ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 41 (2015), No. 5, pp. 1147–1159

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 41 (2015), No. 5, pp. 1147–1159
Online ISSN: 1735-8515

RANDOM APPROXIMATION OF A GENERAL SYMMETRIC EQUATION

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(Communicated by Hamid Pezeshk)

Abstract. In this paper, we prove the Hyers-Ulam stability of the symmetric functional equation $f(\varphi_1(x, y)) = \varphi_2(f(x), f(y))$ in random normed spaces. As a consequence, we obtain some random stability results in the sense of Hyers-Ulam-Rassias.
Keywords: Hyers-Ulam stability, general symmetric equation, random normed space.
MSC(2010): Primary: 39B52, 60H25, 47H40; Secondary: 39B82.

1. Introduction

In 1940, the stability problem of functional equations originated from a question of Ulam [11] concerning the stability of group homomorphisms. In 1941, Hyers [3] gave an affirmative partial answer for the question of Ulam for Banach spaces. Since then, In 1978, Hyers' theorem was generalized by Th.M. Rassias [8] for linear mappings by considering the unbounded Cauchy difference as follows:

Theorem 1.1. *Let f be an approximately additive mapping from a normed vector space E into a Banach space E', i.e., f satisfies the inequality*

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon(\|x\|^r + \|y\|^r)$$

for all $x, y \in E$, where ϵ and r are constants with $\epsilon > 0$ and $0 \leq r < 1$. Then the mapping $L : E \to E'$ defined by $L(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ is the unique additive mapping which satisfies

$$\|f(x + y) - L(x)\| \leq \frac{2\epsilon}{2 - 2^r}\|x\|^r$$

for all $x \in E$.

Article electronically published on October 17, 2015.
Received: 12 February 2013, Accepted: 16 July 2014.
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In 1994, a generalization of the Th. M. Rassias' theorem was obtained by Găvruta [2] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias' approach.

Some of the most famous functional equations such as, additive Cauchy equation, generalized additive Cauchy equation, exponential equation and logarithmic equation (see [5]), satisfy the following general equation:

$$f(\varphi_1(x,y)) = \varphi_2(f(x), f(y)), \tag{1.1}$$

where for $i = 1, 2$, $\varphi_i : X_i \times X_i \to X_i$ is an given mapping $f : X_1 \to X_2$ is a unknown mapping, X_1 is a set and X_2 is a complete metric space. The existence of solutions of (1.1) and a generalization of Hyers' theorem was investigated by Forti [1]. He proved, under certain hypothesis on φ_1 and φ_2, that the existence of solution of the functional inequality

$$d\big(f(\varphi_1(x,y)), \varphi_2(f(x), f(y))\big) \leq h(x,y),$$

where $h : X \times X \to [0, +\infty)$ is a suitable function, implies the existence of a solution of the equation (1.1). This means that the equation (1.1) has Hyers-Ulam stability. The results in [1] are improved and simplified in [9]

The purpose of this paper is to solve the Hyers-Ulam stability problem for symmetric type functional equations (1.1) when the unknown function is with values in a random normed space. In particular, Theorem 2.2 in both [6] and [7] will be obtained. We note that in our proofs you use the direct method.

2. **Random normed spaces**

In the section, we adopt the usual terminology, notions and conventions of the theory of random normed spaces as in [10].

Throughout this paper, let Δ^+ denote the set of all probability distribution functions $F : \mathbb{R} \to [0,1]$ such that F is left-continuous and nondecreasing on $\mathbb{R}$ and $F(0) = 0$. It is clear that the set

$$D^+ = \{F \in \Delta^+ : F(+\infty) = 1\},$$

where $F(+\infty) := \lim_{t\to+\infty} f(t)$, is a subset of Δ^+. The set Δ^+ is partially ordered by the usual point-wise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. For any $a \geq 0$, the element $H_a(t)$ of D^+ is defined by

$$H_a(t) = \begin{cases} 0, & \text{if} \quad t \leq a, \\ 1, & \text{if} \quad t > a. \end{cases} \tag{2.1}$$

We can easily show that the maximal element in Δ^+ is the distribution function $H_0(t)$.

Definition 2.1. A function $T : [0,1]^2 \to [0,1]$ is a *continuous triangular norm* (briefly, a t-norm) if T satisfies the following conditions:

(i) T is commutative and associative;

(ii) T is continuous;
(iii) $T(x,1) = x$ for all $x \in [0,1]$;
(iv) $T(x,y) \leq T(z,w)$ whenever $x \leq z$ and $y \leq w$ for all $x,y,z,w \in [0,1]$.

Three typical examples of continuous t-norms are as follows:

$$T(x,y) = xy, \quad T(x,y) = \max\{x+y-1,0\}, \quad T(x,y) = \min x,y.$$

Recall that, if T is a t-norm and $\{x_n\}$ is a sequence in $[0,1]$, then $T(x_1,x_2,\ldots,x_n)$ is defined recursively by

$$T(x_1,x_2,\ldots,x_n) = T(T(x_1,\ldots,x_{n-1}),x_n).$$

for every integer $n \geq 2$.

Definition 2.2. A *random normed space* (briefly, *RN*-space) is a triple (X,μ,T), where X is a vector space, T is a continuous t-norm and $\mu : X \to D^+$ is a mapping such that the following conditions hold (denote $\mu(x)$ by $\mu[x]$)
(a) $\mu[x](t) = H_0(t)$ for all $t > 0$ if and only if $x = 0$;
(b) $\mu[\alpha x](t) = \mu[x](\frac{t}{|\alpha|})$ for all $\alpha \in \mathbb{R}$ with $\alpha \neq 0$, $x \in X$ and $t \geq 0$;
(c) $\mu[x+y](t+s) \geq T(\mu[x](t),\mu[y](s))$ for all $x,y \in X$ and $t,s \geq 0$.

Moreover,

$$\mu\left[\sum_{k=1}^{n} x_k\right]\left(\sum_{k=1}^{n} t_k\right) \geq T(\mu[x_1](t_1),\ldots,\mu[x_n](t_n)).$$

Every normed space $(X,\|\cdot\|)$ defines a random normed space (X,μ,T_M), where

$$\mu[u](t) = \frac{t}{t+\|u\|}$$

for all $t > 0$ and T_M is the minimum t-norm. This space X is called the *induced random normed space*.

If the t-norm T is such that $\sup_{0<a<1} T(a,a) = 1$, then every *RN*-space (X,μ,T) is a metrizable linear topological space with the topology τ (called the *μ-topology* or the *(ε,δ)-topology*, where $\varepsilon > 0$ and $\lambda \in (0,1)$) induced by the base $\{U(\varepsilon,\lambda)\}$ of neighborhoods of θ, where

$$U(\varepsilon,\lambda) = \{x \in X : \mu[x](\varepsilon) > 1-\lambda\}.$$

Definition 2.3. Let (X,μ,T) be an *RN*-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ (write $x_n \to x$ as $n \to \infty$) if

$$\lim_{n\to\infty} \mu[x_n - x](t) = 1$$

for all $t > 0$.

(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* in X if

$$\lim_{n,m\to\infty} \mu[x_n - x_m](t) = 1$$

for all $t > 0$.

(3) The RN-space (X, μ, T) is said to be complete if every Cauchy sequence in X is convergent.

Theorem 2.4. *If (X, μ, T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then*

$$\lim_{n\to\infty} \mu[x_n](t) = \mu[x](t).$$

3. Random stability of the equation (1.1): the first case

Let X be any set. A mapping $\varphi : X \times X \to X$ is called *diagonal symmetric* if

$$\varphi(\varphi(x, x), \varphi(y, y)) = \varphi(\varphi(x, y), \varphi(x, y)).$$

for all $x, y \in X$. For example, if X is any vector space, and $\varphi : X \times X \to X$ is a mapping such that

$$\varphi(\lambda x, \lambda y) = \lambda\varphi(x, y) \quad (x, y \in X)$$

for all scalar λ and $\varphi(x, x) = \alpha x$ for some scalar α, then φ is diagonal symmetric on X. It is easy to show that if $\Delta x := \varphi(x, x)$ then φ is diagonal symmetric if and only if $\varphi(\Delta x, \Delta y) = \Delta\varphi(x, y)$.

In what follows, all t-norms will be assumed to be the minimum t-norm and for any function $g : X \times X \to D^+$, $g(x, y)$ denoted by $g[x, y]$ for all $x, y \in X$.

Theorem 3.1. *Let X_1 be a real linear space and let (X_2, μ, T) a complete RN-space. Assume that φ_1, φ_2 are two diagonal symmetric mappings on X_1, X_2, respectively, such that φ_1 is continuous, $\Delta_1 x := \varphi_1(x, x)$ is an invertible mapping on X_1, and there exists a $\beta > 0$ such that*

$$\mu\big[\Delta_2 x - \Delta_2 y\big](t) \geq \mu\big[x - y\big](\beta t) \quad (x, y \in X_2), \tag{3.1}$$

where $\Delta_2 x := \varphi_2(x, x)$. Let $h : X_1 \times X_1 \to D^+$ be a mapping for which there exists an $\alpha > \beta^{-1}$ with

$$h\big[\Delta_1 x, \Delta_1 y\big](\alpha t) \leq h[x, y](t) \quad (x, y \in X_1). \tag{3.2}$$

Suppose that $f : X_1 \to X_2$ is a mapping satisfying $f(0) = 0$ and

$$\mu\big[f(\varphi_1(x, y)) - \varphi_2(f(x), f(y))\big](t) \geq h\big[x, y\big](t) \quad (x, y \in X_1, t > 0). \tag{3.3}$$

Then there is a unique mapping $A : X_1 \to X_2$ satisfying

$$A(\varphi_1(x, y)) = \varphi_2(A(x), A(y)) \quad (x, y \in X_1), \tag{3.4}$$

and

$$\mu\big[A(x) - f(x)\big](t) \geq h\big[x, x\big](\frac{\alpha\beta - 1}{\beta} t) \quad (x, y \in X_1, t > 0). \tag{3.5}$$

Proof. Putting $x = y$ in (3.3), we get

$$\mu\big[\Delta_2 f(x) - f(\Delta_1 x)\big](t) \geq h\big[x,x\big](t) \quad (x \in X_1, t > 0). \tag{3.6}$$

Since Δ_1 is invertible on X_1, (3.2) implies that

$$h\big[\Delta_1^{-1}x, \Delta_1^{-1}y\big](t) \geq h\big[x,y\big](\alpha t) \quad (x, y \in X_1, t > 0). \tag{3.7}$$

Let $q_0 := f$ and $q_n := \Delta_2^n f \Delta_1^{-n}$ for $n > 0$. Then $q_n = \Delta_2 q_{n-1}\Delta_1^{-1}$, and we prove by induction that

$$\mu\big[q_n(x) - q_{n-1}(x)\big](t) \geq h\big[x,x\big]\big(\alpha(\alpha\beta)^{n-1}\big) \tag{3.8}$$

for all $x \in X_1$, $t > 0$ and $n \in \mathbb{N}$. Fix $x \in X_1$ and $t > 0$. Using (3.7), we obtain that

$$\begin{aligned}
\mu\big[q_1(x) - q_0(x)\big](t) &= \mu\big[\Delta_2 f(\Delta_1^{-1}x) - f(x)\big](t)\\
&= \mu\big[\Delta_2 f(\Delta_1^{-1}x) - f\Delta_1(\Delta_1^{-1}x)\big](t)\\
&\geq h[\Delta_1^{-1}x, \Delta_1^{-1}x](t) \geq h[x,x](\alpha t).
\end{aligned}$$

Suppose that (3.8) holds for n. Then for $n+1$, applying (3.1) and (3.6), we obtain

$$\begin{aligned}
\mu\big[q_{n+1}(x) - q_n(x)\big](t) &= \mu\big[\Delta_2 q_n \Delta_1^{-1}x - \Delta_2 q_{n-1}\Delta_1^{-1}x\big](t)\\
&\geq \mu\big[q_n\Delta_1^{-1}x - q_{n-1}\Delta_1^{-1}x\big](\beta t)\\
&\geq h[\Delta_1^{-1}x, \Delta_1^{-1}x]\big((\alpha\beta)^n t\big)\\
&\geq h[x,x]\big(\alpha(\alpha\beta)^n t\big).
\end{aligned}$$

From (3.8) and the relation

$$q_{n+m}(x) - q_m(x) = \sum_{k=m}^{n+m-1} q_{k+1}(x) - q_k(x) \quad (x \in X_1),$$

we deduce that

$$\begin{aligned}
&\mu\big[q_{n+m}(x) - q_m(x)\big]\left(t\sum_{k=m}^{n+m-1}\frac{1}{\alpha(\alpha\beta)^k}\right)\\
&= \mu\left[\sum_{k=m}^{n+m-1} q_{k+1}(x) - q_k(x)\right]\left(t\sum_{k=m}^{n+m-1}\frac{1}{\alpha(\alpha\beta)^k}\right)\\
&\geq T\Bigg(\mu[q_{m+1}(x) - q_m(x)]\left(\frac{1}{\alpha(\alpha\beta)^m}t\right), \cdots,\\
&\qquad \mu[q_{n+m}(x) - q_{n+m-1}(x)]\left(\frac{1}{\alpha(\alpha\beta)^{n+m-1}}t\right)\Bigg)\\
&\geq T\Big(h\big[x,x\big](t), \cdots, h\big[x,x\big](t)\Big) = h\big[x,x\big](t).
\end{aligned}$$

Hence
(3.9)
$$\mu\big[q_{n+m}(x)-q_m(x)\big](t)\geq h\big[x,x\big]\left(\frac{t}{\sum_{k=m}^{n+m-1}\frac{1}{\alpha(\alpha\beta)^k}}\right)\quad (x\in X_1; m,n\geq 0).$$

Since $\alpha\beta>1$, $\sum_{k=m}^{n+m-1}\frac{1}{\alpha(\alpha\beta)^k}\to 0$ as $m,n\to+\infty$ and so

$$\lim_{m,n\to+\infty} h\big[x,x\big]\left(\frac{t}{\sum_{k=m}^{n+m-1}\frac{1}{\alpha(\alpha\beta)^k}}\right)=1.$$

Thus

$$\lim_{m,n\to+\infty}\mu\big[q_{n+m}(x)-q_m(x)\big]=1,$$

and for every $x\in X_1$, the sequence $\{q_n(x)\}$ is a Cauchy sequence in (X_2,μ,T). Since (X_2,μ,T) is a complete RN-space, this sequence converges to some point $A(x)\in X_2$. Fix $x\in X_1$ and put $m=0$ in (3.9). Then we obtain

$$\mu\big[q_n(x)-f(x)\big](t)=\mu\big[q_n(x)-q_0(x)\big](t)\geq h\big[x,x\big]\left(\frac{t}{\sum_{k=0}^{n-1}\frac{1}{\alpha(\alpha\beta)^k}}\right)$$

and so for every $\delta>0$, we have

$$\begin{aligned}\mu[A(x)-f(x)](t+\delta) &\geq T\Big(\mu[A(x)-q_n(x)](\delta),\mu[q_n(x)-f(x)](t)\Big)\\ &\geq T\left(\mu[A(x)-q_n(x)](\delta),h[x,x]\left(\frac{t}{\sum_{k=0}^{n-1}\frac{1}{\alpha(\alpha\beta)^k}}\right)\right).\end{aligned}$$

Taking the limit as $n\to+\infty$ and using the last relation, we get

$$\begin{aligned}\mu[A(x)-f(x)](t+\delta) &\geq T\left(\mu[0](\delta),h[x,x]\left(\frac{t}{\sum_{k=0}^{+\infty}\frac{1}{\alpha(\alpha\beta)^k}}\right)\right)\\ &= h[x,x]\left(\frac{t}{\sum_{k=0}^{+\infty}\frac{1}{\alpha(\alpha\beta)^k}}\right),\end{aligned}$$

and so

$$\mu[A(x)-f(x)](t+\delta)\geq h[x,x](\frac{\alpha\beta-1}{\beta}t).$$

Since δ was arbitrary, by taking $\delta\to 0$ in the above relation, we obtain

$$\mu[A(x)-f(x)](t)\geq h[x,x](\frac{\alpha\beta-1}{\beta}t)\quad (x\in X_1, t>0).$$

Now we prove that A satisfies (3.4). To show this, note that

$$\begin{aligned}
&\mu[q_n(\varphi_1(x,y)) - \varphi_2(q_n(x), q_n(y))](t)\\
&= \mu[\Delta_2^n f(\Delta_1^{-n}\varphi_1(x,y)) - \varphi_2(\Delta_2^n f(\Delta_1^{-n}x), \Delta_2^n f(\Delta_1^{-n}y))](t)\\
&\geq \mu[\Delta_2^n f(\Delta_1^{-n}\varphi_1(x,y)) - \Delta_2^n(\varphi_2(f\Delta_1^{-n}x, f\Delta_1^{-n}y))](t)\\
&\geq \mu[f(\varphi_1(\Delta_1^{-n}x, \Delta_1^{-n}y)) - \varphi_2(f\Delta_1^{-n}x, f\Delta_1^{-n}y)](\beta^n t)\\
&\geq h[\Delta_1^{-n}x, \Delta_1^{-n}y](\frac{t}{\beta^n}) \geq h[x,x]((\alpha\beta)^n t),
\end{aligned}$$

where the diagonal property of φ_i is used for $i = 1, 2$. Therefore,

$$\mu[q_n(\varphi_1(x,y)) - \varphi_2(q_n(x), q_n(y))](t) \geq h[x,x]((\alpha\beta)^n t) \tag{3.10}$$

for all $x, y \in X_1$ and all $n \in \mathbb{N}$. Since $\alpha\beta > 1$, and

$$\lim_{n\to+\infty} h[x,x]((\alpha\beta)^n t) = 1,$$

using the continuity of φ_2 and the fact that $q_n(x) \to A(x)$ for all $x \in X_1$ in (3.10), we obtain

$$\mu[A(\varphi_1(x,y)) - \varphi_2(A(x), A(y))] = 1$$

for all $x, y \in X_1$ and every $t > 0$. Hence A satisfies (3.4).

Finally it remains to prove that A is a unique mapping satisfying (3.4) and (3.5). Assume that there exists another mapping $A' : X_1 \to X_2$ satisfying (3.4) and (3.5). Letting $y = x$ in (3.4) for both A and A', respectively, we get $A(\Delta_1 x) = \Delta_2 A(x), A'(\Delta_1 x) = \Delta_2 A'(x)$ and more generally

$$A(\Delta_1^n x) = \Delta_2^n A(x) \text{ and } A'(\Delta_1^n x) = \Delta_2^n A'(x).$$

for all $x \in X_1$ and $n \in \mathbb{N}$. It follows from (3.1), (3.5) and (3.7) that

$$\begin{aligned}
&\mu\big[A(x) - A'(x)\big](t)\\
&= \mu\big[A(\Delta_1^n(\Delta_1^{-n}x)) - A'(\Delta_1^n(\Delta_1^{-n}x))\big](t)\\
&= \mu\big[\Delta_2^n A(\Delta_1^{-n}x) - \Delta_2^n A'(\Delta_1^{-n}x)\big](t)\\
&\geq \mu\big[A(\Delta_1^{-n}x) - A'(\Delta_1^{-n}x)\big](\beta^n t)\\
&\geq T\bigg(\mu\big[A(\Delta_1^{-n}x) - f(\Delta_1^{-n}x)\big](\frac{1}{2}\beta^n t), \mu\big[f(\Delta_1^{-n}x) - A'(\Delta_1^{-n}x)(\frac{1}{2}\beta^n t)\big]\bigg)\\
&\geq T\bigg(h\big[\Delta_1^{-n}x, \Delta_1^{-n}x\big]\bigg(\frac{\alpha\beta-1}{2\beta}\beta^n t\bigg), h\big[\Delta_1^{-n}x, \Delta_1^{-n}x\big]\bigg(\frac{\alpha\beta-1}{2\beta}\beta^n t\bigg)\bigg)\\
&= h\big[\Delta_1^{-n}x, \Delta_1^{-n}x\big]\bigg(\frac{\alpha\beta-1}{2\beta}\beta^n t\bigg)\\
&\geq h[x,x]\bigg(\frac{\alpha\beta-1}{2\beta}(\alpha\beta)^n t\bigg)
\end{aligned}$$

Since $\lim_{n\to+\infty} \frac{\alpha\beta-1}{2\beta}(\alpha\beta)^n t = +\infty$, we get

$$\lim_{n\to+\infty} h[x,x]\left(\frac{\alpha\beta-1}{2\beta}(\alpha\beta)^n t\right) = 1.$$

Therefore, it follows that $\mu\big[A(x) - A'(x)\big](t)$ for all $t > 0$ and so $A(x) = A'(x)$. This completes the proof. $\square$

Corollary 3.2. ([7, Theorem 2.2]) *Let X_1 be a real linear space, (X_2, μ, T) an RN-space, and h a mapping from $X_1 \times X_1$ to D^+ such that, for some $\alpha_0 > 2$,*

$$h[x,y](t) \geq h[2x,2y](\alpha_0 t) \quad (x, y \in X_1, t > 0).$$

If $f : X_1 \to X_2$ is a mapping with $f(0) = 0$ such that

$$\mu[f(x+y) - f(x) + f(y)](t) \geq h[x,y](t) \qquad (x, y \in X_1, t > 0)$$

holds, then there exists a unique additive mapping $A : X_1 \to X_2$ such that

$$\mu[f(x) - A(x)](t) \geq h[x,x]((\alpha_0 - 2)t) \quad (x, y \in X_1, t > 0).$$

Proof. Applying Theorem 3.1, for $\alpha = \alpha_0$, $\varphi_i(x,y) = x + y, i = 1, 2,$ we get $T_i(x) = 2x$, $\beta = \frac{1}{2}$, $\alpha > \beta^{-1}$ and the proof is complete. $\square$

4. Random stability of the equation (1.1): the second case

Theorem 4.1. *Let X_1 be a real linear space and (X_2, μ, T) a complete RN-space. Assume that φ_1, φ_2 are two diagonal symmetric mappings on X_1, X_2, respectively, such that φ_2 is continuous, $\Delta_2 x := \varphi_2(x,x)$ is an invertible mapping on X_2 and there exists a $\beta > 0$ such that*

$$\mu\big[\Delta_2 x - \Delta_2 y\big](t) \leq \mu\big[x - y\big](\beta t) \quad (x, y \in X_2, t > 0). \tag{4.1}$$

Let $h : X_1 \times X_1 \to D^+$ be a mapping for which there exists an $\alpha > \beta$ such that

$$h\big[\Delta_1 x, \Delta_1 y\big](t) \geq h[x,y](\alpha t) \quad (x, y \in X_1, t > 0), \tag{4.2}$$

where $\Delta_1 x := \varphi_1(x,x)$. Suppose that $f : X_1 \to X_2$ is a mapping satisfying $f(0) = 0$ and (3.4). Then there is a unique mapping $A : X_1 \to X_2$ satisfying (3.5) and

$$\mu\big[A(x) - f(x)\big](t) \geq h\big[x,x\big]\left(\frac{\alpha-\beta}{\alpha\beta}t\right). \tag{4.3}$$

Proof. Putting $x = y$ in (3.3), we get

$$\mu\big[f(\Delta_1 x) - \Delta_2 f(x)\big](t) \geq h\big[x,x\big](t) \quad (x \in X_1, t > 0).$$

Since Δ_2 is invertible on X_2, (4.1) implies that

$$\mu\big[\Delta_2^{-1} x - \Delta_2^{-1} y\big](t) \geq \mu\big[x - y\big]\left(\frac{t}{\beta}\right) \quad (x, y \in X_2, t > 0). \tag{4.4}$$

Let $q_0 := f$ and $q_n := \Delta_2^{-n} f \Delta_1^n$ for $n > 0$. Then $\Delta_2 q_n = q_{n-1}\Delta_1$, and we show by induction that

$$\mu\big[q_n(x) - q_{n-1}(x)\big](t) \geq h\big[x, x\big]\left(\frac{\alpha^{n-1}}{\beta^n} t\right) \tag{4.5}$$

for all $x \in X_1$, $t > 0$ and $n \in \mathbb{N}$. Fix $x \in X_1$ and $t > 0$. Using (4.4), we obtain that

$$\begin{aligned}
\mu\big[q_1(x) - q_0(x)\big](t) &= \mu\big[\Delta_2^{-1} f(\Delta_1 x) - f(x)\big](t) \\
&\geq \mu\big[f(\Delta_1 x) - \Delta_2 f(x)\big]\left(\frac{t}{\beta}\right) \\
&\geq h\big[x, x\big]\left(\frac{t}{\beta}\right).
\end{aligned}$$

Suppose that (4.5) holds for n. Then for $n+1$ we have

$$\begin{aligned}
\mu\big[q_{n+1}(x) - q_n(x)\big](t) &= \mu\big[\Delta_2^{-1} q_n(\Delta_1 x) - q_n(x)\big](t) \\
&\geq \mu\big[q_n(\Delta_1 x) - \Delta_2 q_n(x)\big]\left(\frac{t}{\beta}\right) \\
&= \mu\big[q_n(\Delta_1 x) - q_{n-1}(\Delta_1 x)\big]\left(\frac{t}{\beta}\right) \\
&\geq h[\Delta_1 x, \Delta_1 x]\left(\frac{\alpha^{n-1}}{\beta^{n+1}} t\right) \\
&\geq h[x, x]\left(\frac{\alpha^n}{\beta^{n+1}} t\right).
\end{aligned}$$

From (4.5) and the relation

$$q_n(x) - f(x) = q_n(x) - q_0(x) = \sum_{k=0}^{n-1} q_{k+1}(x) - q_k(x) \quad (x \in X_1, t > 0),$$

we obtain that

$$\begin{aligned}
&\mu\big[q_{n+m}(x)-q_m(x)\big]\bigg(t\sum_{k=m}^{n+m-1}\frac{\beta^{k+1}}{\alpha^k}\bigg)\\
&=\mu\bigg[\sum_{k=m}^{n+m-1}q_{k+1}(x)-q_k(x)\bigg]\bigg(t\sum_{k=m}^{n+m-1}\frac{\beta^{k+1}}{\alpha^k}\bigg)\\
&\geq T\bigg(\mu[q_{m+1}(x)-q_m(x)]\left(\frac{\beta^{m+1}}{\alpha^m}t\right),\cdots,\\
&\qquad\mu[q_{n+m}(x)-q_{n+m-1}(x)]\left(\frac{\beta^{n+m+1}}{\alpha^{n+m}}t\right)\bigg)\\
&\geq T\bigg(h\big[x,x\big](t),\cdots,h\big[x,x\big](t)\bigg)=h\big[x,x\big](t).
\end{aligned}$$

Hence

$$\mu\big[q_{n+m}(x)-q_m(x)\big](t)\geq h\big[x,x\big]\left(\frac{t}{\sum_{k=m}^{n+m-1}\frac{\beta^{k+1}}{\alpha^k}}\right)\quad(x\in X_1;m,n\geq 0). \tag{4.6}$$

Since $\alpha>\beta$, $\sum_{k=m}^{n+m-1}\frac{\beta^{k+1}}{\alpha^k}\to 0$ as $m,n\to+\infty$ and so

$$\lim_{m,n\to+\infty}h\big[x,x\big]\left(\frac{t}{\sum_{k=m}^{n+m-1}\frac{\beta^{k+1}}{\alpha^k}}\right)=1.$$

Thus

$$\lim_{m,n\to+\infty}\mu\big[q_{n+m}(x)-q_m(x)\big]=1,$$

and for every $x\in X_1$, the sequence $\{q_n(x)\}$ is a Cauchy sequence in (X_2,μ,T). Since (X_2,μ,T) is a complete RN-space, this sequence converges to some point $A(x)\in X_2$. Fix $x\in X_1$ and put $m=0$ in (4.6). Then we obtain

$$\mu\big[q_n(x)-f(x)\big](t)=\mu\big[q_n(x)-q_0(x)\big](t)\geq h\big[x,x\big]\left(\frac{t}{\sum_{k=0}^{n-1}\frac{\beta^{k+1}}{\alpha^k}}\right)$$

and so for every $\delta>0$, we have

$$\begin{aligned}
\mu[A(x)-f(x)](t+\delta)\quad&\geq\quad T\Big(\mu[A(x)-q_n(x)](\delta),\mu[q_n(x)-f(x)](t)\Big)\\
&\geq\quad T\left(\mu[A(x)-q_n(x)](\delta),h[x,x]\left(\frac{t}{\sum_{k=0}^{n-1}\frac{\beta^{k+1}}{\alpha^k}}\right)\right).
\end{aligned}$$

Taking the limit as $n\to+\infty$ and using the last relation, we get

$$\begin{aligned}
\mu[A(x)-f(x)](t+\delta)\quad&\geq\quad T\left(\mu[0](\delta),h[x,x]\left(\frac{t}{\sum_{k=0}^{+\infty}\frac{\beta^{k+1}}{\alpha^k}}\right)\right)\\
&=\quad h[x,x]\left(\frac{t}{\sum_{k=0}^{+\infty}\frac{\beta^{k+1}}{\alpha^k}}\right)
\end{aligned}$$

and so

$$\mu[A(x) - f(x)](t + \delta) \geq h[x, x](\frac{\alpha - \beta}{\alpha\beta} t).$$

Since δ was arbitrary, by taking $\delta \to 0$ in the above relation, we obtain (4.3), i.e.,

$$\mu[A(x) - f(x)](t) \geq h[x, x](\frac{\alpha - \beta}{\alpha\beta} t) \quad (x \in X_1, t > 0).$$

Now we prove that A satisfies (3.4). The diagonal symmetric property of φ_i for $i = 1, 2$ implies that

$$\varphi_i(\Delta_i x, \Delta_i y) = \Delta_i \varphi_i(x, y)$$

for all $x, y \in X_i$. Hence

$$\begin{aligned}
&\mu[q_n(\varphi_1(x, y)) - \varphi_2(q_n(x), q_n(y))](t) \\
&= \mu[\Delta_2^{-n} f(\Delta_1^n \varphi_1(x, y)) - \varphi_2(q_n(x), q_n(y))](t) \\
&\geq \mu[f(\Delta_1^n \varphi_1(x, y)) - \Delta_2^n(\varphi_2(q_n(x), q_n(y))] \left(\frac{t}{\beta^n}\right) \\
&\geq \mu[f(\varphi_1(\Delta_1^n x, \Delta_1^n y)) - \varphi_2(\Delta_2^n q_n(x), \Delta_2^n q_n(y))] \left(\frac{t}{\beta^n}\right) \\
&\geq \mu[f(\varphi_1(\Delta_1^n x, \Delta_1^n y)) - \varphi_2(f(\Delta_1^n x), f(\Delta_1^n y))] \left(\frac{t}{\beta^n}\right) \\
&\geq h[\Delta_1^n x, \Delta_1^n y](\frac{t}{\beta^n}) \geq h[x, x] \left(\frac{\alpha^n}{\beta^n} t\right)
\end{aligned}$$

Therefore,

$$\mu[q_n(\varphi_1(x, y)) - \varphi_2(q_n(x), q_n(y))](t) \geq h[x, x] \left(\left(\frac{\alpha}{\beta}\right)^n t\right). \tag{4.7}$$

for all $x, y \in X_1$ and all $n \in \mathbb{N}$. Since $\alpha > \beta$ and

$$\lim_{n \to +\infty} h[x, x]((\frac{\alpha}{\beta})^n t) = 1,$$

applying the continuity of φ_2 and the fact that $q_n(x) \to A(x)$ for all $x \in X_1$ in (4.7), we obtain

$$\mu[A(\varphi_1(x, y)) - \varphi_2(A(x), A(y))] = 1$$

for all $x, y \in X_1$ and all $t > 0$. Hence A satisfies (3.4). Now we prove that A is a unique mapping satisfying (3.4) and (4.3). Assume that there exists another mapping $A' : X_1 \to X_2$ satisfying (3.4) and (4.3). Letting $y = x$ in (3.4) for A and A', we get $A(\Delta_1 x) = \Delta_2 A(x), A'(\Delta_1 x) = \Delta_2 A'(x)$ and more generally

$$A(\Delta_1^n x) = \Delta_2^n A(x) \text{ and } A'(\Delta_1^n x) = \Delta_2^n A'(x),$$

for all $x \in X_1$ and $n \in \mathbb{N}$. It follows from (4.2), (3.4) and (4.4) that

$$\begin{aligned}
&\mu\big[A(x) - A'(x)\big](t)\\
&= \mu\big[\Delta_2^{-n}\Delta_2^n A(x) - \Delta_2^{-n}\Delta_2^n A'(x)\big](t)\\
&\geq \mu\big[\Delta_2^n A(x) - \Delta_2^n A'(x)\big]\left(\frac{t}{\beta^n}\right)\\
&= \mu\big[A(\Delta_1^n x) - A'(\Delta_1^n x)\big]\left(\frac{t}{\beta^n}\right)\\
&\geq T\left(\mu\big[A(\Delta_1^n x) - f(\Delta_1^n x)\big]\left(\frac{t}{2\beta^n}\right), \mu\big[f(\Delta_1^n x) - A'(\Delta_1^n x)\big]\left(\frac{t}{2\beta^n}\right)\right)\\
&\geq T\left(h[\Delta_1^n x, \Delta_1^n x]\left(\frac{(\alpha-\beta)}{2\alpha\beta^{n+1}}t\right), h[\Delta_1^n x, \Delta_1^n x]\left(\frac{(\alpha-\beta)}{2\alpha\beta^{n+1}}t\right)\right)\\
&= h[\Delta_1^n x, \Delta_1^n x]\left(\frac{(\alpha-\beta)}{2\alpha\beta^{n+1}}t\right) \geq h[x,x]\left(\frac{(\alpha-\beta)\alpha^n}{2\alpha\beta^{n+1}}t\right).
\end{aligned}$$

Since

$$\lim_{n\to+\infty} \frac{(\alpha-\beta)\alpha^n}{2\alpha\beta^{n+1}}t = +\infty,$$

we get

$$\lim_{n\to+\infty} h[x,x]\left(\frac{(\alpha-\beta)\alpha^n}{2\alpha\beta^{n+1}}t\right) = 1.$$

Therefore, it follows that $\mu\big[A(x) - A'(x)\big](t)$ for all $t > 0$ and so $A(x) = A'(x)$. This completes the proof. $\square$

Corollary 4.2. ([6, Theorem 2.2]) *Let X_1 be a real linear space, (X_2, μ, T) an RN-space, and h a mapping from $X_1 \times X_1$ to D^+ such that, for some $\alpha_0 > 2$,*

$$h[2x, 2y](\alpha_0 t) \geq h[x, y](t) \quad (x, y \in X_1, t > 0).$$

If $f : X_1 \to X_2$ is a mapping with $f(0) = 0$ such that

$$\mu[f(x+y) - f(x) + f(y)](t) \geq h[x, y](t) \qquad (x, y \in X_1, t > 0)$$

holds, then there exists a unique additive mapping $A : X_1 \to X_2$ such that

$$\mu[f(x) - A(x)](t) \geq h[x, x]((2 - \alpha_0)t) \quad (x, y \in X_1, t > 0).$$

Proof. Applying Theorem 4.1, for $\alpha = \frac{1}{\alpha_0}$, $\varphi_i(x, y) = x + y, i = 1, 2$, we obtain $T_i(x) = 2x, i = 1, 2$, $\beta = \frac{1}{2}$ and $\alpha > \beta$. Now the proof is complete. $\square$

Acknowledgments

The authors are very grateful to the referee for many helpful suggestions and interesting comments about the paper.

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