***}$. Then, we have the following Theorem.

Theorem 1.4. *Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}-$module. Suppose that $Z_{\mathcal{A}}(X^{**}) = X^{**}$. Then the following assertions hold.*

(i) $P : X^{****} \to X^{**}$ is an $\mathcal{A}^{**}-$module morphism.
(ii) If $D : \mathcal{A} \to \mathcal{X}^{**}$ is a derivation, then there exists a derivation $\tilde{D} : \mathcal{A}^{**} \to X^{**}$ with $\tilde{D}$ an extension of D.

Proof. (i) conclude from Proposition 1.8 of [3]. For (ii), we know that $D'' : \mathcal{A}^{**} \to X^{****}$, the second adjoint of D, is a derivation (see for example Proposition 1.7 of [3]). By (i), $P \circ D''$ is a derivation from $\mathcal{A}^{**}$ into X^{**}.

Corollary 1.5. *Let $\mathcal{A}$ be an Arens regular Banach algebra. If for every ideal I of $\mathcal{A}^{**}$, $H^1(\mathcal{A}^{**}, I^{**}) = \{0\}$, then $\mathcal{A}$ is 2-ideally amenable.*

For convenience, we will write $x \mapsto J(x)$ for the canonical embedding of a Banach space into its second dual. We find some relations between m and n-ideal amenabilities of a Banach algebra.

Theorem 1.6. *Let $\mathcal{A}$ be a Banach algebra and I be a closed ideal of $\mathcal{A}$. For each $n \in \mathbb{N}$, if $\mathcal{A}$ is $n{+}2{-}I{-}$weakly amenable then $\mathcal{A}$ is $n{-}I{-}$weakly amenable.*

Proof. Let $D : \mathcal{A} \to I^{(n)}$ be a derivation. Since $J : I^{(n)} \to I^{(n+2)}$ is an $\mathcal{A}$-module homomorphism, then $J \circ D : \mathcal{A} \to I^{(n+2)}$ is a derivation. Therefore, there exists $F \in I^{(n+2)}$ such that $J \circ D = \delta_F$. Let $P : I^{(n+2)} \to I^{(n)}$ be the above projective. Then, for every $a \in \mathcal{A}$, we have $D(a) = P \circ J \circ D(a) = a \cdot P(F) - P(F) \cdot a$. Thus, $D = \delta_{P(F)}$.

Corollary 1.7. *Let $\mathcal{A}$ be a Banach algebra, and $n \in \mathbb{N}$. If $\mathcal{A}$ is $n + 2-$ideally amenable then $\mathcal{A}$ is $n-$ideally amenable.*

Theorem 1.8. *Let $\mathcal{A}$ be a Banach algebra and I be a closed two sided ideal of $\mathcal{A}$ with a bounded approximate identity. If $\mathcal{A}$ is n-ideally amenable (or permanently ideally amenable) then I is n-ideally amenable (or permanently ideally amenable).*

Proof. Since I has bounded approximate identity, then by Cohen factorization Theorem for every closed ideal J of I, we have $JI = IJ = J$. Then, J is an ideal of $\mathcal{A}$. Let $D : I \longrightarrow J^{(n)}$ be a derivation. By [22, Proposition 2.1.6], D can be extend to a derivation $\tilde{\mathbf{D}} : \mathcal{A} \longrightarrow J^{(n)}$. So, there is $a_n m \in J^{(n)}$ such that $\tilde{\mathbf{D}} = \delta_m$. Then, $D(i) = \tilde{\mathbf{D}}(i) = \delta_m(i)$ for each $i \in I$. Thus, D is inner.

Theorem 1.9. *Let $\mathcal{A}$ be a Banach algebra and $n \in \mathbb{N}$. Let I be a closed ideal of $\mathcal{A}$ with $Z_{\mathcal{A}}(I^{(2n)}) = I^{(2n)}$. Suppose that $H^1(\mathcal{A}^{**}, I^{(2n)}) = \{0\}$. Then $\mathcal{A}$ is 2n-2-I-weakly amenable.*

Proof. Let $D : \mathcal{A} \longrightarrow I^{(2n-2)}$ be a derivation. Then, by Theorem 1.3, there exists an extension $\tilde{D} : \mathcal{A}^{**} \longrightarrow I^{(2n)}$ such that $\tilde{D}$ is a (bounded) derivation. Thus, $\tilde{D}$ is inner, and so is D.

Theorem 1.10. *Let $\mathcal{A}$ be a Banach algebra with a left bounded approximate identity. Let I be a closed ideal of $\mathcal{A}$ and $\mathcal{A}$ be an ideal of $\mathcal{A}^{**}$. If I is left strongly irregular (i.e., $Z_t(I^{**}) = I$) and $\mathcal{A}$ is $I-$weakly amenable then $\mathcal{A}$ is $3 - I-$weakly amenable.*

Proof. First, since we have the following $\mathcal{A}-$module direct sum decomposition,

$$I^{***} = \widehat{I^{*}} \oplus \widehat{I}^{\perp},$$

then we have,

$$H^1(\mathcal{A}, \mathcal{I}^{***}) = H^1(\mathcal{A}, \widehat{\mathcal{I}^{*}}) + H^1(\mathcal{A}, \widehat{\mathcal{I}}^{\perp}).$$

We have to show that $H^1(\mathcal{A}, \widehat{\mathcal{I}}^{\perp}) = \{0\}$. To this end, let $a \in \mathcal{A}$, $i'' \in I^{**}$ and $\pi : I \to \mathcal{A}$ be the inclusion map. Then, by Lemma 3.3 of [9],

$$i''a = \pi''(i'')\widehat{a} \in \pi''(I^{**}) \cap \widehat{\mathcal{A}} = \widehat{I}.$$

Then, the right module action of $\mathcal{A}$ on $\widehat{I}^{\perp}$ is trivial. Now, let $D : \mathcal{A} \to \widehat{\mathcal{I}}^{\perp}$ be a derivation. Suppose (e_α) is a left bounded approximate identity for $\mathcal{A}$. Since $\widehat{I}^{\perp}$ is a $weak^{*}-$closed subspace of I^{***}, then we take $weak^{*} - \lim_\alpha D(e_\alpha) = F \in \widehat{I}^{\perp}$. Thus, for every $a \in \mathcal{A}$, we have

$$D(a) = \lim_{\alpha} D(e_\alpha a) = Fa = Fa - aF = \delta_F(a).$$

Let $\mathcal{A}^{\#}$ be the unitization of $\mathcal{A}$. We know that $\mathcal{A}$ is amenable if and only if $\mathcal{A}^{\#}$ is amenable. If $\mathcal{A}$ is weakly amenable then $\mathcal{A}^{\#}$ is weakly amenable [3], and the weak amenability of $\mathcal{A}^{\#}$ does not imply the weak amenability of $\mathcal{A}$ [21]. Also, Gordji and Yazdanpanah have shown that $\mathcal{A}$ is ideally amenable if and only if $\mathcal{A}^{\#}$ is ideally amenable [8]. In the following, we will take the same result for n-ideal amenability.

Proposition 1.11. *Let $\mathcal{A}$ be a Banach algebra, and $n \in \mathbb{N}$. Then the following assertions hold.*
(i) *If $\mathcal{A}^{\#}$ is n-ideally amenable, then $\mathcal{A}$ is n-ideally amenable.*
(ii) *If $\mathcal{A}$ is $2n-1-$ideally amenable, then $\mathcal{A}^{\#}$ is $2n-1-$ideally amenable.*
(iii) *If $\mathcal{A}$ is commutative and n-ideally amenable, then $\mathcal{A}^{\#}$ is n-ideally amenable.*

Proof. (i): Let $\mathcal{A}^{\#}$ be n-ideally amenable, and I be a closed ideal of $\mathcal{A}$. Let $D : \mathcal{A} \longrightarrow I^{(n)}$ be a derivation. It is easy to show that I is an ideal of $\mathcal{A}^{\#}$. We define $\tilde{\mathbf{D}} : \mathcal{A}^{\#} \longrightarrow I^{(n)}$ by $\tilde{\mathbf{D}}(a+\alpha) = D(a)$, $(a \in \mathcal{A},\ \alpha \in \mathbb{C})$. Then $\tilde{\mathbf{D}}$ is a derivation. Since $\mathcal{A}^{\#}$ is n-ideally amenable, then $\tilde{\mathbf{D}}$ is inner, and hence D is inner. For (ii), let $\mathcal{A}$ be $2n-1-$ideally amenable and I be a closed ideal of $\mathcal{A}^{\#}$. First, we know that $\mathcal{A}$ is $2n-1-$weakly amenable. Then, by proposition 1.4 of [3], $\mathcal{A}^{\#}$ is $2n-1-$weakly amenable. Thus, $\mathcal{A}^{\#}$ is $2n-1-I-$weakly amenable whenever $I = \mathcal{A}^{\#}$. Let $I \neq \mathcal{A}^{\#}$. Then $1 \notin I$ and I is an ideal of $\mathcal{A}$. If $D : \mathcal{A}^{\#} \longrightarrow I^{(n)}$ is a derivation, then $D(1) = 0$ and D drops to a derivation from $\mathcal{A}$ into $I^{(n)}$, and hence D is inner. The proof of (iii) is similar to the one given for (ii).

2. Commutative Banach algebras

We know that a commutative Banach algebra $\mathcal{A}$ is weakly amenable if and only if every derivation from $\mathcal{A}$ into a symmetric Banach $\mathcal{A}$-module is zero (Theorem 1.5 of [2]). Thus, we have the following Theorem.

Theorem 2.1. *Let $\mathcal{A}$ be a commutative Banach algebra. Then the following assertions are equivalent.*
(i) *$\mathcal{A}$ is weakly amenable.*
(ii) *$\mathcal{A}$ is 2k+1-weakly amenable for some $k \in \mathbb{N} \cup \{0\}$.*
(iii) *$\mathcal{A}$ is ideally amenable.*
(iv) *$\mathcal{A}$ is 2k+1-ideally amenable for some $k \in \mathbb{N} \cup \{0\}$.*
(v) *$\mathcal{A}$ is permanently ideally amenable.*

Theorem 2.2. *Let $\mathcal{A}$ be a commutative Banach algebra and let $n \in \mathbb{N}$. Then the following assertions are equivalent.*
(i) *$\mathcal{A}$ is 2n-weakly amenable.*
(ii) *$\mathcal{A}$ is 2n-ideally amenable.*

Proof. (ii) $\Rightarrow$ (i): This is obvious. For (i) $\Rightarrow$ (ii), let $\mathcal{A}$ be $2n$-weakly amenable and I be a closed two sided ideal of $\mathcal{A}$. We let $\pi : I \longrightarrow \mathcal{A}$ be the natural inclusion map. Then $\pi^{(2n)} : I^{(2n)} \longrightarrow \mathcal{A}^{(2n)}$, the $2n$-th adjoint of π, is $\mathcal{A}-$module morphism. Let $D : \mathcal{A} \longrightarrow I^{(2n)}$ be a derivation. Then $\pi^{(2n)} oD : \mathcal{A} \longrightarrow \mathcal{A}^{(2n)}$ is a derivation. Since $\mathcal{A}^{(2n)}$ is symmetric $\mathcal{A}-$module and $H^1(\mathcal{A}, \mathcal{A}^{(2n)}) = \{0\}$, then $\pi^{(2n)} oD = 0$. Therefore, $D = 0$.

Corollary 2.3. *Let $\mathcal{A}$ be a commutative Banach algebra which is Arens regular, and suppose that $\mathcal{A}^{**}$ is semisimple. Then $\mathcal{A}$ is 2-ideally amenable.*

Proof. By Corollary 1.11 of [3], $\mathcal{A}$ is 2-weakly amenable. Then, by Theorem 2.2, $\mathcal{A}$ is 2-ideally amenable.

Corollary 2.4. *Let $\mathcal{A}$ be a commutative Banach algebra such that $\mathcal{A}^{(2n)}$ is Arens regular, and $H^1(\mathcal{A}^{(2n+2)}, \mathcal{A}^{(2n+2)}) = \{0\}$ for each $n \in \mathbb{N}$. Then, $\mathcal{A}$ is 2n-ideally amenable for each $n \in \mathbb{N}$.*

Proof. By Corollary 1.12 of [3], $\mathcal{A}$ is $2n$-weakly amenable for each $n \in \mathbb{N}$. Then, by Theorem 2.2, $\mathcal{A}$ is $2n$-ideally amenable for each $n \in \mathbb{N}$.

Corollary 2.5. *Every uniform Banach algebra is 2n-ideally amenable for each $n \in \mathbb{N}$.*

Proof. Applying Theorem 2.2 above and Theorem 3.1 of [3], the proof is easily obtained.

Let $\mathbb{D} = \{F \in \mathbb{C} : |F| <1\}$ be the open unit disc and $A(\overline{\mathbb{D}})$ be the disc algebra. It follows from Corollary 2.5 above and page 35 of [3] that $A(\overline{\mathbb{D}})$ is a 2-ideally amenable Banach function algebra which is not ideally amenable.

3. $\mathcal{C}^*-$algebras

It is well known that every $\mathcal{C}^*-$algebra is ideally amenable [8, Corollary 2.2]. Also, a $\mathcal{C}^*-$algebra is amenable if and only if it is nuclear ([16]).
As in [3, Theorem 2.1], we have the following theorem.

Theorem 3.1. *Every C^*-algebra is permanently weakly amenable.*

We can not show that every C^*-algebra is permanently ideally amenable, but we have the following theorem.

Theorem 3.2. *Let $n = 2k+1$ $(k \in \mathbb{N}\bigcup\{0\})$. Then every C^*-algebra is n-ideally amenable.*

Proof. Let $\mathcal{A}$ be a C^*-algebra and I be a closed ideal of $\mathcal{A}$. Since $\mathcal{A}$ is ideally amenable, then $H^1(\mathcal{A}, I^*) = \{0\}$. Now, we will show that $\mathcal{A}$ is $n+2-I-$ weakly amenable if it is $n-I-$ weakly amenable $(n = 2k+1)$. Let $D : \mathcal{A} \to I^{(n+2)}$ be a derivation. First, we show that D'' is a derivation. Let $a'', b'' \in \mathcal{A}^{**}$. Then, there are nets (a_α) and (b_β) in $\mathcal{A}$ such that they converge respectively to a'' and b'' in the *weak**-topology of $\mathcal{A}^{**}$. Then,

$$\begin{aligned} D''(a''b'') &= weak^* lim_\alpha lim_\beta D(a_\alpha b_\beta) \\ &= weak^* lim_\alpha lim_\beta D(a_\alpha) b_\beta + weak^* lim_\alpha lim_\beta a_\alpha D(b_\beta) \\ &= D''(a'').b'' + lim_\alpha a_\alpha . D''(b''). \end{aligned} \tag{3.1}$$

Let $x'' \in I^{(n+3)}$, $\pi : I \to \mathcal{A}$ be the inclusion map and $i : \mathcal{A} \to \mathcal{A}^{**}$ be the natural embedding. The maps $i'', i^{(4)}, ..., i^{(n+3)}$ are $weak^* - weak^*-$continuous. Then, $weak^* - \lim_\alpha i^{(n+3)}(a_\alpha) = i^{(n+3)}(a'')$. On the other hand, $\mathcal{A}^{(n+3)}$ is a C^*-algebra, and thus is Arens regular. then,

$$\lim_\alpha x'' a_\alpha = \lim_\alpha \pi^{(n+3)}(x'') i^{(n+3)}(a_\alpha) = \pi^{(n+3)}(x'') i^{(n+3)}(a'') = x'' a''.$$

Since $I^{(n+1)}$ is a C^*-algebra, then by Corollary 3.2.43 of [D], D is weakly compact. Thus, $D''(b'') \in \widehat{I^{(n+2)}}$, and for each $x'' \in I^{(n+3)}$, we have

$$\begin{aligned} lim_\alpha \langle a_\alpha . D''(b''), x'' \rangle &= lim_\alpha \langle x'' a_\alpha, D''(b'') \rangle \\ &= \langle x'' a'', D''(b'') \rangle \\ &= \langle a'' . D''(b''), x'' \rangle . \end{aligned}$$

Then,

$$a''.D''(b'') = lim_\alpha a_\alpha . D''(b''),$$

and by (3.1), D'' is a derivation. Since D is weakly compact, then $D''(\mathcal{A}^{**}) \subseteq \widehat{I^{(n+2)}}$. We can suppose that D'' is a derivation from $\mathcal{A}^{**}$ into $I^{(n+2)}$. Similarly, $D^4 := (D'')''$ is a derivation from $\mathcal{A}^{(4)}$ into $\widehat{I^{(n+2)}}$.

We can suppose that $D^{(2k+2)}$ (the $2k+2-$th conjugate of D) is a derivation from $\mathcal{A}^{(2k+2)}$ into $\widehat{I^{(n+2)}}$. On the other hand, $I^{(n+1)}$ is a closed ideal of $\mathcal{A}^{(2k+2)}$ and $\mathcal{A}^{(2k+2)}$ is ideally amenable (since it is a $C^{*}-$algebra). Thus, $D^{(2k+2)}$ is inner. Since $D^{(2k+2)}$ is an extension of D, then it is easy to see that D is inner.

4. **Codimension one ideals**

Let $\mathcal{A}$ be a Banach algebra and I be a closed ideal of $\mathcal{A}$, with codimension one. We find the relationship between n-weak amenability of I and $n-I$-weak amenability of $\mathcal{A}$. As [15, Theorem 2.3], we have the following result.

Theorem 4.1. *Let $\mathcal{A}$ be a Banach algebra with bounded approximate identity and I be a codimension one closed two sided ideal of $\mathcal{A}$. Then, $H^1(\mathcal{A}, X^*) \cong H^1(\mathcal{I}, X^*)$ for every neo unital Banach $\mathcal{A}$-module X.*

Let G be a discrete group, and I_0 be a codimension 1 closed two sided ideal of $l^1(G)$. Then, $l^1(G)$ is $n-I_0-$weakly amenable for every $n \in \mathbb{N}$.

Corollary 4.2. *Let $\mathcal{A}$ be a $C^{*}-$algebra and I be a codimension one closed two sided ideal of $\mathcal{A}$. Then, for every $n \in \mathbb{N}$, $\mathcal{A}$ is $n-I-$weakly amenable.*

Proof. Let $n = 2k+1$. Then, by Theorem 3.2, $H^1(\mathcal{A}, I^{(n)}) = \{0\}$. Let $n = 2k$. Then, we have $\mathcal{A}I^{(n-1)} = II^{(n-1)} = I^{(n-1)}$ and $I^{(n-1)}\mathcal{A} = I^{(n-1)}I = I^{(n-1)}$. Then, by Theorem 4.1, $\mathcal{A}$ is $n-I-$weakly amenable.

Lemma 4.3. *Let $\mathcal{A}$ be a Banach algebra with bounded approximate identity and I be a closed two sided ideal of $\mathcal{A}$. For every $n \geq 0$, the following assertions hold.*
(i) *If $\pi_r : I^{(n)} \times \mathcal{A} \longrightarrow I^{(n)}$ is Arens regular, then $I^{(n+1)}$ factors on the right.*
(ii) *If $\pi_l : \mathcal{A} \times I^{(n)} \longrightarrow I^{(n)}$ is Arens regular, then $I^{(n+1)}$ factors on the left.*

Proof. Let (e_α) be a bounded approximate identity for $\mathcal{A}$ with cluster point E. Suppose that π_r is Arens regular and let $i^{n+2} \in I^{(n+2)}$ be the cluster point of (i^n_β) in the weak$^*-$ topology of $I^{(n+2)}$ ((i^n_β) is a net in $I^{(n)}$). Then, we have

$$\begin{aligned} i^{n+2} &= weak^* - \lim_\beta i^n_\beta \\ &= weak^* - \lim_\beta \lim_\alpha i^n_\beta e_\alpha \\ &= weak^* - \lim_\alpha \lim_\beta i^n_\beta e_\alpha \\ &= weak^* - \lim_\alpha i^{(n+2)} e_\alpha = i^{(n+2)} E. \end{aligned}$$

Then, for $i^{n+1} \in I^{(n+1)}$, we have

$$\begin{aligned} \lim_\alpha \langle e_\alpha i^{n+1}, i^{n+2} \rangle &= \lim_\alpha \langle i^{n+1}, i^{n+2} e_\alpha \rangle \\ &= \langle i^{n+1}, i^{n+2} E \rangle \\ &= \langle i^{n+1}, i^{n+2} \rangle. \end{aligned}$$

Thus, $e_\alpha i^{n+1} \to i^{n+1}$ weakly in $I^{(n+1)}$. Since $e_\alpha i^{n+1} \in \mathcal{A} I^{(n+1)}$ for every α, then by Cohen-Hewit Factorization Theorem, we know that $\mathcal{A} I^{(n+1)}$ is closed in $I^{(n+1)}$, and thus $i^{n+1} \in \mathcal{A} I^{(n+1)}$. Thus, the proof of (i) is complete. For (ii), let $i^{n+2} \in I^{(n+2)}$ be the cluster point of (i^n_β). For each β, we know that $E i^n_\beta = i^n_\beta$. Since π_l is Arens regular, then $e_\alpha i^{n+2} \to E i^{n+2} = i^{n+2}$ by $weak^*$ topology of $I^{(n+2)}$. Then, for every $i^{n+1} \in I^{(n+1)}$,

$$\begin{aligned} \lim_\alpha \langle i^{n+1} e_\alpha, i^{n+2} \rangle &= \lim_\alpha \langle i^{n+1}, e_\alpha i^{n+2} \rangle \\ &= \langle i^{n+1}, E i^{n+2} \rangle \\ &= \langle i^{n+1}, i^{n+2} \rangle. \end{aligned}$$

Therefore, $i^{n+1} e_\alpha \to i^{n+1}$ weakly. Again, by Cohen-Hewit Factorization Theorem, we conclude that $I^{(n+1)}$ factors on the left.
By Theorem 4.1 and Lemma 4.3, we have the following result.

Proposition 4.4. *Let $\mathcal{A}$ be a Banach algebra with bounded approximate identity and I be a codimension one closed two sided ideal of $\mathcal{A}$. For every $n \geq 1$, if the module actions of $\mathcal{A}$ on $I^{(n-1)}$ are Arens regular, then I is $n-$weakly amenable if and only if $\mathcal{A}$ is $n-I-$weakly amenable.*

5. **Open Problems**

- We do not know whether or not 2-ideal amenability implies 4-ideal amenability for an arbitrary Banach algebra.
- We do not know whether or not 1-ideal amenability implies 3-ideal amenability for an arbitrary (non commutative) Banach algebra.
- We know that every C^*-algebra is $2k+1$ ideally amenable for every $k \in \mathbb{N}\bigcup\{0\}$. But, we do not know whether or not every C^*-algebra is permanently ideally amenable.
- The group algebras $L^1(G)$ are n-weakly amenable for each odd n [3], but we do not know for which G and which $n \in \mathbb{N}$, the algebra $L^1(G)$ is n-ideally amenable. We know that $l^1(\ F_2)$, the group algebra of free group F_2, is permanent weakly amenable [19], but we do not know whether or not $l^1(\ F_2)$ is permanent ideally amenable.
- It is known that in some cases, a Banach algebra A inherits weak amenability from A^{**} (see [11], [9], [D-G-V] and [7]). But we do not know whether or not $\mathcal{A}$ inherits the ideal amenability from $\mathcal{A}^{**}$.

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