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Author(s):

B. Liu and F. Li

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A NOTE ON CRITICAL POINT AND BLOW-UP RATES FOR SINGULAR AND DEGENERATE PARABOLIC EQUATIONS

B. LIU* AND F. LI

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Abstract. In this paper, we consider singular and degenerate parabolic equations

$$u_t = (x^\alpha u_x)_x + u^m(x_0,t)v^n(x_0,t), \quad v_t = (x^\beta v_x)_x + u^q(x_0,t)v^p(x_0,t),$$

in $(0,a)\times(0,T)$, subject to null Dirichlet boundary conditions, where $m,n,p,q \geq 0$, $\alpha,\beta \in [0,2)$ and $x_0 \in (0,a)$. The optimal classification of non-simultaneous and simultaneous blow-up solutions is determined. Additionally, we obtain blow-up rates and sets for the solutions. The singular rates for the derivation of the solutions are given.

Keywords: singular and degenerate parabolic equations, blow-up classification, simultaneous blow-up rates.
MSC(2010): Primary: 35K55; Secondary: 35K57; 35K65.

1. Introduction

In this note, we consider the singular and degenerate parabolic system

$$(1.1)\quad \begin{cases} u_t = (x^\alpha u_x)_x + u^m(x_0,t)v^n(x_0,t), & (x,t)\in(0,a)\times(0,T),\\ v_t = (x^\beta v_x)_x + u^q(x_0,t)v^p(x_0,t), & (x,t)\in(0,a)\times(0,T),\\ u(0,t)=v(0,t)=u(a,t)=v(a,t)=0, & t\in(0,T),\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x), & x\in(0,a), \end{cases}$$

where $\alpha,\beta \in [0,2)$ and $m,n,p,q \geq 0$; x_0 is any fixed point in $(0,a)$; T represents the maximal existence time of the solutions; $u_0(x), v_0(x) \in C^{2+\gamma}[0,a]$ for $\gamma \in (0,1)$. The system, like (1.1), describes some heat conduction corresponding to the geometric shape of the body [4]. The coefficients of u_{xx}, v_{xx} and u_x, v_x may tend to 0 or $+\infty$ as $x \to 0$, which is called degenerate or singular, respectively.

For the system (1.1) with $\alpha=\beta=0$, Li and Wang [11], Zhao and Zheng [16] discussed the existence of simultaneous blow-up solutions (that is, u and v blow up at the same time in L^∞ norm) and even the uniform blow-up profile,

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*Corresponding author.

respectively. Moreover, the boundary layer sizes are given in some exponent regions. The parabolic equations with power or exponential sources also can be seen from [1, 9, 12, 16] and the papers cited therein.

The singular and degenerate parabolic system

$$(1.2)\qquad \begin{cases} x^{\gamma_1}u_t = (x^\alpha u_x)_x + f, \quad (x,t) \in (0,a) \times (0,T), \\ x^{\gamma_2}v_t = (x^\beta v_x)_x + g, \quad (x,t) \in (0,a) \times (0,T), \\ u(0,t) = v(0,t) = u(a,t) = v(a,t) = 0, \quad t \in (0,T), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), \quad x \in (0,a), \end{cases}$$

has aroused much attention in recent years. For the scalar cases of (1.2), the readers refer to [2, 5, 7, 14] for $\gamma_1 > 0$, $\alpha \in [0,2)$, $f = u^p$ $(p > 1)$ or e^{qu} $(q > 0)$, and to [3, 6, 10, 19] for $\gamma_1 > 0$, $\alpha \in [0,2)$, $f = \int_0^a u^p(x,t)dx$ $(p > 1)$, respectively. For the coupled system (1.2), one refers to [8, 17] for the existence and uniqueness of classical solutions and the blow-up criteria of positive solutions. Especially, Zhou and Mu [18] considered the parabolic equations (1.2) with

$$\gamma_1 = \gamma_2 = 0, \quad \alpha, \beta \in [0,2),$$
$$f = u^m(x_0(t),t)v^n(x_0(t),t), \quad g = u^q(x_0(t),t)v^p(x_0(t),t),$$

where $x_0(t) : R^+ \to (0,a)$ is Hölder continuous. They proved that, if $m > 1$ or $p > 1$ or $nq > (m-1)(p-1)$, the classical solutions blow up for large initial data and remain global for small initial data (see Theorem 1.2 of [18]), respectively. Conversely, for $m < 1$, $p < 1$, and $nq < (m-1)(p-1)$, all solutions are global (see Theorem 1.1 of [18]). Moreover, under the assumption

$$\text{(H)}\qquad \alpha = \beta \in (0,1),\ (x^\alpha u_0'(x))' \le M_1,\ (x^\alpha v_0'(x))' \le M_2,\ x \in (0,a),$$

for some positive constants M_1 and M_2. Three results are obtained:

- Theorem 1.3: Suppose that u and v blow up simultaneously and (H) holds. The blow-up set is any closed subset of $(0,a)$.
- Theorem 1.4: Suppose (H) holds. If $q \ge m-1 > 0$ and $n \ge p-1 > 0$, or $q > m-1$, $n > p-1$, and $nq > (m-1)(p-1)$, u and v blow up simultaneously for large initial data.
- Theorem 1.5: Suppose (H) and Theorem 1.4 hold. The uniform blow-up profiles are obtained with precise coefficients for simultaneous blow-up solutions. It is interesting that such results are same with the ones of [11, 16].

To our knowledge, the system (1.1) has not been discussed before. Considering the results of [18] with $x_0(t) \equiv x_0$, one may obtain parts of the results on simultaneous blow-up for the special case $\alpha = \beta \in (0,1)$. Since parabolic system (1.1) is degenerate and singular, the traditional Green's identity method (see, for example, [9]) fails to discuss the classification of simultaneous versus non-simultaneous blow-up of (1.1). Inspired by [9], we will use some new methods, which are also different from the ones used in [18], to study the optimal classification for all of the blow-up solutions, and the blow-up rates and sets.

This paper is arranged as follows. In the next section, we obtain the *critical point* for non-simultaneous versus simultaneous blow-up of (1.1) is

$$(q - m + 1, n - p + 1) = (0, 0).$$

At Section 3, the blow-up rates and sets are given. It is interesting that such results for the singular and degenerate parabolic system (1.1) are the same with the ones for the semilinear parabolic equations

$$\begin{aligned} &u_t = u_{xx} + u^m(x_0, t)v^n(x_0, t), \\ &v_t = v_{xx} + u^q(x_0, t)v^p(x_0, t),\ (x, t) \in (0, a) \times (0, T), \end{aligned}$$

subject to homogeneous Dirichlet boundary conditions.

2. **Critical point**

The local existence of classical solutions and the comparison principle for system (1.1) can be obtained by the similar procedure of Theorem 2.6 and Lemma 2.3 of [18], respectively. There exist blow-up solutions of (1.1) if $m > 1$ or $p > 1$ or $nq > (m - 1)(p - 1)$.

Define the set of initial data of (1.1) as

$$\begin{aligned} \mathbb{V}_0 = \Big\{ &(x^\alpha u_0')' + (1 - \varepsilon\varphi_1)u_0^m(x_0)v_0^n(x_0) \geq 0, \\ &(x^\beta v_0')' + (1 - \varepsilon\varphi_2)u_0^q(x_0)v_0^p(x_0) \geq 0,\ \varepsilon \in (0, 1), \\ &(x^\alpha u_0')' \leq M,\ (x^\beta v_0')' \leq M,\ x \in (0, a) \Big\}, \end{aligned}$$

where M is a positive constant; $\varphi_1(x)$ and $\varphi_2(x)$ are the first eigenfunctions of

$$\left\{ \begin{array}{ll} -(x^\alpha \varphi_1'(x))' = \lambda_1 \varphi_1(x), & x \in (0, a), \\ \varphi_1(0) = \varphi_1(a) = 0; \end{array} \right. \tag{2.1}$$

$$\text{and} \quad \left\{ \begin{array}{ll} -(x^\beta \varphi_2'(x))' = \lambda_2 \varphi_2(x), & x \in (0, a), \\ \varphi_2(0) = \varphi_2(a) = 0, \end{array} \right. \tag{2.2}$$

respectively, normalized by $\|\varphi_1\|_\infty = \|\varphi_2\|_\infty = 1$. The conditions in $\mathbb{V}_0$ are natural for the localized parabolic equations. The similar conditions also can be found in [9, 11], etc.

The main results of the present paper are given as follows.

Theorem 2.1. *Assume* $(u_0, v_0) \in \mathbb{V}_0$.

(i) *There exist non-simultaneous blow-up solutions if and only if*

$$m > q + 1 \text{ (for } u \text{ blowing up alone) or } p > n + 1 \text{ (for } v \text{ blowing up alone).}$$

(ii) *Any blow-up is simultaneous if and only if*

$$m \le q+1 \quad \textit{and} \quad p \le n+1.$$

(iii) *Any blow-up is non-simultaneous if and only if*

$$m > q+1 \textit{ and } \ p \le n+1 \ (\textit{for } u \textit{ blowing up alone}),$$
$$\textit{or} \quad m \le q+1 \ \ \textit{and} \ \ p > n+1 \ (\textit{for } v \textit{ blowing up alone}).$$

(iv) *Both simultaneous and non-simultaneous blow-up may occur if and only if*

$$m > q+1 \quad \textit{and} \quad p > n+1.$$

It can be checked that Theorem 2.1-(ii) holds by using case (i) directly. The results complete the ones of Theorem 1.4 in [18] with $x_0(t) = x_0$, and extend $\alpha, \beta \in (0,1)$ to $\alpha, \beta \in [0,2)$. Moreover, the theorem shows another exponent region $\{m > q+1,\ p > n+1\}$, where simultaneous blow-up may occur. The critical point for non-simultaneous and simultaneous blow-up is $(q-m+1, n-p+1) = (0,0)$.

Now, we give the proof of Theorem 2.1. In [9], the relationship between u and v plays an important role, built by using the method developed from the Souplet's method [15]. It can be found in [18] that such relationship holds under the condition $\alpha = \beta \in (0,1)$. In this paper, we will use new methods (different from [9,18]) to establish such relationship for $\alpha, \beta \in [0,2)$.

Using the methods in Lemma 4.1 of [18], we obtain the solution (u,v) satisfies that

$$(x^\alpha u_x(x,t))_x \le M,\ (x^\beta v_x(x,t))_x \le M, \quad (x,t) \in (0,a)\times(0,T), \tag{2.3}$$

provided that $(x^\alpha u_0')', (x^\beta v_0')' \le M$ in $(0,a)$. For the blow-up solutions, there exists some constant $C > 0$ such that

$$\begin{aligned} u_t &= (x^\alpha u_x)_x + u^m(x_0,t)v^n(x_0,t) \\ &\le Cu^m(x_0,t)v^n(x_0,t), \quad (x,t) \in (0,a)\times(0,T), \end{aligned} \tag{2.4}$$

$$\begin{aligned} v_t &= (x^\beta v_x)_x + u^q(x_0,t)v^p(x_0,t) \\ &\le Cu^q(x_0,t)v^p(x_0,t), \quad (x,t) \in (0,a)\times(0,T). \end{aligned} \tag{2.5}$$

In the sequel, we use the notations C and c to denote positive constants which can be different from line to line.

We introduce a lemma to show the converse inequalities of (2.4) and (2.5). Define two function ϕ and ψ as the classical solutions of the following two problems, respectively,

$$(2.6)\qquad \begin{cases} \phi_t = (x^\alpha \phi_x)_x, & (x,t) \in (0,a) \times (0,T), \\ \phi(0,t) = \phi(a,t) = 0, & t \in (0,T), \\ \phi(x,0) = \varphi_1(x), & x \in (0,a), \end{cases}$$

$$(2.7)\qquad \begin{cases} \psi_t = (x^\beta \psi_x)_x, & (x,t) \in (0,a) \times (0,T), \\ \psi(0,t) = \psi(a,t) = 0, & t \in (0,T), \\ \psi(x,0) = \varphi_2(x), & x \in (0,a), \end{cases}$$

where φ_1 and φ_2 are defined in (2.1) and (2.2), respectively; $\alpha, \beta \in [0,2)$. In fact, the existence and uniqueness of the classical solutions can be obtained by the similar method of [18].

Lemma 2.2. *For small constant $\varepsilon > 0$, there are*

$$(2.8)\qquad u_t(x,t) \geq \varepsilon\phi(x,t)u^m(x_0,t)v^n(x_0,t), \quad (x,t) \in (0,a) \times (0,T),$$

$$(2.9)\qquad v_t(x,t) \geq \varepsilon\psi(x,t)u^q(x_0,t)v^p(x_0,t), \quad (x,t) \in (0,a) \times (0,T),$$

where ϕ and ψ are defined in (2.6) and (2.7), respectively.

Proof. Define

$$J(x,t) = u_t(x,t) - \varepsilon\phi(x,t)u^m(x_0,t)v^n(x_0,t),$$
$$K(x,t) = v_t(x,t) - \varepsilon\psi(x,t)u^q(x_0,t)v^p(x_0,t).$$

By a series computation and for small $\varepsilon > 0$, we get

$$\begin{aligned} J_t - (x^\alpha J_x)_x = {} & u_{tt} - (x^\alpha u_{tx})_x \\ & - \varepsilon\phi m u^{m-1}(x_0,t)v^n(x_0,t)u_t(x_0,t) \\ & - \varepsilon\phi n u^m(x_0,t)v^{n-1}(x_0,t)v_t(x_0,t) \\ & - \varepsilon\phi_t u^m(x_0,t)v^n(x_0,t) + \varepsilon(x^\alpha\phi_x)_x u^m(x_0,t)v^n(x_0,t) \\ \geq {} & 0, \quad (x,t) \in (0,a) \times (0,T), \end{aligned}$$

similarly,

$$K_t - (x^\beta K_x)_x \geq 0, \quad (x,t) \in (0,a) \times (0,T).$$

It can be checked that

$$J(0,t) = K(0,t) = J(a,t) = K(a,t) = 0, \quad t \in (0,T),$$
$$J(x,0) = (x^\alpha u_0')' + (1 - \varepsilon\varphi_1)u_0^m(x_0)v_0^n(x_0) \geq 0, \quad x \in (0,a),$$
$$K(x,0) = (x^\beta v_0')' + (1 - \varepsilon\varphi_2)u_0^q(x_0)v_0^p(x_0) \geq 0, \quad x \in (0,a).$$

By the comparison principle (see Lemma 2.3 of [18]), we obtain (2.8) and (2.9). □

By the way, due to $(x^\alpha u_0')' + u_0^m(x_0)v_0^n(x_0) \geq 0$ and $(x^\beta v_0')' + u_0^q(x_0)v_0^p(x_0) \geq 0$, the classical components u and v are nondecreasing in t by the comparison principle.

Since (2.8) holds and by integration, there exists some positive constant C_u such that

$$u(x_0,t) \le C_u^{-\frac{1}{m-1}}(T-t)^{-\frac{1}{m-1}} \quad \text{for } m>1. \tag{2.10}$$

Even if u do not blow up, the upper bound estimate (2.10) still holds.
Proof of Theorem 2.1-(i). Without loss of generality, we consider the case for u blowing up alone.

Assume $m>q+1$. Let $(\tilde{u}_0,\tilde{v}_0)$ be a pair of initial data in $\mathbb{V}_0$ such that the solution of (1.1) blows up. Fix $v_0=\tilde{v}_0$ and $M_0>\|v_0\|_\infty$. Let $u_0\ge\tilde{u}_0$ be so large such that the blow-up time T satisfies

$$M_0 \ge \|v_0\|_\infty + MT + \frac{m-1}{m-q-1}C_u^{-\frac{q}{m-1}}M_0^pT^{\frac{m-q-1}{m-1}},$$

with M defined in (2.3).

Introduce an auxiliary problem

$$\begin{cases} \bar{v}_t=(x^\beta\bar{v}_x)_x+C_u^{-\frac{q}{m-1}}M_0^p(T-t)^{-\frac{q}{m-1}}, \\ \qquad\qquad (x,t)\in(0,a)\times(0,T), \\ \bar{v}(0,t)=\bar{v}(a,t)=0,\ t\in(0,T), \\ \bar{v}(x,0)=v_0(x),\ x\in(0,a). \end{cases} \tag{2.11}$$

Integrating the equation of (2.11) from 0 to t and using (2.3), we have

$$\begin{aligned}\int_0^t \bar{v}_t(x,t)dt &= \int_0^t (x^\beta\bar{v}_x)_x(x,t)dt + C_u^{-\frac{q}{m-1}}M_0^p\int_0^t(T-t)^{-\frac{q}{m-1}}dt \\ &\le MT+\frac{m-1}{m-q-1}C_u^{-\frac{q}{m-1}}M_0^pT^{\frac{m-q-1}{m-1}},\end{aligned}$$

hence,

$$\bar{v}(x,t)\le\|v_0\|_\infty+MT+\frac{m-1}{m-q-1}C_u^{-\frac{q}{m-1}}M_0^pT^{\frac{m-q-1}{m-1}}\le M_0.$$

So $\bar{v}$ satisfies that

$$\bar{v}_t\ge(x^\beta\bar{v}_x)_x+C_u^{-\frac{q}{m-1}}(T-t)^{-\frac{q}{m-1}}\bar{v}^p(x_0,t), \quad (x,t)\in(0,a)\times(0,T).$$

Combining (1.1) with the estimate (2.10), we have

$$v_t\le(x^\beta v_x)_x+C_u^{-\frac{q}{m-1}}(T-t)^{-\frac{q}{m-1}}v^p(x_0,t), \quad (x,t)\in(0,a)\times(0,T),$$

and hence $v\le\bar{v}\le M_0$ by the comparison principle.

Next, assume that u blows up at time T and v still remains bounded. Using (2.4), we obtain

$$u_t(x_0,t)\le Cu^m(x_0,t), \quad t\in(0,T).$$

By integration, we have $u(x_0,t)\ge c(T-t)^{-\frac{1}{m-1}}$. By (2.9), v satisfies that

$$v_t(x,t)\ge cv_0^p(x_0)(T-t)^{-\frac{q}{m-1}}, \quad t\in[0,T),$$

and consequently,

$$v(x,t) \geq c v_0^p(x_0) \int_0^t (T-\tau)^{-\frac{q}{m-1}} d\tau + v_0(x_0).$$

The boundedness of v requires that $m > q+1$. $\square$

Proof of Theorem 2.1-(iii). Without loss of generality, treat the case for v blowing up alone. Theorem 2.1-(i) implies non-simultaneous blow-up for v blowing up alone requires that $p > n+1$, and u remaining bounded yields that $m \leq q+1$. Assume $p > n+1$ and $m \leq q+1$. Due to (2.4), (2.5), (2.8), and (2.9), there exists some small constant $\delta > 0$ such that

(2.12)

$$\delta u^{q-m}(x_0,t) u_t(x_0,t) \leq v^{n-p}(x_0,t) v_t(x_0,t) \leq \frac{1}{\delta} u^{q-m}(x_0,t) u_t(x_0,t), \ t \in [0,T).$$

We only need to prove that any simultaneous blow-up cannot occur. Otherwise, if $p > n+1$ with $m < q+1$, then by integrating the first inequality of (2.12) from 0 to t, one gets

$$c u^{q+1-m}(x_0,t) \leq C - v^{-(p-n-1)}(x_0,t)/(p-n-1),$$

a contradiction to simultaneous blow-up; if $p > n+1$ with $m = q+1$, a similar contradiction also can be obtained. $\square$

Proof of Theorem 2.1-(iv). By using the idea of Lemma 3 in [9] and combining the methods in the proof of Theorem 2.1-(i), we obtain the set of (u_0, v_0) in $\mathbb{V}_0$ such that u (v) blows up while v (u) remains bounded is open in L^∞-topology. Similarly to the discussion of Theorem 3 of [9], one can obtain both simultaneous and non-simultaneous blow-up may occur if $m > q+1$ and $p > n+1$.

Now show the coexistence of simultaneous and non-simultaneous blow-up of solutions does not holds without $m > q+1$ and $p > n+1$. Indeed, the non-simultaneous blow-up with u (or v) blowing up alone requires $m > q+1$ (or $p > n+1$) by the item (i) of the theorem, while the only simultaneous blow-up needs $p \leq n+1$ and $m \leq q+1$ by (ii), and the sole non-simultaneous blow-up requires $p \leq n+1$ or $m \leq q+1$ by (iii). $\square$

3. **Blow-up rates and sets**

Integrating the equations of (1.1) from 0 to t, we have, for example,

$$u(x,t) = u_0(x) + \int_0^t (x^\alpha u_x)_x dt + \int_0^t u^m(x_0,t) v^n(x_0,t) dt.$$

Due to the blow-up of u and $(x^\alpha u_x)_x \leq M$, blow-up occurs everywhere in $(0,a)$.

It can be understood that the non-simultaneous blow-up rates are equivalent to the related scalar cases, for example, $u(x,t) = O((T-t)^{-\frac{1}{m-1}})$ for u blowing up alone.

The following theorem gives all possible simultaneous blow-up rates for (1.1). Due to Theorem 2.1, there are two exponent regions, where simultaneous blow-up may occur,

$$\{m \le q+1,\ p \le n+1\}; \quad \{m > q+1,\ p > n+1\}.$$

We divide the blow-up rates of (u,v) into four subcases in the following theorem.

Theorem 3.1. *Assume* $(u_0, v_0) \in \mathbb{V}_0$.

(i) *If* $m > q+1$ *and* $p > n+1$ *with simultaneous blow-up occurring, or* $m < q+1$ *and* $p < n+1$*, then*

$$\begin{cases} u(x_0,t) = O\big((T-t)^{-\frac{n+1-p}{nq-(m-1)(p-1)}}\big), \\ v(x_0,t) = O\big((T-t)^{-\frac{q+1-m}{nq-(m-1)(p-1)}}\big). \end{cases} \tag{3.1}$$

(ii) *If* $m < q+1$ *and* $p = n+1$*, then*

$$\begin{cases} u^{q+1-m}(x_0,t) = O\big(|\log(T-t)|\big), \\ v^{p-1}(x_0,t)(\log v(x_0,t))^{\frac{q}{q+1-m}} = O\big((T-t)^{-1}\big). \end{cases} \tag{3.2}$$

(iii) *If* $m = q+1$ *and* $p < n+1$*, then*

$$u^{m-1}(x_0,t)(\log u(x_0,t))^{\frac{n}{n+1-p}} = O\big((T-t)^{-1}\big),$$
$$v^{n+1-p}(x_0,t) = O\big(|\log(T-t)|\big).$$

(iv) *If* $m = q+1$ *and* $p = n+1$*, then*

$$\log u(x_0,t) = O\big(|\log(T-t)|\big),\ \log v(x_0,t) = O\big(|\log(T-t)|\big).$$

Proof. Inspired by [20] and followed by the similar procedure of [13], we obtain the simultaneous blow-up rates in Theorem 3.1. In fact, the key relationships between u and v can be established at the point $x = x_0$ by using the inequalities (2.4), (2.5), (2.8) and (2.9). □

We also determine the blow-up rates for u_t and v_t. It is so interesting that, as u and v blow up simultaneously, u_t and v_t also blow up simultaneously, and the blow-up sets of u_t and v_t are the whole interval $(0,a)$.

Theorem 3.2. *Assume* $(u_0, v_0) \in \mathbb{V}_0$.

(i) *If* $m > q+1$ *and* $p > n+1$ *with simultaneous blow-up happening, or* $m < q+1$ *and* $p < n+1$*, then*

$$u_t(x,t) = O\big((T-t)^{\frac{-m(n+1-p)-n(q+1-m)}{nq-(m-1)(p-1)}}\big),$$
$$v_t(x,t) = O\big((T-t)^{\frac{-q(n+1-p)-p(q+1-m)}{nq-(m-1)(p-1)}}\big).$$

Consequently,

$$u(x,t) = O\big((T-t)^{-\frac{n+1-p}{nq-(m-1)(p-1)}}\big), \quad v(x,t) = O\big((T-t)^{-\frac{q+1-m}{nq-(m-1)(p-1)}}\big).$$

(ii) *If $m < q+1$ and $p = n+1$, then*

$$u_t(x,t) = O\big(|\log(T-t)|^{\frac{m-q}{q+1-m}}(T-t)^{-1}\big),$$
$$v_t(x,t) = O\big(|\log(T-t)|^{\frac{-q}{n(q+1-m)}}(T-t)^{-\frac{p}{n}}\big).$$

(iii) *If $m = q+1$ and $p < n+1$, then*

$$u_t(x,t) = O\big(|\log(T-t)|^{\frac{-n}{q(n+1-p)}}(T-t)^{-\frac{m}{q}}\big),$$
$$v_t(x,t) = O\big(|\log(T-t)|^{\frac{p-n}{n+1-p}}(T-t)^{-1}\big).$$

(iv) *If $m = q+1$ and $p = n+1$, then*

$$\log u_t(x,t) = O\big(|\log(T-t)|\big), \ \log v_t(x,t) = O\big(|\log(T-t)|\big).$$

Proof. (i) By using the inequalities (2.4), (2.5), (2.8) and (2.9), we get from (3.1) that

$$u_t(x,t) \sim u^m(x_0,t)v^n(x_0,t) \sim (T-t)^{\frac{-m(n+1-p)-n(q+1-m)}{nq-(m-1)(p-1)}},$$
$$v_t(x,t) \sim u^q(x_0,t)v^p(x_0,t) \sim (T-t)^{\frac{-q(n+1-p)-p(q+1-m)}{nq-(m-1)(p-1)}}.$$

The notation $f \sim g$ denote there exist positive constants c and C such that $cf \le g \le Cf$.

(ii) If $m < q+1$ and $p = n+1$, then there exists some constant $\delta \in (0,1)$ such that

$$\delta \log v(x_0,t) \le u^{q+1-m}(x_0,t) \le \frac{1}{\delta}\log v(x_0,t), \quad t \in [0,T). \tag{3.3}$$

Due to (3.2) and (3.3), there are

$$u_t(x,t) \sim u^m(x_0,t)v^n(x_0,t) \sim |\log(T-t)|^{\frac{m-q}{q+1-m}}(T-t)^{-1},$$
$$v_t(x,t) \sim u^q(x_0,t)v^p(x_0,t) \sim |\log(T-t)|^{\frac{-q}{n(q+1-m)}}(T-t)^{-\frac{p}{n}}.$$

Case (iii) can be proved similarly.

(iv) If $m = q+1$ and $p = n+1$, then there are some constants $\delta \in (0,1)$ and $T_0 \in (0,T)$ such that

$$\delta \log v(x_0,t) \le \log u(x_0,t) \le \frac{1}{\delta}\log v(x_0,t), \quad t \in [T_0,T). \tag{3.4}$$

Combining (2.4), (2.5), (2.8), and (2.9) with (3.4), we get

$$\log u_t(x,t) \sim \log[u^m(x_0,t)v^n(x_0,t)] \sim |\log(T-t)|,$$
$$\log v_t(x,t) \sim \log[u^q(x_0,t)v^p(x_0,t)] \sim |\log(T-t)|.$$

The proof is completed. □

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(Bingchen Liu) College of Science, China University of Petroleum, Qingdao 266580, Shandong Province, P.R. China
E-mail address: bcliu@aliyun.com

(Fengjie Li) College of Science, China University of Petroleum, Qingdao 266580, Shandong Province, P.R. China
E-mail address: fjli@upc.edu.cn