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ON PROPERTIES OF DEPENDENT GENERAL PROGRESSIVELY TYPE-II CENSORED ORDER STATISTICS

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Abstract. In the literature of life-testing, general progressive censoring has been studied extensively. But, all the results have been developed under the key assumption that the units under test are independently distributed. In this paper, we study general progressively Type-II censored order statistics arising from identical units under test which are jointly distributed according to an Archimedean copula with completely monotone generator (GPCOSARCM-II). Density, distribution and joint density functions of GPCOSARCM-II are all derived. Finally, some examples of GPCOSARCM-II are also provided.
Keywords: Stochastic ordering, Archimedean copula, order statistics, general progressive censoring, reliability systems.
MSC(2010): Primary: 62E15; Secondary: 60K10.

1. Introduction

Many papers and several monographs have appeared on the theory of order statistics and as noted by Kamps [6] there are several other models of ordered random variables with different interpretations and interesting applications in many fields, for example, in reliability theory, survival analysis, financial economics, etc. One of the interesting modifications of order statistics is the concept of Progressive Type II censored-order statistics, which is very useful in reliability and lifetime studies. Several authors have discussed properties of progressively censored order statistics and their applications to inference. For a comprehensive review on various developments concerning progressive censoring, one may refer to Balakrishnan and Aggarwala [2] and the recent discussion paper Balakrishnan [3]. In all these works, progressively censored order statistics are assumed to arise from N independent and identically distributed

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random variables. Here, we study general progressively censored order statistics arising from N dependent random variables. This situation would arise when the N units under test are dependent and have a joint distribution with a special structure to be defined through a copula.

Copulas are convenient tools for modeling general dependence structure of variables. A very convenient subclass of copulas is the Archimedean copula, which has a close connection to Laplace transforms. For a thorough discussion on Archimedean copulas, one may refer to Joe [5] and Nelsen [9]. This class is used in Rezapour et. al. [11] and Rezapour and Alamatsaz [14] to study a $(n-k+1)$-out-of-n system with dependent components. Rezapour et. al. [12] considered some reliability property of a system whose components distributed according to an Archimedean copula. To be specific, we assume that the N underling variables are jointly distributed according to an Archimedean copula. First, recall that a function $\psi : \Re_+ \mapsto [0,1]$ is said to be *d-monotone* if $(-1)^k \psi^{(k)} \geq 0$ for $k \in \{1, ..., d-2\}$ and $(-1)^{d-2}\psi^{(d-2)}$ is a decreasing and convex function. If a function is d-monotone for all $d \in \mathbf{N}$, then it is said to be *completely monotone*. If a copula C_ψ has the form

$$C_\psi(u_1, ..., u_N) = \psi\left(\sum_{i=1}^{N} \psi^{-1}(u_i)\right), \tag{1.1}$$

where $\psi : \Re_+ \mapsto [0,1]$ is a N-monotone ($N \geq 2$) function such that $\psi(0) = 1$ and $\lim_{x\to\infty} \psi(x) = 0$, then it is called an Archimedean copula (see e.g. McNeil and Nèslehovà [8]). The function ψ is said to be the generator function of this Archimedean copula.

In this work, we concentrate on a special case of the Archimedean copula whose generator function ψ is completely monotone. Let $G(u) = \exp\{-\psi^{-1}(u)\}$ and M_ψ be the distribution function with Laplace Transform ψ. Then, we can obtain an equivalent representation of (1.1) as

$$C_\psi(u_1, \ldots, u_N) = \int_0^\infty \prod_{j=1}^{N} G^\alpha(u_i) dM_\psi(\alpha). \tag{1.2}$$

This representation is the key to all subsequent developments.

Now, let us consider the following progressive Type-II censoring scheme arising from independent or dependent samples. First, we consider an independent sample. Suppose $X^{\mathbf{R}}_{1:m:N}, \ldots, X^{\mathbf{R}}_{r:m:N}$ are the progressively Type-II censored order statistics (PCOS-II) of size m from an independent sample of size N with a progressive censoring scheme $\mathbf{R} = (R_1, \ldots, R_m)$. Assuming that X_i's have a common absolutely continuous cumulative distribution function F with density function f, the joint density function of the first r PCOS-II is given by

$$f_{X^{\mathbf{R}}_{1:m:N}, \ldots, X^{\mathbf{R}}_{r:m:N}}(x_1, \ldots, x_r) = \left(\prod_{j=1}^{r} \gamma_j\right) \prod_{j=1}^{r} f(x_j)\{\bar{F}(x_j)\}^{R_j}\{\bar{F}(x_r)\}^{\gamma_r - R_r - 1} \tag{1.3}$$

for $x_1 \le x_2 \le \cdots \le x_r$, where $\gamma_j = N - \sum_{v=1}^{j-1}(R_v+1) = \sum_{v=j}^{m}(R_v+1)$ and $\gamma_1 = N$. We refer the reader to Balakrishnan and Aggarwala [2] and the references therein for a comprehensive discussion and inferential procedures based on progressive censoring.

Now, consider a random vector $\mathbf{X} = (X_1, \ldots, X_N)$ with joint distribution function

$$\psi\left(\sum_{i=1}^{N}\psi^{-1}(F(x_i))\right) = \int_0^\infty \prod_{i=1}^{N} G^\alpha(F(x_i))dM_\psi(\alpha)\,, \tag{1.4}$$

where F is the marginal distribution function. Let us further assume that the function G has the first derivative g. Then, the joint density function of $\mathbf{X}$ equals

$$\int_0^\infty \prod_{i=1}^{N} \alpha g(F(x_i))f(x_i)G^{\alpha-1}(F(x_i))dM_\psi(\alpha)\,, \tag{1.5}$$

which yields

$$\begin{aligned}
&\Pr\Big(x_1 < X_1 \le y_1, \ldots, x_n < X_n \le y_n | X_{n+1} = x_{n+1}, \ldots, X_N = x_N\Big) \\
&\quad \times f_{X_{n+1},\ldots,X_N}(x_{n+1}, \ldots, x_N) \\
&= \int_0^\infty \int_{x_1}^{y_1} \cdots \int_{x_n}^{y_n} \prod_{s=1}^{n} \alpha g(F(w_s))G^{\alpha-1}(F(w_s))f(w_s)dw_s \\
&\quad \times \prod_{s=n+1}^{N} \alpha g(F(v_s))G^{\alpha-1}(F(v_s))f(v_s)dM_\psi(\alpha) \\
&= \int_0^\infty \prod_{s=1}^{n} \Big\{G^\alpha(F(y_s)) - G^\alpha(F(x_s))\Big\} \\
&\qquad \times \prod_{s=n+1}^{N} \alpha g(F(x_s))G^{\alpha-1}(F(x_s))f(x_s)dM_\psi(\alpha),
\end{aligned} \tag{1.6}$$

where $x_i < y_i$ for $i = 1, 2, \ldots, n$. For more details, one may refer to Joe [5] or Nelsen [9].

At the time of the first failure (i.e., when the minimum lifetime occurs), R_1 surviving units are withdrawn from the test; next, at the time of the i-th failure, R_i $(i = 2, \ldots, m)$ surviving units are withdrawn from the test. Let $X_{j:m:N}^{(R_1,\ldots,R_m)}$ denote the j-th dependent PCOS-II from m unit failures observed from N units on test with progressive censoring scheme $\mathbf{R} = (R_1, R_2, \ldots, R_m)$. It is evident that $N = m + R_1 + R_2 + ... + R_m$. Then, by using (1.6), we can express the joint density function as

$$\int_0^\infty c_{m-1} \prod_{i=1}^{m} \alpha G^{\alpha-1}(F(x_i))g(F(x_i))f(x_i)\Big\{1 - G^\alpha(F(x_i))\Big\}^{R_i} dM_\psi(\alpha)\,, \tag{1.7}$$

where $c_{m-1} = \prod_{i=1}^{m} \gamma_i$, $\gamma_i = \sum_{v=i}^{m}(R_v + 1)$ for $i = 2, \ldots, m-1$, $\gamma_1 = N$ and $\gamma_m = 1$. But, we can obviously show that $G(F(x))$ is a distribution function, and consequently, the representation of the density function $f_{X^{\mathbf{R}}_{1:m:N},\ldots,X^{\mathbf{R}}_{m:m:N}}$ in (1.7) can be written as

$$f_{\mathbf{X}^{\mathbf{R}}}(t_1, \ldots t_m) = \int_0^{\infty} f_{\mathbf{X}^{\mathbf{R}}_{\alpha}}(t_1, \ldots t_m) dM_{\psi}(\alpha), \quad t_1 \leq \cdots \leq t_m, \tag{1.8}$$

where $f_{\mathbf{X}^{\mathbf{R}}_{\alpha}}(t_1, \ldots t_m)$ is defined as the integrand in (1.7). Notice that $f_{\mathbf{X}^{\mathbf{R}}_{\alpha}}$ is the joint density function of PCOS-II arising from an independent sample $\mathbf{X}^{\mathbf{R}}_{\alpha} = (X^{\mathbf{R}}_{1:m:N;\alpha}, \ldots, X^{\mathbf{R}}_{m:m:N;\alpha})$ with baseline distribution $G^{\alpha}(F(\cdot))$, $\alpha \geq 0$ (see Balakrishnan and Cramer [4], Theorem 1). Rezapour et.al. [10] drive the marginal density and distribution functions of $X^{\mathbf{R}}_{r:m:N}$. PCOS-II arising from non-identical but independent units considered by Balakrishnan and Cramer [4]. Rezapour et. al. [13] extend their results for the units under test which are jointly distributed according to an Archimedean copula with completely monotone generator. Here, we consider GPCOSARCHM-II and obtain their marginal density and distributions.

The rest of the paper is organized as follows. In Section 2, we consider the normalized spacing of general progressively Type-II censored order statistics arising from an iid random sample with standard exponential distribution (GPCOSEXP-II) and show that they are independent and obtain their density function. Then, we obtain the density and distribution function of GPCOSEXP-II and show that this results reduces to the corresponding results in Kamps and Cramer [7] for special cases. Moreover, we obtain the marginal density and distribution functions of GPCOS-II with a similar tact used in Kamps and Cramer [7] for obtaining the corresponding results for PCOS-II. In Section 3, we obtain the marginal density and distribution functions of GPCOSARCHM-II. Finally, an example of a special case of GPCOSARCHM-II is provided for illustrative purpose.

2. General progressively Type-II censored order statistics arising from iid units

In this section, we consider the general progressive Type-II censoring considered by [1] and [2] and obtain some distributional results in this case. Specifically, we shall consider the following scheme. Suppose that N iid units are under a life-test and the first r units to fail are not observed; then, at the time of the $(r+1)$-th failure, R_1 surviving units are censored from the test; next, at the time of the $(r+i)$-th failure, R_i surviving units are censored from the test (for $i = 2, \ldots, m$). Let $\mathbf{R}_1 = (R_{r+1}, \ldots, R_m)$ and $X^{\mathbf{R}_1}_{j:m:N}$ denote the j-th GPCOS-II from m unit failures observed from N iid random variables with progressive censoring scheme $\mathbf{R}_1$. It is evident that $N = \sum_{j=r+1}^{m}(R_i + 1)$ and the joint density function of $(X^{\mathbf{R}_1}_{r+1:m:N}, \dots, X^{\mathbf{R}_1}_{r:m:N})$ equals

$$f_{X^{\mathbf{R}_1}_{r+1:m:N},\dots,X^{\mathbf{R}_1}_{m:m:N}}(x_{r+1},\dots,x_m) = d_m \prod_{j=r+1}^{m} f(x_j)\{\bar{F}(x_j)\}^{R_j}\{F(x_{r+1})\}^r \tag{2.1}$$

for $x_1 \le x_2 \le \cdots \le x_r$, where $d_m = \binom{N}{r}(N-r)\Big(\prod_{k=r+2}^{m}\gamma_k\Big)$ and $\gamma_j = \sum_{v=j}^{m}(R_v+1)$. We refer the reader to Balakrishnan and Aggarwala [2] and the references therein for a comprehensive discussion and inferential procedures based on progressive censoring. Now, consider the normalized spacing $Y^{\mathbf{R}_1}_{v:m:N} = \gamma_v(X^{\mathbf{R}_1}_{v:m:N} - X^{\mathbf{R}_1}_{v-1:m:N})$, for $v = r+2,\dots,m$, where $X^{\mathbf{R}_1}_{r+1:m:N},\dots,X^{\mathbf{R}_1}_{m:m:N}$ are GPCOSEXP-II. The following result shows that the first $m-r-2$ spacing from GPCOSEXP-II are iid random variables with standard exponential distribution and independent of the last one. They are immediate consequences of Theorem 3.4 in [2].

Proposition 2.1. *Y_v $(v = r+2,\dots,m)$'s are iid random variables with standard exponential distribution and $Y^{\mathbf{R}_1}_{r+1:m:N} = \gamma_{r+1}x^{\mathbf{R}_1}_{r+1:m:N} = (N-r)x^{\mathbf{R}_1}_{r+1:m:N}$ is independent of $Y^{\mathbf{R}_1}_{v:m:N}$, $v = r+2,\dots,m$ with density function*

$$\binom{N}{r}(1-e^{-\frac{y_{r+1}}{N-r}})^r \exp\Big(-y_{r+1}\Big). \tag{2.2}$$

Now, using Proposition 2.1, we can obtain the marginal density and distribution function of GPCOSEXP-II.

Proposition 2.2. *Under the assumptions of Proposition 2.1, the density and distribution functions of $X^{\mathbf{R}_1}_{r+1:m:N}$ are given by*

$$(N-r)\binom{N}{r}(1-e^{-x_{r+1}})^r \exp\Big(-(N-r)x_{r+1}\Big) \tag{2.3}$$

and

$$\sum_{v=r+1}^{N}\binom{N}{v}(1-e^{-x_{r+1}})^v \exp\Big(-(N-v)x_{r+1}\Big), \tag{2.4}$$

respectively, and the density and distribution functions of $X^{\mathbf{R}_1}_{j:m:N}$, $r+1 < j \le m$, are

$$(N-r)\binom{N}{r}\Big(\prod_{v=r+2}^{j}\gamma_v\Big)\sum_{v=r+2}^{j}\frac{a_v^{r+1}(j)e^{-\gamma_v x}}{\beta(r+1, N-r-\gamma_v)} \tag{2.5}$$

and

$$1-(N-r)\binom{N}{r}\Big(\prod_{v=r+2}^{j}\gamma_v\Big)\sum_{v=r+2}^{j}\frac{a_v^{r+1}(j)}{\gamma_v}\frac{e^{-\gamma_v x}}{\beta(r+1, N-r-\gamma_v)}, \tag{2.6}$$

respectively, where $a_v^{r+1}(j) = \prod_{k=r+2,k\neq v}^{j} \frac{1}{\gamma_k - \gamma_v}$ *and*

$$\beta(r+1, N-r-\gamma_v) = \frac{(N-\gamma_v)!}{r!(N-r-\gamma_v-1)!} = \int_0^1 (1-u)^r u^{N-r-\gamma_v-1} du\,.$$

Remark 2.3. Under the assumptions of Proposition 2.2, if we set $r = 0$, we have $\frac{a_v^1(j)}{\beta(1,N-\gamma_v)} = \prod_{k=1,k\neq v}^{j} \frac{1}{\gamma_k-\gamma_v}$. Then, the expressions in (2.5) and (2.6) reduce to those in (6) and (5) of Kamps and Cramer [7], respectively.

Suppose that G is the standard exponential distribution and F is an arbitrary distribution function. Then, following theorems follow using Propositions 2.1 and 2.2, applying the identity $U_{v:m:N}^{\mathbf{R},\mathbf{P}}$ and quantile transformation $Z_{v:m:N}^{\mathbf{R},\mathbf{P}} = F^{-1}(U_{v:m:N}^{\mathbf{R},\mathbf{P}})$, $v = r+1, \ldots, m\,.$

Theorem 2.4. *The density and distribution functions of the* $(r+1)$*-th GPCOS-II arising from* N *iid random variables based on a distribution function* F *equals*

$$\frac{N!}{(N-r-1)!r!} f(x_{r+1}) F^r(x_{r+1}) \Big(1 - F(x_{r+1})\Big)^{N-r-1} \tag{2.7}$$

and

$$\sum_{v=r+1}^{N} \frac{N!}{(N-v-1)!v!} F^v(x_{r+1}) \Big(1 - F(x_{r+1})\Big)^{N-v}, \tag{2.8}$$

respectively, which are indeed the density and distribution functions of the usual $(r+1)$*-th order statistic from* N *iid random variables from distribution* F*.*

Theorem 2.5. *The density and distribution function of the* j*-th GPCOS-II arising from* N *iid random variables based on a distribution function* F *(for* $r+1 < j \leq m$*) equals*

$$(N-r)\binom{N}{r}\left(\prod_{v=r+2}^{j} \gamma_v\right) f(x) \sum_{v=r+2}^{j} \frac{a_v^{r+1}(j)(1-F(x))^{\gamma_v-1}}{\beta(r+1, N-r-\gamma_v)} \tag{2.9}$$

and

$$1-(N-r)\binom{N}{r}\left(\prod_{v=r+2}^{j} \gamma_v\right) \sum_{v=r+2}^{j} \frac{a_v^{r+1}(j)(1-F(x))^{\gamma_v}}{\gamma_v \beta(r+1, N-r-\gamma_v)}\,, \tag{2.10}$$

respectively, where $a_v^{r+1}(\cdot)$ *and* $\beta(\cdot,\cdot)$ *are as defined earlier in Proposition 2.1.*

3. General progressively Type-II censored order statistics arising from dependent units

In this section, we consider GPCOSARCHM-II and obtain the marginal density and distribution functions. Let us consider the progressive Type-II censoring scheme arising from arising from identical units under test which are jointly distributed according to an Archimedean copula with completely monotone generator, i.e., the random vector $(X_1,\dots,X_N)$ with joint density function (1.5). Let $X^{\mathbf{R}_1}_{j:m:N}$, $j=r+1,\dots,m$ be the j-th GPCOSARCHM-II, where $\mathbf{R}_1=(R_{r+1},\dots,R_m)$. Therefore, using the sampling scheme described in Section 2, we can obtain the joint density function of $(X^{\mathbf{R}_1}_{1:m:N},\dots,X^{\mathbf{R}_1}_{m:m:N})$ as

$$\begin{aligned}(3.1)\quad &\sum_{D_N} Pr\Big\{X_{i_1}\le x_{r+1},\dots,X_{i_r}\le x_{r+1}, x_{r+1}<X_{i_{r+1}}\le x_{r+1}+dx,\\ &X_{i_{r+2}}>x_{r+1},\dots,X_{i_{r+R_{r+1}+1}}>x_{r+1}, x_{r+2}<X_{i_{r+R_{r+1}+2}}\le x_{r+2}+dx,\\ &X_{i_{r+R_{r+1}+3}}>x_{r+2},\dots,X_{i_{r+R_{r+1}+R_{r+2}+2}}>x_{r+2}, x_{r+3}<X_{i_{r+R_{r+1}+R_{r+2}+3}}\\ &\le x_{r+3}+dx,\dots,X_{i_{r+R_1+\cdots+R_m+(m-r)}}>x_m\Big\}\end{aligned}$$

for $x_{r+1}<x_{r+2}<\cdots<x_m$, where the summation D_N extends over all permutations $(i_1,\dots,i_N)$ of $1,\dots,N$. Let $G(x_j,\alpha)=G^{\alpha}(F(x_j))$ and $g(x_j,\alpha)=\frac{d}{dx_j}G(x_j,\alpha)$. Then, by using (1.6), we can express the joint density function as

$$(3.2)\quad \int_0^{\infty} d_m\{G(x_{r+1},\alpha)\}^r\prod_{k=r+1}^{m} g(x_k,\alpha)\Big\{1-G(x_k,\alpha)\Big\}^{R_k} dM_{\psi}(\alpha)\,.$$

where $d_m=\binom{N}{r}(N-r)\prod_{k=r+2}^{m}\Big(N-r-\sum_{v=r+1}^{k-1}(R_v+1)\Big)=\binom{N}{r}(N-r)\prod_{k=r+2}^{m}\Big(\sum_{v=k}^{m}(R_v+1)\Big)=\binom{N}{r}(N-r)\prod_{k=r+2}^{m}\gamma_k$. We can represent the density function $f_{X^{\mathbf{R}}_{r+1:m:N},\dots,X^{\mathbf{R}}_{m:m:N}}$ given in (3.2) as

$$(3.3)\quad f_{\mathbf{X}^{\mathbf{R},\mathbf{P}}}(t_{r+1},\dots t_m)=\int_0^{\infty} f_{\mathbf{X}^{\mathbf{R},\mathbf{P}}_{\alpha}}(t_{r+1},\dots t_m)dM_{\psi}(\alpha),\quad t_{r+1}\le\cdots\le t_m,$$

where $f_{\mathbf{X}^{\mathbf{R},\mathbf{P}}_{\alpha}}(t_{r+1},\dots t_m)$ is defined as the integrand in (3.2). Notice that $f_{\mathbf{X}^{\mathbf{R},\mathbf{P}}_{\alpha}}$ is the joint density function of GPCOS-II arising from an independent sample, $\mathbf{X}^{\mathbf{R},\mathbf{P}}_{\alpha}=(X^{\mathbf{R},\mathbf{P}}_{r+1:m:N;\alpha},\dots,X^{\mathbf{R},\mathbf{P}}_{m:m:N;\alpha})$, with baseline distributions $G^{\alpha}(F(\cdot))$, $\alpha\ge 0$ (see [1] and [2]).

Now, we can obtain the marginal density and distribution function of the $(r+1)$-th GPCOSARCHM-II by using the representation in (3.3) and Theorem 2.4.

Theorem 3.1. *The marginal density and distribution functions of* $X^{\mathbf{R},\mathbf{P}}_{r+1:m:N}$ *are given, respectively, by*

$$(3.4)\quad \frac{N!\sum_{i=0}^{N-r-1}\binom{N-r-1}{i}(-1)^{i+1}\psi'\Big((r+i+1)\psi^{-1}(F(x_{r+1}))\Big)f(x_{r+1})}{(N-r-1)!r!\psi'\Big(\psi^{-1}(F(x_{r+1}))\Big)}$$

and

$$(3.5)\quad \sum_{i=0}^{N-r-1}\frac{N!(-1)^{i+1}}{(N-r-1-i)!i!r!(r+i+1)}\psi\Big((r+i+1)\psi^{-1}(F(x_{r+1}))\Big)\,.$$

Proof. From (3.3) and (2.9), we can obtain the marginal density function of $X^{\mathbf{R},\mathbf{P}}_{r+1:m:N}$ as

$$
\begin{aligned}
&(N-r)\binom{N}{r}\int_0^\infty g(x_{r+1},\alpha)\Big\{G(x_{r+1},\alpha)\Big\}^r\Big\{1-G(x_{r+1},\alpha)\Big\}^{N-r-1}dM_\psi(\alpha)\\
&=N\binom{N-1}{r}\sum_{i=0}^{N-r-1}\binom{N-r-1}{i}(-1)^i\int_0^\infty g(x_{r+1},\alpha)\Big\{G(x_{r+1},\alpha)\Big\}^{r+i}dM_\psi(\alpha)\\
&=\sum_{i=0}^{N-r-1}\frac{N\binom{N-1}{r}\binom{N-r-1}{i}(-1)^i f(x_{r+1})}{\psi'\Big(\psi^{-1}(F(x_{r+1}))\Big)}\int_0^\infty \alpha e^{-\alpha\Big((r+i+1)\psi^{-1}(F(x_{r+1}))\Big)}dM_\psi(\alpha)\\
&=\frac{N!\sum_{i=0}^{N-r-1}\binom{N-r-1}{i}(-1)^{i+1}\psi'\Big((r+i+1)\psi^{-1}(F(x_{r+1}))\Big)f(x_{r+1})}{(N-r-1)!r!\psi'\Big(\psi^{-1}(F(x_{r+1}))\Big)},
\end{aligned}
$$

as required in (3.4), where the second equality follows from $G(x,\alpha)=\exp(-\alpha\psi^{-1}(F(x)))$ and $g(x,\alpha)=\frac{\alpha\exp(-\alpha\psi^{-1}(F(x)))f(x)}{\psi'(\psi^{-1}(F(x)))}$, and the last equality follows from $\psi'(x)=-\int_0^\infty \alpha e^{-\alpha x}dM_\psi(\alpha)$. Integrating the density in (3.4), we obtain the cumulative distribution function as in (3.5). □

Remark 3.2. Since the GPCOSARCHM-II $X^{R_{r+1},\ldots,R_m}_{r+1:m:n}$ is the usual $(r+1)$-th order statistic, the expressions in (3.4) and (3.5) are indeed the density and distribution functions of the usual $(r+1)$-th order statistic from N dependent variables.

Next, the marginal density and distribution functions of the j-th (for $r+1<j\le m$) GPCOSARCHM-II are derived in the following theorem.

Theorem 3.3. *The marginal density and distribution functions of* $X^{\mathbf{R},\mathbf{P}}_{j:m:N}$ *(for* $r+1<j\le m$*) are given, by*

$$
\begin{aligned}
&\frac{a(N,r,j)f(x)}{\psi'(\psi^{-1}(F(x)))}\sum_{v=r+2}^{j}\sum_{i=0}^{\gamma_v-1}\frac{a_v^{r+1}(j)\binom{\gamma_v-1}{i}(-1)^i}{\beta(r+1,N-r-\gamma_v)}\\
&\qquad\times\psi'\Big((i+1)\psi^{-1}\Big(F(x)\Big)\Big)
\end{aligned}
\tag{3.6}
$$

and

$$
a(N,r,j)\sum_{v=r+2}^{j}\sum_{i=0}^{\gamma_v-1}\frac{a_v^{r+1}(j)\binom{\gamma_v-1}{i}\frac{(-1)^i}{i+1}}{\beta(r+1,N-r-\gamma_v)}\psi\Big((i+1)\psi^{-1}\Big(F(x)\Big)\Big),
\tag{3.7}
$$

respectively, where $a(N,r,j)=(N-r)\binom{N}{r}\Big(\prod_{v=r+2}^{j}\gamma_v\Big)$.

Proof. Using (3.3) and (2.9), we can obtain the density function of $X^{\mathbf{R},\mathbf{P}}_{j:m:N}$ (for $r+1<j\leq m$) as

$$a(N,r,j)\sum_{v=r+2}^{j}\frac{a_v^{r+1}(j)}{\beta(r+1,N-r-\gamma_v)}\int_0^\infty g(x,\alpha)(1-G(x,\alpha))^{\gamma_v-1}dM_\psi(\alpha)$$
$$=a(N,r,j)\sum_{v=r+2}^{j}\sum_{i=0}^{\gamma_v-1}\frac{a_v^{r+1}(j)\binom{\gamma_v-1}{i}(-1)^i}{\beta(r+1,N-r-\gamma_v)}\int_0^\infty g(x,\alpha)G^i(x,\alpha)dM_\psi(\alpha)\,.$$

Since $G(x,\alpha)=\exp\Big(-\alpha\psi^{-1}\Big(F(x)\Big)\Big)$ and $g(x,\alpha)=-\frac{\alpha\exp(-\alpha\psi^{-1}(F(x)))f(x)}{\psi'(\psi^{-1}(F(x)))}$, we can write the marginal density function of $X^{\mathbf{R},\mathbf{P}}_{j:m:N}$ as

$$\frac{a(N,r,j)f(x)}{\psi'(\psi^{-1}(F(x)))}\sum_{v=r+2}^{j}\sum_{i=0}^{\gamma_v-1}\frac{a_v^{r+1}(j)\binom{\gamma_v-1}{i}(-1)^{i+1}}{\beta(r+1,N-r-\gamma_v)}$$
$$\times\int_0^\infty\alpha\exp\Big\{-\alpha(i+1)\psi^{-1}\Big(F(x)\Big)\Big\}dM_\psi(\alpha)\,.$$

Then, the required expression for the marginal density function in (3.6) follows from the fact that

$$\int_0^\infty\alpha\exp(-\alpha x)dM_\psi(\alpha)=-\psi'(x)\,.$$

Finally, the expression for the cumulative distribution function in (3.7) is obtained by integrating (3.6). □

In the following we provide an illustrative example in which we consider GPCOS-II arising from dependent variables distributed according to the generalized Gumbel-Hougaard family of Archimedean copula.

Example 3.4. Let $\psi(s)=\exp(-s^{1/\theta})$ for $\theta\geq 1$. Then, by Theorem 3.1, the marginal density function of $X^{\mathbf{R}_1}_{r+1:m:N}$ is given by

$$\frac{f(x)}{F(x)}N\binom{N-1}{r}\sum_{i=0}^{N-r-1}\binom{N-r-1}{i}(-1)^i(r+i+1)^{1/\theta-1}(F(x))^{(r+i+1)^{(1/\theta)}}\,.$$

Under the assumptions of Theorem 3.3, the marginal density function of $X^{\mathbf{R}_1}_{j:m:N}$, for $r+1<j\leq m$, is given by

$$\frac{a(N,r,j)f(x)}{F(x)}\sum_{v=r+2}^{j}\sum_{i=0}^{\gamma_v-1}\frac{a_v^{r+1}(j)\binom{\gamma_v-1}{i}(-1)^i}{\beta(r+1,N-r-\gamma_v)}(i+1)^{1/\theta-1}(F(x))^{(i+1)^{(1/\theta)}}$$

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