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ON SANDWICH THEOREMS FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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Abstract. The purpose of this paper is to derive some subordination and superordination results for certain analytic functions in the open unit disk.
Keywords: Subordination, superordination, integral operators, Hadamard product.
MSC(2010): Primary 30C45; Secondary 30C80.

1. Introduction and preliminaries

Let H be the class of all functions analytic in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ and for $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let $H[a, n] = \{f \in H : f(z) = a + a_n z^n + ...\}$. Let

$$\mathcal{A}_n = \{f \in H : f(z) = z + a_{n+1}z^{n+1} + ...\},$$

and set $\mathcal{A} := \mathcal{A}_1$. We say that the function $f \in H$ is subordinate to the function $g \in H$ in the unit disc U (written $f \prec g$) if there exists an analytic function w in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$ such that $f(z) = g(w(z))$ in U. If g is univalent in U, then the following equivalence relationship holds true:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \; and \; f(U) \subset g(U).$$

For $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Recently, Komatu [5] introduced a certain integral operator ${I_a}^{\lambda}$ defined by

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$$(1.1)\quad {I_a}^{\lambda} f(z) = \frac{a^{\lambda}}{\Gamma(\lambda)} \int_0^1 t^{a-2} (\log \frac{1}{t})^{\lambda-1} f(tz) dt \quad (z \in U, a > 0, \lambda \geq 0, f \in \mathcal{A}).$$

If $f \in \mathcal{A}_n$ is of the form $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$, then it is easy to see from (1.1) that

$${I_a}^{\lambda} f(z) = z + \sum_{k=n+1}^{\infty} (\frac{a}{a+k-1})^{\lambda} a_k z^k.$$

We now introduce the operator $L_a^{\lambda}(b,c,z) : \mathcal{A}_n \to \mathcal{A}_n$ by

$$(1.2)\qquad L_a^{\lambda}(b,c,z)f(z) = \left[z + \sum_{k=n+1}^{\infty} \frac{(b)_{k-1}}{(c)_{k-1}} (\frac{a+1}{a+k})^{\lambda} z^k \right] * f(z)$$

where $\lambda \geq 0, b, a \in \mathbb{C}, Re(a) > -1, c \in \mathbb{C} - \{0, -1, -2, ...\}$. In view of (1.2), for $f \in \mathcal{A}_n$ and $z \in U$ it follows that

$$(1.3)\qquad z(L_a^{\lambda+1}(b,c,z)f)'(z) = (1+a) L_a^{\lambda}(b,c,z)f(z) - a L_a^{\lambda+1}(b,c,z)f(z),$$

and

$$(1.4)\qquad z(L_a^{\lambda}(b,c,z)f)'(z) = b L_a^{\lambda}(b+1,c,z)f(z) - (b-1) L_a^{\lambda}(b,c,z)f(z).$$

Furthermore let Q be the set of all analytic and univalent functions on the set $\overline{U}/E(f)$, such that $f'(\zeta) \neq 0$ for all $\zeta \in \partial U / E(f)$. where

$$E(f) := \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\},$$

The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$. We shall need the following definition and lemmas to prove our results.

Definition 1.1. A function $p(z,t) : U \times [0, \infty) \to \mathbb{C}$ is called a subordination (or Loewner) chain if $p(\cdot, t)$ is analytic and univalent in U, for all $t \geq 0$ and $p(\cdot, t_1) \prec p(\cdot, t_2)$, for all $0 \leq t_1 \leq t_2$.

The next lemma gives a sufficient condition so that the function $p(\cdot, t)$ to be a subordination chain.

Lemma 1.2. *(see [9, p.159]) Let*

$$p(z,t) = a_1(t) z + a_2(t) z^2 + ... \quad (a_1(t) \neq 0, t \geq 0),$$

with $a_1(t) \neq 0$ *for all* $t \geq 0$ *and* $\lim_{t \to \infty} |a_1(t)| = \infty$.
Suppose that $p(.,t)$ *is analytic in* U *for all* $t \geq 0$, *and* $p(z, \cdot)$ *is continuously differentiable on* $[0, \infty)$ *for any* $z \in U$. *If* $p(z,t)$ *satisfies*

$$Re \left(z \frac{\partial p(z,t)/\partial z}{\partial p(z,t)/\partial t} \right) > 0 \qquad (z \in U, t \geq 0),$$

and

$$|p(z,t)| \leq k_0 |a_1(t)|, \qquad |z| < r_0 < 1, t \geq 0$$

for some positive constants k_0 *and* r_0*, then,* $p(z,t)$ *is a subordination chain.*

Lemma 1.3. *(see [7]) Suppose that the function* $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ *satisfies the condition*

$$\psi(ix, y; z) \leq \delta,$$

for real $x,\ y \leq -n(\frac{1+x^2}{2})$ *and for all* $z \in U$*. If the function* $\varphi(z) = 1 + a_n z^n + ...$ *is analytic in* U *and*

$$Re\{\psi(\varphi(z), z\varphi'(z); z)\} > \delta,$$

then $Re\{\varphi(z)\} > 0$ *in* U*.*

Lemma 1.4. *(see [8]) Let* $q \in H[a,1]$ *and* $\phi : \mathbb{C}^2 \to \mathbb{C}$*. Also set*

$$\phi(q(z), zq'(z)) \equiv h(z) \quad (z \in U).$$

If $P(z,t) := \phi(q(z), tzq'(z))$ *is subordination chain and* $p \in H[a,1] \cap Q(a)$*, then*

$$h(z) \prec \phi(p(z), zp'(z))$$

implies that

$$q(z) \prec p(z).$$

Furthermore, if $\phi(q(z), zq'(z)) = h(z)$ *has a univalent solution* $q \in Q(a)$*, then* q *is the best subordinate.*

Making use of the principle of subordination between analytic functions, many authors (see for example [4,6–8]) obtained some interesting subordination-preserving properties for certain subclasses of the class $\mathcal{A}_1$. In the present paper we aim to investigate the subordination and superordination-preserving properties of the integral operator $L_a^{\lambda}(b,c,z)f$ in the class $\mathcal{A}_n$. Our results extends many results obtained by other authors (see for example [1–3]). Throughout the following section we assume that $b \in \mathbb{C}, \lambda \geq 0$ and $c \in \mathbb{C} - \{0, -1, -2, -3, ...\}$.

2. Main results

Our first result involve the integral operator $L_a^{\lambda}(b,c,z)$.

Lemma 2.1. *Let* f *be in the class* $\mathcal{A}_n, (L_a^{\lambda+1}(b,c,z)f)'(z) \neq 0$ *for all* $z \in U$ *and* $a \in \mathbb{C}$ *with* $Re(a) > 0$*. If*

$$Re\left\{1 + \frac{z(L_a^{\lambda}(b,c,z)f)''(z)}{(L_a^{\lambda}(b,c,z)f)'(z)}\right\} > -\delta \tag{2.1}$$

then

$$Re\left\{1 + \frac{z(L_a^{\lambda+1}(b,c,z)f)''(z)}{(L_a^{\lambda+1}(b,c,z)f)'(z)}\right\} > 0 \tag{2.2}$$

where

$$\delta = n\frac{1 + |a|^2 - |1 - a^2|}{4Re(a)}. \tag{2.3}$$

Proof. Let $f \in \mathcal{A}_n$ and

$$p(z) = 1 + \frac{z(L_a^{\lambda+1}(b,c,z)f)''(z)}{(L_a^{\lambda+1}(b,c,z)f)'(z)}. \tag{2.4}$$

It is easy to see that p is analytic in U and $p(z) = 1 + b_n z^n + b_{n+1}z^{n+1} + ...$ Taking the logarithmic differentiation in (2.4) and using identity (1.3) in the resulting equation, we get that

$$p(z) + \frac{zp'(z)}{a + p(z)} = 1 + \frac{z(L_a^{\lambda}(b,c,z)f)''(z)}{(L_a^{\lambda}(b,c,z)f)'(z)}. \tag{2.5}$$

To make use of Lemma 1.3 we consider the function

$$\psi(r,s) = r + \frac{s}{r+a} + \delta.$$

In view of (2.1) and (2.5) we have $Re\{\psi(p(z), zp'(z)\} > 0$. Furthermore for $x \in \mathbb{R}$ and $y < \frac{-n}{2}(1+x^2)$ we have

$$Re\{\psi(ix,y)\} = Re\{ix + \frac{y}{a+ix} + \delta\} = \frac{yRe(a)}{|a+ix|^2} + \delta \leq -\frac{k}{2|a+ix|^2},$$

where

$$k = (nRe(a) - 2\delta)x^2 - 4\delta Im(a)x - 2\delta|a|^2 + nRe(a).$$

But in view of the value of δ given by (2.1), we know that k is a perfect square, which implies that

$$Re\{\psi(ix,y)\} \leq 0 \quad (x \in \mathbb{R}, y \leq -n\frac{1+x^2}{2}).$$

Now by Lemma 1.3, we deduce that

$$Re(p(z)) > 0 \quad (z \in U),$$

this completes the proof. $\square$

Lemma 2.2. *Let f be in the class $\mathcal{A}_n, (L_a^{\lambda+1}(b,c,z)f)'(z) \neq 0$ for all $z \in U$ and $a > -1$. If*

$$Re\left\{1 + \frac{z(L_a^{\lambda}(b,c,z)f)''(z)}{(L_a^{\lambda}(b,c,z)f)'(z)}\right\} > -(n+a) + \frac{(1+a)}{4} \tag{2.6}$$

then

$$Re\left\{\sqrt{\frac{z(L_a^{\lambda+1}(b,c,z)f)''(z)}{(L_a^{\lambda+1}(b,c,z)f)'(z)} + (1+a)}\right\} > \frac{\sqrt{1+a}}{2}. \tag{2.7}$$

Proof. Let $f \in \mathcal{A}_n$ and set

$$p(z) = \frac{2}{\sqrt{1+a}}\sqrt{1 + a + \frac{z(L_a^{\lambda+1}(b,c,z)f)''(z)}{(L_a^{\lambda+1}(b,c,z)f)'(z)}} - 1. \tag{2.8}$$

Then it is clear that p is analytic in U and $p(z) = 1 + b_n z^n + b_{n+1}z^{n+1} + ...$. Taking the logarithmic differentiation in (2.8) and using identity (1.3) in the resulting equation, we obtain

$$\frac{(1+a)}{4}(1+p(z))^2 + 2\frac{zp'(z)}{1+p(z)} - a = 1 + \frac{z(L_a^\lambda(b,c,z)f)''(z)}{(L_a^\lambda(b,c,z)f)'(z)}. \tag{2.9}$$

To make use of Lemma 1.3 we consider the function

$$\psi(r,s) = \frac{(1+a)}{4}(1+r)^2 + \frac{2s}{1+r} + n - \frac{(1+a)}{4},$$

then by (2.6), (2.9) we have $Re\{\psi(p(z), zp'(z)\} > 0$. Furthermore for $x \in \mathbb{R}$ and $y < \frac{-n}{2}(1+x^2)$ we have

$$\begin{aligned} Re\{\psi(ix,y)\} &= \frac{(1+a)}{4}(1-x^2) + 2Re\frac{y}{1+ix} + n - \frac{(1+a)}{4} \\ &< \frac{(1+a)}{4}(1-x^2) - 2\frac{n(1+x^2)}{2(1+x^2)} + n - \frac{(1+a)}{4} \leq 0 \end{aligned}$$

Now in view of Lemma 1.3, we get that

$$Re(p(z)) > 0 \quad (z \in U),$$

this completes the proof. $\square$

By putting $\lambda = 0, b = c = 1$ and $a = 0$ in the Lemma 2.2 we obtain the following corollary which is generalization of well known result in the literature.

Corollary 2.3. *Let f be in the class $\mathcal{A}_n$ and $f(z) \neq 0$ for all $z \in U$. If*

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > -n + \frac{1}{4},$$

then

$$Re\sqrt{\frac{zf'(z)}{f(z)}} > \frac{1}{2}.$$

Theorem 2.4. *Let $f, g \in \mathcal{A}_n, a \in \mathbb{C}, Re(a) > 0$ and $(L_a^{\lambda+1}(b,c,z)g)'(z) \neq 0$. If*

$$Re\{1 + \frac{z(L_a^\lambda(b,c,z)g)''(z)}{(L_a^\lambda(b,c,z)g)'(z)}\} > -\delta,$$

where δ is defined in (2.3). Then the relation

$$L_a^\lambda(b,c,z)f \prec L_a^\lambda(b,c,z)g$$

implies

$$L_a^{\lambda+1}(b,c,z)f \prec L_a^{\lambda+1}(b,c,z)g.$$

Proof. Let F, G and p be defined by

$$F(z) := L_a^{\lambda+1}(b,c,z)f,\ G(z) := L_a^{\lambda+1}(b,c,z)g,\ p(z) := 1 + \frac{G''(z)}{G'(z)}. \tag{2.10}$$

In view of Lemma 2.1, we have $Rep(z) > 0$. From definition of G and $L_a^\lambda(b,c,z)g$ and taking differentiation of G we conclude

$$G(z)(z\frac{G'(z)}{G(z)}) + a = (1+a)L_a^\lambda(b,c,z)g(z). \tag{2.11}$$

But $Rep(z) > 0$ implies that the function G is convex (univalent) in U. We can assume, without loss of generality, that G is univalent on $\overline{U}$ and that $G'(\zeta) \neq 0 (|\zeta| = 1)$. To prove $F \prec G$, we let function $L(z,t)$ be defined by

$$L(z,t) = \frac{a}{1+a}G(z) + \frac{1+t}{1+a}zG'(z) = a_1(t)z + ... \tag{2.12}$$

It is easy to see that $L(z,t)$ is analytic in $|z| < 1$, for all $t \geq 0$. It is also continuously differentiable on $[0,\infty)$, for each $|z| < 1$. According to (2.11), we have $L(z,0) = L_a^\lambda(b,c,z)g(z)$. From (2.12) we obtain

$$a_1(t) = \frac{\partial L}{\partial z}(0,t) = [\frac{a}{a+1} + \frac{1+t}{a+1}]G'(0) = 1 + \frac{t}{a+1} \neq 0 \tag{2.13}$$

and $\lim_{t\to\infty} |a_1(t)| = \infty$. A simple calculation yields

$$Re\big(z\frac{\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\big) = Rea + (1+t)Re(1 + \frac{G''(z)}{G'(z)}) = Rea + (1+t)Rep(z) > 0.$$

Since G is convex and normalized in the unit disk, therefore the following well-known growth and distortion sharp inequalities (see [9, p.45,48]) are true:

$$\frac{r}{1+r} \leq |G(z)| \leq \frac{r}{1-r}, \qquad if \qquad |z| \leq r \tag{2.14}$$

and

$$\frac{r}{(1+r)^2} \leq |G'(z)| \leq \frac{r}{(1-r)^2}, \qquad if \qquad |z| \leq r. \tag{2.15}$$

Now by making use of (2.14) and (2.15) in the relations (2.12) and (2.13) we obtain

$$\begin{aligned}\frac{|L(z,t)|}{|a_1(t)|} &\leq \frac{|a||G(z)| + (1+t)|zG'(z)|}{|a+1+t|}\\ &\leq \frac{r}{(1-r)^2}\frac{|a|+1+t}{|a+1+t|} \leq \frac{r}{(1-r)^2}\frac{|a|+1+t}{Re(a)+1+t}.\end{aligned}$$

If we define $h(t) = \frac{|a|+1+t}{Re(a)+1+t}$ it is easy to see that h is a decreasing function for $t \geq 0$, and so

$$h(t) \leq \frac{|a|+1}{Re(a)+1}.$$

Therefore from above inequality we conclude that

$$\frac{|L(z,t)|}{|a_1(t)|} \le \frac{|a|+1}{Re(a)+1}\frac{r}{(1-r)^2} \qquad |z| \le r, t \ge 0.$$

Hence both assumptions of Lemma 1.2 hold. Now by Lemma 1.2 we obtain that $L(z,t)$ is a subordination chain. From Definition 1.1, it follows that

$$L_a^{\lambda}(b,c,z)g(z) = L(z,0) \prec L(z,t) \ (t \ge 0),$$

which implies

$$L(\zeta,t) \notin L(U,0) = L_a^{\lambda}(b,c,z)g(U) \ (\zeta \in \partial U, t \ge 0). \tag{2.16}$$

Suppose that $F \not\prec G$. Then there exists $z_0 \in U, \zeta_0 \in \partial U$ such that $F(z_0) = G(\zeta_0)$ and $F(|z| < |z_0|) \subset G(U)$. Hence by Jack's Lemma we have $z_0F'(z_0) = (1+t)\zeta_0 G'(\zeta_0)$ with $t \ge 0$. From (2.12), we obtain

$$\begin{aligned} L(\zeta_0,t) &= \frac{a}{1+a}G(\zeta_0) + \zeta_0\frac{1+t}{1+a}G'(\zeta_0) \\ &= \frac{a}{1+a}F(z_0) + z_0\frac{1}{1+a}F'(z_0) \\ &= L_a^{\lambda}(b,c,z)f(z_0) \in L_a^{\lambda}(b,c,z)g(U). \end{aligned}$$

But this is contradicts (2.16), and thus we deduce that $F \prec G$. □

By setting $b = c = 1$ and $\lambda = 0$ in the Theorem 2.4 we obtain.

Corollary 2.5. *Let $a \in \mathbb{C}, Re(a) > 0$ and $f, g \in \mathcal{A}_n$. If*

$$Re(1 + \frac{g''(z)}{g'(z)}) > -n\frac{1+|a|^2-|1-a^2|}{4Re(a)},$$

then

$$f \prec g \Rightarrow \frac{1+a}{z^a}\int_0^z t^{a-1}f(t)dt \prec \frac{1+a}{z^a}\int_0^z t^{a-1}g(t)dt.$$

Corollary 2.6. *Let $a > 0$ and $f, g \in \mathcal{A}_n$. If*

$$Re(1 + \frac{z(L_a^{\lambda}(b,c,z)g)''(z)}{(L_a^{\lambda}(b,c,z)g)'(z)}) > -\delta_1,$$

then

$$L_a^{\lambda}(b,c,z)f \prec L_a^{\lambda}(b,c,z)g \Rightarrow L_a^{\lambda+1}(b,c,z)f \prec L_a^{\lambda+1}(b,c,z)g,$$

where

$$\delta_1 = \begin{cases} -\frac{an}{2} & 0 < a < 1 \\ -\frac{n}{2a} & a \ge 1. \end{cases}$$

By putting $b = c = 1$ and $\lambda = 0$ in the Corollary 2.6 we have

Corollary 2.7. *Let $a > 0$ and $f, g \in \mathcal{A}_n$. If*

$$Re(1 + \frac{zg''(z)}{g'(z)}) > -\delta_1,$$

then

$$f \prec g \Rightarrow \frac{1+a}{z^a}\int_0^z t^{a-1}f(t)dt \prec \frac{1+a}{z^a}\int_0^z t^{a-1}g(t)dt,$$

where

$$\delta_1 = \begin{cases} -\frac{an}{2} & 0 < a < 1 \\ \\ -\frac{n}{2a} & a \geq 1. \end{cases}$$

By using similar arguments as in the proofs of Lemma 2.1 and Theorem 2.4 we obtain the following results and we omit details.

Lemma 2.8. *Let f be in the class $\mathcal{A}_n, (L_a^\lambda(b,c,z)f)'(z) \neq 0$ for all $z \in U$ and $b \in \mathbb{C}, Re(b) > 1$. If*

$$Re\left\{1 + \frac{z(L_a^\lambda(b+1,c,z)f)''(z)}{(L_a^\lambda(b+1,c,z)f)'(z)}\right\} > -\delta_2 \tag{2.17}$$

then

$$Re\left\{1 + \frac{z(L_a^\lambda(b,c,z)f)''(z)}{(L_a^\lambda(b,c,z)f)'(z)}\right\} > 0 \tag{2.18}$$

where

$$\delta_2 = n\frac{1+|b-1|^2 - |b(2-b)|}{4Re(b-1)}. \tag{2.19}$$

Theorem 2.9. *Let $f, g \in \mathcal{A}_n, (L_a^\lambda(b,c,z)g)'(z) \neq 0$ and $b \in \mathbb{C}, Re(b) > 1$. If*

$$Re\{1 + \frac{z(L_a^\lambda(b+1,c,z)g)''(z)}{(L_a^\lambda(b+1,c,z)g)'(z)}\} > -\delta_2,$$

where δ_2 is defined in (2.16). Then

$$L_a^\lambda(b+1,c,z)f \prec L_a^\lambda(b+1,c,z)g \Rightarrow L_a^\lambda(b,c,z)f \prec L_a^\lambda(b,c,z)g.$$

By putting $b = c = 2$ and $\lambda = a = 1$ in the Lemma 2.8 we get the following result.

Corollary 2.10. *Let $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in \mathcal{A}_n$ satisfies the condition*

$$Re(1 + \frac{zf''(z)}{f'(z)}) > \frac{-n}{2},$$

then the function $h(z) = \frac{2}{z}\int_0^z f(t)dt = z + \sum_{k=n+1}^{\infty} \frac{2}{k+1}a_k z^k$ is convex.

Theorem 2.11. *Let $f, g \in \mathcal{A}_n$, $(L_a^{\lambda+1}(b,c,z)g)'(z) \neq 0$ and $a \in \mathbb{C}$ with $Re(a) > 0$. If*

$$Re\{1 + \frac{z(L_a^{\lambda}(b,c,z)g)''(z)}{(L_a^{\lambda}(b,c,z)g)'(z)}\} > -\delta$$

where δ is defined in (2.3) and if $L_a^{\lambda}(b,c,z)f$ is univalent and $L_a^{\lambda+1}(b,c,z)f \in Q(0)$ then

$$L_a^{\lambda}(b,c,z)g \prec L_a^{\lambda}(b,c,z)f \Rightarrow L_a^{\lambda+1}(b,c,z)g \prec L_a^{\lambda+1}(b,c,z)f.$$

and $L_a^{\lambda+1}(b,c,z)g$ is the best subordinate.

Proof. Let the functions F, G and p be defined by (2.10). From Lemma 2.1 we get

$$Re(p(z)) > 0 \ (z \in U).$$

Next, to arrive at our desired result, we show that $G \prec F$. For this, let the functions $L(z,t)$ and $\phi(r,s)$ are defined by

$$\begin{aligned} L(z,t) &= \phi(G(z), tzG'(z)) \\ &= \frac{a}{1+a}G(z) + \frac{t}{1+a}zG'(z) = a_1(t)z + \ldots \ (z \in U). \end{aligned} \tag{2.20}$$

Then it is easy to see that $L(z,t)$ is analytic in $|z| < 1$, for all $t \geq 0$. It is also continuously differentiable on $[0, \infty)$, for each $|z| < 1$. From (2.20) we obtain

$$a_1(t) = \frac{\partial L}{\partial z}(0,t) = [\frac{a}{a+1} + \frac{t}{a+1}]G'(0) = \frac{a+t}{a+1} \neq 0 \tag{2.21}$$

and $\lim_{t\to\infty} |a_1(t)| = \infty$. A simple calculation yields

$$Re\big(z\frac{\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\big) = Rea + tRe(1 + \frac{G''(z)}{G'(z)}) = Rea + tRep(z) > 0.$$

Since G is convex and normalized in the unit disk, therefore by using similar argument of Theorem 2.4 we obtain

$$\begin{aligned} \frac{|L(z,t)|}{|a_1(t)|} &\leq \frac{|a||G(z)| + t|zG'(z)|}{|a+t|} \\ &\leq \frac{r}{(1-r)^2}\frac{|a|+t}{|a+t|} \leq \frac{r}{(1-r)^2}\frac{|a|+t}{Re(a)+t}. \end{aligned}$$

If we define $k(t) = \frac{|a|+t}{Re(a)+t}$ it is easy to see that k is a decreasing function for $t \geq 0$, and so

$$k(t) \leq \frac{|a|+1}{Re(a)+1}.$$

Therefore from above inequality we conclude that

$$\frac{|L(z,t)|}{|a_1(t)|} \leq \frac{|a|}{Re(a)}\frac{r}{(1-r)^2} \qquad |z| \leq r, t \geq 0.$$

Hence both assumptions of Lemma 1.2 hold. Now by Lemma 1.2 we obtain that $L(z,t)$ is a subordination chain. Now following the same as proof of Theorem 2.4 we conclude $G \prec F$ and Moreover, G is best subordinate. $\square$

By suitably combining Theorems 2.4 and 2.11 we obtain the following Sandwich type theorem.

Corollary 2.12. *Let $f, g_k \in \mathcal{A}_n (k = 1, 2)$ and $a \in \mathbb{C}$ with $Re(a) > 0$. If*

$$Re(1 + \frac{z(L_a^\lambda(b,c,z)g_k)''(z)}{(L_a^\lambda(b,c,z)g_k)'(z)}) > -\delta$$

where δ is given by (2.3) and if the function $(L_a^\lambda(b,c,z)f)(z)$ is univalent and $(L_a^{\lambda+1}(b,c,z)f)(z) \in Q(0)$, then

$$L_a^\lambda(b,c,z)g_1 \prec L_a^\lambda(b,c,z)f \prec L_a^\lambda(b,c,z)g_2$$

implies

$$L_a^{\lambda+1}(b,c,z)g_1 \prec L_a^{\lambda+1}(b,c,z)f \prec L_a^{\lambda+1}(b,c,z)g_2.$$

By setting $b = c = 1$ and $\lambda = 0$ in the Corollary 2.12 we obtain

Corollary 2.13. *Let $a \in \mathbb{C}, Re(a) > 0$ and $f, g_k \in \mathcal{A}_n(k = 1, 2)$. If f is univalent and $L_a^1(1,1,z)f \in Q(0)$. Then*

$$Re(1 + \frac{zg_k''(z)}{g_k'(z)}) > -n\frac{1 + |a|^2 - |1 - a^2|}{4Re(a)}$$

implies

$$g_1 \prec f \prec g_2 \Rightarrow \frac{1+a}{z^a}\int_0^z t^{a-1}g_1(t)dt \prec \frac{1+a}{z^a}\int_0^z t^{a-1}f(t)dt \prec \frac{1+a}{z^a}\int_0^z t^{a-1}g_2(t)dt.$$

Corollary 2.14. *Let $a > 0$ and $f, g_k \in \mathcal{A}_n, k = 1, 2$. If f is univalent and $L_a^1(1,1,z)f \in Q(0)$. Then*

$$Re(1 + \frac{zg_k''(z)}{g_k'(z)}) > \begin{cases} -\frac{an}{2} & 0 < a < 1 \\ -\frac{n}{2a} & a \geq 1, \end{cases}$$

implies

$$g_1 \prec f \prec g_2 \Rightarrow \frac{1+a}{z^a}\int_0^z t^{a-1}g_1(t)dt \prec \frac{1+a}{z^a}\int_0^z t^{a-1}f(t)dt \prec \frac{1+a}{z^a}\int_0^z t^{a-1}g_2(t)dt.$$

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