Title:
Automatic continuity of surjective $n$-homomorphisms on Banach algebras

Author(s):
M. Eshaghi Gordji, A. Jabbari and E. Karapınar
AUTOMATIC CONTINUITY OF SURJECTIVE
\(n\)-HOMOMORPHISMS ON BANACH ALGEBRAS

M. ESHAGHI GORDJI, A. JABBARI* AND E. KARAPINAR

(Communicated by Antony To-Ming Lau)

Abstract. In this paper, we show that every surjective \(n\)-homomorphism (\(n\)-anti-homomorphism) from a Banach algebra \(A\) into a semisimple Banach algebra \(B\) is continuous.

Keywords: Banach algebra, \(n\)-homomorphism, semisimple algebra.

MSC(2010): Primary: 46H05.

1. Introduction

Let \(A\) and \(B\) be complex Banach algebras. The linear mapping \(\theta : A \rightarrow B\) is called an \(n\)-homomorphism, if \(\theta(a_1a_2\cdots a_n) = \theta(a_1)\theta(a_2)\cdots \theta(a_n)\), for all \(a_1a_2\cdots a_n \in A\). A linear mapping \(\theta : A \rightarrow B\) is called an \(n\)-anti-homomorphism if \(\theta(a_1a_2\cdots a_n) = \theta(a_n)\cdots \theta(a_2)\theta(a_1)\), for all \(a_1a_2\cdots a_n \in A\). The algebra \(B\) is called factorizable if for every \(a \in B\) there are \(b, c \in B\) such that \(a = bc\). The concept of \(n\)-homomorphisms was studied for complex algebras by Hejazian, Mirzavaziri, and Moslehian [6]. Bračić and Moslehian [1] investigated. \(3\)-homomorphisms on Banach algebras with bounded approximate identities and established that every involution preserving \(3\)-homomorphism between \(C^*\)-algebras is continuous and norm decreasing. It is due to Park and Trout that every \(*\)-preserving \(n\)-homomorphism between \(C^*\)-algebras is continuous [7]. Automatic continuity of \(n\)-homomorphisms considered for factorizable Banach algebras in [4]. A similar problem was studied for topological algebras in [5]. A linear mapping \(\theta : A \rightarrow B\) is called an \(n\)-Jordan homomorphism if \(\theta(a^n) = [\theta(a)]^n\) for all \(a \in A\). Some results about automatic continuity of \(n\)-Jordan homomorphisms on Banach algebras and \(C^*\)-algebras are investigated in [3].
Let $A$ be a Banach algebra. If $A$ has a unit $e_A$, the spectrum of $a \in A$ is defined by
\[ sp_A(a) = \{ \lambda \in \mathbb{C} : \lambda e_A - a \notin Inv\ A \}, \]
and the spectral radius of $a$ is defined as follows
\[ \rho_A(a) = \sup \{ |\lambda| : \lambda \in sp_A(a) \}, \]
where $Inv A$ is the set of all invertible elements of $A$.

If $A$ is non-unital, we consider the quasi product $\circ$ on $A$ as follows
\[ a \circ b = a + b - ab \quad (a, b \in A). \]

An element $a \in A$ is called left (right) quasi-invertible if there is $b \in A$ such that $b \circ a = 0$ ($a \circ b = 0$). Then an element $a \in A$ is quasi-invertible if it is both left and right quasi-invertible. The set of all quasi-invertible elements of $A$ denoted by $qInv A$.

Let $A$ be a non-unital (complex) Banach algebra. Then $A^\# = A \oplus \mathbb{C}$ is a unital Banach algebra with the product and norm given by
\[ (a, \alpha)(b, \beta) = (ab + \beta a + \alpha b, \alpha \beta), \]
\[ \| (a, \alpha) \| = \| a \| + |\alpha| \]
for all $a, b \in A$ and $\alpha, \beta \in \mathbb{C}$. We denote the identity of $A^\#$ by $e_{A^\#} (= (0, 1))$.

Let $A$ be a non-unital (complex) Banach algebra. Then, obviously for every $x, y \in A$, $a \circ b = 0$ if and only if
\[ (e_{A^\#} - a)(e_{A^\#} - b) = e_{A^\#}. \]

The definition of spectrum in the non-unital Banach algebras is different from the unital case, and we define it as follows
\[ sp_A(a) = \{ 0 \} \cup \{ \lambda \in \mathbb{C} \setminus \{ 0 \} : \frac{1}{\lambda} a \notin q - Inv A \}, \]
and it is easy to see that $sp_A(a) = sp_{A^\#}((a, 0))$ and $\rho_A(a) = \rho_{A^\#}((a, 0))$. By $\partial sp(a)$, we mean the boundary set of $sp(a)$. The radical of $A$, denoted by $Rad(A)$, is the intersection of all maximal left ideals of $A$. The algebra $A$ is called semisimple if $Rad(A) = \{ 0 \}$ (for more details see Section 1.5 of [2]). If $A$ is a semisimple Banach algebra, given $a \in A$, if $axy = 0$ (or $xay = 0$ or $xya = 0$) for all $x, y \in A$, then it is easy to show that $a = 0$.

2. Automatic continuity

In this section we extend Johnson’s techniques [8] for $n$-homomorphism on non-unital Banach algebras. Our results differ from those obtained in [4, 5, 8] and [7].

We state [8, Lemma 1], which is valid for non-unital Banach algebras.

Lemma 2.1. (8, Lemma 1) Let $A$ be a Banach algebra, $a \in A$, and suppose that $\rho_A(a_1 a) = 0$ for all $a_1 \in A$. Then $a \in Rad(A)$.
We can generalize this result for non-unital Banach algebras as follows:

**Lemma 2.2.** Let $A$ be a Banach algebra. Then

1. given $a \in A$ satisfies $\rho_A(a_1a_2 \cdots a_{n-1}a) = 0$ for all $a_1, a_2, \ldots, a_{n-1} \in A$, then $a \in \text{Rad}(A)$.
2. given $a \in A$ satisfies $\rho_A(aa_1a_2 \cdots a_{n-1}) = 0$ for all $a_1, a_2, \ldots, a_{n-1} \in A$, then $a \in \text{Rad}(A)$.

**Proof.** (1) Suppose that $\rho_A(a_1a_2 \cdots a_{n-1}a) = 0$ for all $a_1, a_2, \ldots, a_{n-1} \in A$. By Lemma 2.1, $a_2 \cdots a_{n-1}a \in \text{Rad}(A)$. Since the radical of any normed algebra is a topologically nil ideal ([9, Theorem 2.3.4]), $\rho_A(a_2 \cdots a_{n-1}) = 0$ for all $a_2, \ldots, a_{n-1} \in A$. Repeating the argument we get $\rho_A(a_{n-1}) = 0$ for all $a_{n-1} \in A$, which, by Lemma 2.1, assures that $a \in \text{Rad}(A)$. Part (2) needs a similar argument. □

Let $T : A \rightarrow B$ be a linear mapping between Banach algebras. The separating space of $T$ is defined by

$$G(T) = \{ b \in B : \text{there exists } (a_n) \subseteq A \text{ such that } a_n \rightarrow 0 \text{ and } T(a_n) \rightarrow b \}. $$

We know that $G(T)$ is a closed linear subspace of $B$. By the closed graph theorem, $T$ is continuous if and only if $G(T) = \{0\}$ ([10, Lemma 1.2]). The proof of the following lemma is clear and lefts to the reader.

**Lemma 2.3.** Let $\theta : A \rightarrow B$ be an $n$-homomorphism between Banach algebras. The following statements hold:

1. Given $b_1, \ldots, b_{n-1}$ in $\theta(A)$ and $b \in G(\theta)$, the product 
   $$b_1 \cdots b_{i-1}bb_{i+1} \cdots b_{n-1}$$
   lies in $G(\theta)$.
2. When $\theta$ has dense range and $b \in G(\theta)$, then 
   $$b_1 \cdots b_{i-1}bb_{i+1} \cdots b_{n-1} \in G(\theta),$$
   for $b_1, \ldots, b_{n-1} \in B$.
3. When $\theta$ has dense range, then 
   $$b_1 \cdots b_{n-1}b, \; bb_1 \cdots b_{n-1} \in G(\theta),$$
   for $b_1, \ldots, b_{n-1} \in G(\theta)$ and $b \in B$.

Now, we consider our main result. Note that the first part of the proof is taken from [4, Theorem 2.7] see also [8], and for completeness we include the proof.

**Theorem 2.4.** Let $A$ and $B$ be Banach algebras (non-unital) which $B$ is semisimple. Then every surjective $n$-homomorphism $\theta : A \rightarrow B$ is automatically continuous.
Proof. Suppose that \((a_n) \subseteq A\) such that \(a_n \to 0\) and \(\theta(a_m) \to b\) in \(B\). Our aim is showing that \(b = 0\). Since \(\theta\) is surjective, there exists \(a \in A\) such that \(\theta(a) = b\). For \(m \geq 1\), we define
\[
P_m(z) = z\theta(a_m) + (\theta(a) - \theta(a_m)) \quad (z \in \mathbb{C}).
\]
Then for every \(z \in \mathbb{C}\), we have
\[
\rho_B(P_m(z)) \leq \|P_m(z)\| \leq \|z\|\|\theta(a_m)\| + \|\theta(a) - \theta(a_m)\|.
\]
In light of [4, Lemma 2.6], we have
\[
\rho_B(P_m(z)^{n-1}) \leq \rho_A((z\theta(a_m) + (a - a_m))^{n-1}) \leq \|(z\theta(a_m) + (a - a_m))^{n-1}\|
\]
(2.1)
\[
\leq (|z|\|a_m\| + |a - a_m\|)^{n-1}.
\]
By [8, Lemma 2], we have
\[
(2.2) \quad \rho_B(b)^2 \leq (R\|a_m\| + \|a - a_m\|)(R^{-1}\|\theta(a_m)\| + \|\theta(a) - \theta(a_m)\|) \to 0,
\]
as \(m \to \infty\) and \(R \to \infty\). This implies that \(\rho_B(b) = 0\). Choose nonzero elements \(b_1, b_2, \ldots, b_{n-1}\) in \(B\). There are \(a_1, a_2, \ldots, a_{n-1} \in A\) such that \(\theta(a_1) = b_1, \theta(a_2) = b_2, \ldots, \theta(a_{n-1}) = b_{n-1}\). By Lemma 2.3 (3), \(b_1 \cdots b_{n-1} b \in \mathfrak{S}(\theta)\) and by the first part of the proof, \(\rho_B(b_1 b_2 \cdots b_{n-1} b) = 0\), and Lemma 2.2 implies that \(b \in \text{Rad}(B)\). Since \(B\) is semisimple, we get \(b = 0\).

The next result is devoted to the automatic continuity of \(n\)-Jordan homomorphisms.

**Corollary 2.5.** Let \(A\) be a Banach algebra and \(B\) be a semisimple Banach algebra. Then every surjective \(n\)-Jordan homomorphism \(\theta : A \to B\) that satisfies \(\partial(s_B(\theta(a)^n)) \subseteq s_B(a^{n-1}) \cup \{0\}\), for all \(a \in A\), is automatically continuous.

**Proof.** Similar to the proof of Theorem 2.4, suppose that \((a_m) \subseteq A\) such that \(a_m \to 0\) and \(\theta(a_m) \to b\) in \(B\). As well as, there exists \(a \in A\) such that \(\theta(a) = b\). Since \(\partial(s_B(\theta(a)^n)) \subseteq s_B(a^{n-1}) \cup \{0\}\), for \(a \in A\), the relations (2.1) and (2.2) hold. Therefore \(\rho_B(b) = 0\). Clearly, \(a_m a_m \cdots a_m \to 0\) and \(\theta(a_m a_m \cdots a_m) = \theta(a_m)^n = b^n\). This follows that \(b^n \in \mathfrak{S}(\theta)\) and \(\rho_B(b b \cdots b) = 0\). By Lemma 2.2, \(b \in \text{Rad}(B)\). Then \(b = 0\) and this completes the proof.

By a similar argument as Theorem 2.4, we have the following result for \(n\)-anti-homomorphisms.

**Theorem 2.6.** Let \(A\) and \(B\) be Banach algebras (non-unital) which \(B\) is semisimple. Then every surjective \(n\)-anti-homomorphism \(\theta : A \to B\) is automatically continuous.
Acknowledgments

The authors wish to thank the referee for his/her invaluable comments and pointing out a correction in Corollary 2.5.

REFERENCES


(Madjid Eshaghi Gordji) DEPARTMENT OF MATHEMATICS, SEMNAN UNIVERSITY, P. O. Box 35195-363, SEMNAN, IRAN
CENTER OF EXCELLENCE IN NONLINEAR ANALYSIS AND APPLICATIONS (CENAA), SEMNAN UNIVERSITY, IRAN
E-mail address: madjid.eshaghi@gmail.com, meshaghi@semnan.ac.ir

(Ali Jabbari) YOUNG RESEARCHERS AND ELITE CLUB, ARDABIL BRANCH, ISLAMIC AZAD UNIVERSITY, ARDABIL, IRAN
E-mail address: jabbari_al@yahoo.com, ali.jabbari@iauardabil.ac.ir

(Erdal Karapınar) DEPARTMENT OF MATHEMATICS, ATILIM UNIVERSITY, 06836, INCEK, ANKARA, TURKEY
E-mail address: erdalkarapinar@yahoo.com, ekarapinar@atilim.edu.tr