1. Introduction

A classical question in the theory of functional equations is the following: When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation? If the problem accepts a solution, we say that the equation is stable. The first stability problem concerning group homomorphisms was raised by Ulam [45] in 1940. In the next year, Hyers [23] gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces. In 1978, Rassias [38] proved a generalization of Hyers’s theorem for additive mappings.

Theorem 1.1. (Th. M. Rassias): Let $f : E \rightarrow E'$ be a mapping from a normed vector space $E$ into a Banach space $E'$ subject to the inequality

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon (\|x\|^p + \|y\|^p)$$

for all $x, y \in E$, where $\epsilon$ and $p$ are constants with $\epsilon > 0$ and $0 \leq p < 1$. Then the limit $L(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and $L : E \rightarrow E'$ is the
unique linear mapping which satisfies
\[ \| f(x) - L(x) \| \leq \frac{2\epsilon}{2^p - 2p} \| x \|^p \]
for all \( x \in E \). Also, if for each \( x \in E \) the function \( f(tx) \) is continuous in \( t \in \mathbb{R} \), then \( L \) is linear.

This new concept is known as generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. Furthermore, in 1994, a generalization of Rassias’s theorem was obtained by Găvruţa [21] by replacing the bound \( \epsilon(\|x\|^p + \|y\|^p) \) by a general control function \( \varphi(x, y) \).

In 1983, a generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [44] for mapping \( f : X \to Y \), where \( X \) is a normed space and \( Y \) is a Banach space. In 1984, Cholewa [11] noticed that the theorem of Skof is still true if the relevant domain \( X \) is replaced by an Abelian group and, in 2002, Czerwik [12] proved the generalized Hyers-Ulam stability of the quadratic functional equation. The reader is referred to ( [1]- [42]) and references therein for detailed information on stability of functional equations.

In 1897, Hensel [22] has introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications (see [13, 27, 30, 31, 36]).

Katsaras [26] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view (see [20, 29]). In particular, Bag and Samanta [5], following Cheng and Mordeson [10], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Karmosil and Michalek type [28]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [6].

**Definition 1.2.** By a non-Archimedean field we mean a field \( K \) equipped with a function (valuation) \( |\cdot| : K \to [0, \infty) \) such that, for all \( r, s \in K \), the following conditions hold:

(a) \( |r| = 0 \) if and only if \( r = 0 \);
(b) \( |rs| = |r||s| \);
(c) \( |r + s| \leq \max\{|r|, |s|\} \).

Clearly, by (b), \( |1| = |-1| = 1 \) and so, by induction, it follows from (c) that \( |n| \leq 1 \) for all \( n \geq 1 \).

**Definition 1.3.** Let \( X \) be a vector space over a scalar field \( K \) with a non-Archimedean non-trivial valuation \( |\cdot| \).

(1) A function \( \| \cdot \| : X \to \mathbb{R} \) is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(a) \( \|x\| = 0 \) if and only if \( x = 0 \) for all \( x \in X \);
\[ \|rx\| = |r|\|x\| \text{ for all } r \in \mathbb{R} \text{ and } x \in X; \]

(c) the strong triangle inequality (ultra-metric) holds, that is,
\[ \|x + y\| \leq \max\{\|x\|, \|y\|\} \]
for all \( x, y \in X. \)

(2) The space \((X, \| \cdot \|)\) is called a non-Archimedean normed space.

Note that \( \|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n - 1\} \) for all \( m, n \in \mathbb{N} \) with \( n > m \).

**Definition 1.4.** Let \((X, \| \cdot \|)\) be a non-Archimedean normed space.

(a) A sequence \(\{x_n\}\) is a Cauchy sequence in \(X\) if \(\{x_{n+1} - x_n\}\) converges to zero in \(X.\)

(b) The non-Archimedean normed space \((X, \| \cdot \|)\) is said to be complete if every Cauchy sequence in \(X\) is convergent.

The most important examples of non-Archimedean spaces are \(p\)-adic numbers. A key property of \(p\)-adic numbers is that they do not satisfy the Archimedean axiom: for all \(x, y > 0\), there exists a positive integer \(n\) such that \(x < ny.\)

**Example 1.5.** Fix a prime number \(p.\) For any nonzero rational number \(x,\) there exists a unique positive integer \(n_x\) such that \(x = \frac{a}{b}p^{n_x},\) where \(a\) and \(b\) are positive integers not divisible by \(p.\) Then \(|x|_p := p^{-n_x}\) defines a non-Archimedean norm on \(\mathbb{Q}.\) The completion of \(\mathbb{Q}\) with respect to the metric \(d(x, y) = |x - y|_p\) is denoted by \(\mathbb{Q}_p,\) which is called the \(p\)-adic number field. In fact, \(\mathbb{Q}_p\) is the set of all formal series \(x = \sum_{k \geq n_x} a_kp^k,\) where \(|a_k| \leq p - 1.\)

The addition and multiplication between any two elements of \(\mathbb{Q}_p\) are defined naturally. The norm \(\|\sum_{k \geq n_x} a_kp^k\|_p = p^{-n_x}\) is a non-Archimedean norm on \(\mathbb{Q}_p\) and \(\mathbb{Q}_p\) is a locally compact filed.

In random stability, we adopt the usual terminology, notions and conventions of the theory of random normed spaces as in [43]. Throughout this paper, let \(\Delta^+\) denote the set of all probability distribution functions \(F : \mathbb{R} \cup [-\infty, +\infty] \to [0, 1]\) such that \(F\) is left-continuous and nondecreasing on \(\mathbb{R}\) and \(F(0) = 0, F(+\infty) = 1.\) It is clear that the set \(D^+ = \{F \in \Delta^+ : l^-F(-\infty) = 1\},\) where \(l^-F(x) = \lim_{t \to -x-} F(t),\) is a subset of \(\Delta^+.\) The set \(\Delta^+\) is partially ordered by the usual point-wise ordering of functions, that is, \(F \leq G\) if and only if \(F(t) \leq G(t)\) for all \(t \in \mathbb{R}.\) For any \(a \geq 0,\) the element \(H_a(t)\) of \(D^+\) is defined by
\[
H_a(t) = \begin{cases} 
0, & \text{if } t \leq a, \\
1, & \text{if } t > a.
\end{cases}
\]

We can easily show that the maximal element in \(\Delta^+\) is the distribution function \(H_0(t).\)
**Definition 1.6.** A function \( T : [0, 1]^2 \to [0, 1] \) is a **continuous triangular norm** (briefly, a \( t \)-norm) if \( T \) satisfies the following conditions:

(a) \( T \) is commutative and associative;

(b) \( T \) is continuous;

(c) \( T(x, 1) = x \) for all \( x \in [0, 1] \);

(d) \( T(x, y) \leq T(z, w) \) whenever \( x \leq z \) and \( y \leq w \) for all \( x, y, z, w \in [0, 1] \).

Three typical examples of continuous \( t \)-norms are as follows: \( T(x, y) = xy \), \( T(x, y) = \max\{x + y - 1, 0\} \), \( T(x, y) = \min(x, y) \).

Recall that, if \( T \) is a \( t \)-norm and \( \{x_n\} \) is a sequence in \([0, 1]\), then \( T^{n}_{i=1} x_i \) is defined recursively by \( T^{n}_{i=1} x_1 = x_1 \) and \( T^{n}_{i=1} x_i = T(T^{n-1}_{i=1} x_i, x_n) \) for all \( n \geq 2 \). \( T^{\infty}_{i=1} x_i \) is defined by \( T^{\infty}_{i=1} x_{n+i} \).

**Definition 1.7.** A **random normed space** (briefly, RN-space) is a triple \((X, \mu, T)\), where \( X \) is a vector space, \( T \) is a continuous \( t \)-norm and \( \mu : X \to D^+ \) is a mapping such that the following conditions hold:

(a) \( \mu_x(t) = H_0(t) \) for all \( t > 0 \) if and only if \( x = 0 \);

(b) \( \mu_{\alpha x}(t) = \mu_x \left( \frac{t}{|\alpha|} \right) \) for all \( \alpha \in \mathbb{R} \) with \( \alpha \neq 0 \), \( x \in X \) and \( t \geq 0 \);

(c) \( \mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s)) \) for all \( x, y \in X \) and \( t, s \geq 0 \).

Every normed space \((X, \| \cdot \|)\) defines a random normed space \((X, \mu, T_M)\), where \( \mu_x(t) = \frac{1}{1 + \|x\|} \) for all \( t > 0 \) and \( T_M \) is the minimum \( t \)-norm. This space \( X \) is called the **induced random normed space**.

If the \( t \)-norm \( T \) is such that \( \sup_{0 < a < 1} T(a, a) = 1 \), then every RN-space \((X, \mu, T)\) is a metrizable linear topological space with the topology \( \tau \) (called the \( \mu \)-topology or the \((\epsilon, \delta)\)-topology, where \( \epsilon > 0 \) and \( \lambda \in (0, 1) \)) induced by the base \( \{U(\epsilon, \lambda)\} \) of neighborhoods of \( \theta \), where

\[ U(\epsilon, \lambda) = \{x \in X : \mu_x(\epsilon) > 1 - \lambda\} \]

**Definition 1.8.** Let \((X, \mu, T)\) be an RN-space.

(a) A sequence \( \{x_n\} \) in \( X \) is said to be **convergent** to a point \( x \in X \) (write \( x_n \to x \) as \( n \to \infty \)) if

\[ \lim_{n \to \infty} \mu_{x_n - x}(t) = 1 \]

for all \( t > 0 \).

(b) A sequence \( \{x_n\} \) in \( X \) is called a **Cauchy sequence** in \( X \) if

\[ \lim_{n \to \infty} \mu_{x_n - x_m}(t) = 1 \]

for all \( t > 0 \).

(c) The RN-space \((X, \mu, T)\) is said to be **complete** if every Cauchy sequence in \( X \) is convergent.

**Theorem 1.9.** If \((X, \mu, T)\) is an RN-space and \( \{x_n\} \) is a sequence such that \( x_n \to x \), then \( \lim_{n \to \infty} \mu_{x_n}(t) = \mu_x(t) \).
Definition 1.10. Let $X$ be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on $X$ if $d$ satisfies the following conditions:

(a) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;
(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(c) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.11. Let $(X, d)$ be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either $d(J^n x, J^{n+1} x) = \infty$ for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that

(a) $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;
(b) the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$;
(c) $y^*$ is the unique fixed point of $J$ in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;
(d) $d(y, y^*) \leq \frac{d(y, Jy)}{1-L}$ for all $y \in Y$.

2. Non-Archimedean Stability of Functional Equation (0.1)

In this section, we deal with the stability problem for the Cauchy-Jensen additive functional equation (0.1) in non-Archimedean normed spaces.

2.1. A Fixed Point Approach.

Theorem 2.1. Let $X$ be a non-Archimedean normed space and that $Y$ be a complete non-Archimedean space. Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with

\begin{equation}
\varphi \left( \frac{x}{2}, \frac{y}{2}, \frac{z}{2} \right) \leq \frac{\alpha \varphi(x, y, z)}{|2|}
\end{equation}

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying

\begin{equation}
\| f \left( \frac{x+y+z}{2} \right) + f \left( \frac{x-y+z}{2} \right) - f(x) - f(z) \|_Y \leq \varphi(x, y, z)
\end{equation}

for all $x, y, z \in X$. Then there exists a unique additive mapping $L : X \to Y$ such that

\begin{equation}
\|f(x) - L(x)\|_Y \leq \frac{\alpha \varphi(x, 2x, x)}{|2| - |2|\alpha}
\end{equation}

for all $x \in X$.

Proof. Putting $y = 2x$ and $z = x$ in (2.2), we get

\begin{equation}
\|f(2x) - 2f(x)\|_Y \leq \varphi \left( \frac{x}{2}, x, \frac{x}{2} \right) \leq \frac{\alpha \varphi(x, 2x, x)}{|2|}
\end{equation}

for all $x \in X$. So
for all \( x \in X \). Consider the set \( S := \{ h : X \to Y \} \) and introduce the generalized metric on \( S \):

\[
   d(g, h) = \inf \left\{ \mu \in (0, +\infty) : \| g(x) - h(x) \|_Y \leq \mu \varphi(x, 2x, x), \ \forall x \in X \right\},
\]

where, as usual, \( \inf \phi = +\infty \). It is easy to show that \((S, d)\) is complete (see [33]). Now we consider the linear mapping \( J : S \to S \) such that

\[
   Jg(x) := 2g\left( \frac{x}{2} \right)
\]

for all \( x \in X \). Let \( g, h \in S \) be given such that \( d(g, h) = \varepsilon \). Then

\[
   \| g(x) - h(x) \|_Y \leq \varepsilon \varphi(x, 2x, x)
\]

for all \( x \in X \). Hence

\[
   \| Jg(x) - Jh(x) \|_Y = \left\| 2g\left( \frac{x}{2} \right) - 2h\left( \frac{x}{2} \right) \right\|_Y = |2| \left\| g\left( \frac{x}{2} \right) - h\left( \frac{x}{2} \right) \right\|_Y \leq \left| 2 \right| \varepsilon \varphi\left( \frac{x}{2}, x, \frac{x}{2} \right) \leq \alpha \cdot \varepsilon \varphi(x, 2x, x)
\]

for all \( x \in X \). So \( d(g, h) = \varepsilon \) implies that \( d(Jg, Jh) \leq \alpha \varepsilon \). This means that

\[
   d(Jg, Jh) \leq \alpha d(g, h)
\]

for all \( g, h \in S \). It follows from (2.5) that \( d(f, Jf) \leq \frac{\alpha}{2} \). By Theorem 1.11, there exists a mapping \( L : X \to Y \) satisfying the following:

1. \( L \) is a fixed point of \( J \), i.e.,

\[
   \frac{L(x)}{2} = L\left( \frac{x}{2} \right)
\]

for all \( x \in X \). The mapping \( L \) is a unique fixed point of \( J \) in the set \( M = \{ g \in S : d(h, g) < \infty \} \). This implies that \( L \) is a unique mapping satisfying (2.6) such that there exists a \( \mu \in (0, \infty) \) satisfying \( \| f(x) - L(x) \|_Y \leq \mu \varphi(x, 2x, x) \) for all \( x \in X \);

2. \( d(J^n f, L) \to 0 \) as \( n \to \infty \). This implies the equality

\[
   \lim_{n \to \infty} 2^n f\left( \frac{x}{2^n} \right) = L(x)
\]

for all \( x \in X \);

3. \( d(f, L) \leq \frac{1}{1 - \alpha} d(f, Jf) \), which implies the inequality \( d(f, L) \leq \frac{\alpha}{|2| - |2\alpha|} \). This implies that the inequalities (2.3) holds.

It follows from (2.1) and (2.2) that

\[
   \| L\left( \frac{x + y + z}{2} \right) + L\left( \frac{x - y + z}{2} \right) - L(x) - L(z) \|_Y
\]

\[
   = \lim_{n \to \infty} |2^n|^n \varphi\left( \frac{x + y + z}{2^n}, \frac{y + z}{2^n}, \frac{z}{2^n} \right) + f\left( \frac{x + y + z}{2^n} \right) - f\left( \frac{x}{2^n} \right) - f\left( \frac{z}{2^n} \right)
\]

\[
   \leq \lim_{n \to \infty} |2^n|^n \varphi\left( \frac{x + y + z}{2^n}, \frac{y + z}{2^n}, \frac{z}{2^n} \right) \leq \lim_{n \to \infty} |2^n|^n \frac{\alpha^n \varphi(x, y, z)}{|2|^n} = 0
\]
for all $x, y, z \in X$. So

$$L \left( \frac{x + y + z}{2} \right) + L \left( \frac{x - y + z}{2} \right) = L(x) + L(z)$$

for all $x, y, z \in X$. □

**Corollary 2.2.** Let $\theta$ be a positive real number and $r$ is a real number with $0 < r < 1$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying

$$\left\| f \left( \frac{x + y + z}{2} \right) + f \left( \frac{x - y + z}{2} \right) - f(x) - f(z) \right\|_Y \leq \theta \left( \|x\|^r + \|y\|^r + \|z\|^r \right)$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq \frac{|2| \theta (2 + |2|^r) \|x\|^r}{2r + 1 - |2|^r}$$

for all $x \in X$.

*Proof.* The proof follows from Theorem 2.1 by taking

$$\varphi(x, y, z) = \theta \left( \|x\|^r + \|y\|^r + \|z\|^r \right)$$

for all $x, y, z \in X$. Then we can choose $\alpha = |2|^{1-r}$ and we get the desired result. □

**Theorem 2.3.** Let $X$ be a non-Archimedean normed space and that $Y$ be a complete non-Archimedean space. Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with

$$\varphi(x, y, z) \leq |2| \alpha \varphi \left( \frac{x}{2}, \frac{y}{2}, \frac{z}{2} \right)$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying (2.2). Then there exists a unique additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq \frac{\varphi(x, 2x, x)}{|2| - |2|\alpha}$$

for all $x \in X$.

*Proof.* Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 2.1. Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{g(2x)}{2}$$

for all $x \in X$. Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\|g(x) - h(x)\|_Y \leq \varepsilon \varphi(x, 2x, x)$$

for all $x \in X$. □
for all $x \in X$. Hence
\[
\| Jg(x) - Jh(x) \|_Y = \left\| \frac{g(2x)}{2} - \frac{h(2x)}{2} \right\|_Y = \frac{\| g(2x) - h(2x) \|_Y}{|2|} \leq \frac{\varepsilon \Phi(2x, 4x, 2x)}{|2|} \leq \frac{2|\alpha| \cdot \varepsilon \Phi(x, 2x, x)}{|2|}
\]
for all $x \in X$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq \alpha \varepsilon$. This means that $d(Jg, Jh) \leq \alpha d(g, h)$ for all $g, h \in S$. It follows from (2.4) that $d(f, Jf) \leq \frac{1}{1 - |2|}$. By Theorem 1.11, there exists a mapping $L : X \to Y$ satisfying the following:

1. $L$ is a fixed point of $J$, i.e.,
   \[
   L(2x) = 2L(x)
   \]
   for all $x \in X$. The mapping $L$ is a unique fixed point of $J$ in the set $M = \{ g \in S : d(h, g) < \infty \}$. This implies that $L$ is a unique mapping satisfying (2.9) such that there exists a $\mu \in (0, \infty)$ satisfying $\| f(x) - L(x) \|_Y \leq \mu \Phi(x, 2x, x)$ for all $x \in X$;
2. $d(J^n f, L) \to 0$ as $n \to \infty$. This implies the equality
   \[
   \lim_{n \to \infty} \frac{f(2^n x)}{2^n} = L(x)
   \]
   for all $x \in X$;
3. $d(f, L) \leq \frac{1}{1 - |2|} d(f, Jf)$, which implies the inequality $d(f, L) \leq \frac{1}{1 - |2|}$. This implies that the inequalities (2.8) holds. The rest of the proof is similar to the proof of Theorem 2.1.

\begin{corollary}
Let $\theta$ be a positive real number and let $r$ be a real number with $r > 1$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying
\[
\| f \left( \frac{x + y + z}{2} \right) + f \left( \frac{x - y + z}{2} \right) - f(x) - f(z) \|_Y \leq \theta \left( \| x \|^r + \| y \|^r + \| z \|^r \right)
\]
for all $x, y, z \in X$. Then there exists a unique additive mapping $L : X \to Y$ such that
\[
\| f(x) - L(x) \|_Y \leq \frac{\theta(2 + |2|^r)}{|2| - |2|^r} \| x \|^r
\]
for all $x \in X$.
\end{corollary}

\textit{Proof.} The proof follows from Theorem 2.3 by taking
\[
\Phi(x, y, z) = \theta \left( \| x \|^r + \| y \|^r + \| z \|^r \right)
\]
for all $x, y, z \in X$. Then we can choose $\alpha = \frac{1}{|2|^r - |2|^r}$ and we get the desired result. \hfill \Box
2.2. A Direct Method. In this section, using a direct method, we prove the generalized Hyers-Ulam-Rassias stability of the Cauchy-Jensen additive functional equation (0.1) in non-Archimedean space.

**Theorem 2.5.** Let $G$ be an additive semigroup and that $X$ is a non-Archimedean Banach space. Assume that $\zeta : G^3 \to [0, +\infty)$ be a function such that

$$\lim_{n \to \infty} |2|^n \zeta \left( \frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n} \right) = 0$$

for all $x, y, z \in G$. Suppose that, for any $x \in G$, the limit

$$\mathcal{E}(x) = \lim_{n \to \infty} \max_{0 \leq k < n} |2|^k \zeta \left( \frac{x}{2^k+1}, \frac{x}{2^k}, \frac{x}{2^k+1} \right)$$

exists. Let $f : G \to X$ be a mapping with $f(0) = 0$ satisfying

$$\left\| f \left( \frac{x+y+z}{2} \right) + f \left( \frac{x-y+z}{2} \right) - f(x) - f(z) \right\|_X \leq \zeta(x, y, z), \text{ (for all } x, y, z \in G)$$

Then the limit $A(x) := \lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right)$ exists for all $x \in G$ and defines an additive mapping $A : G \to X$ such that

$$\|f(x) - A(x)\| \leq \mathcal{E}(x).$$

Moreover, if

$$\lim_{j \to \infty} \lim_{n \to \infty} \max_{0 \leq k < n+j} |2|^k \zeta \left( \frac{x}{2^k+1}, \frac{x}{2^k}, \frac{x}{2^k+1} \right) = 0$$

then $A$ is the unique additive mapping satisfying (2.14).

**Proof.** Putting $y = 2x$ and $z = x$ in (2.13), we get

$$\|f(2x) - 2f(x)\|_Y \leq \zeta(x, 2x, x)$$

for all $x \in G$. Replacing $x$ by $\frac{x}{2^n+1}$ in (2.15), we obtain

$$\left\| 2^{n+1} f \left( \frac{x}{2^{n+1}} \right) - 2^n f \left( \frac{x}{2^n} \right) \right\| \leq |2|^n \zeta \left( \frac{x}{2^{n+1}}, \frac{x}{2^n}, \frac{x}{2^{n+1}} \right).$$

Thus, it follows from (2.11) and (2.16) that the sequence $\{2^n f \left( \frac{x}{2^n} \right)\}_{n \geq 1}$ is a Cauchy sequence. Since $X$ is complete, it follows that $\{2^n f \left( \frac{x}{2^n} \right)\}_{n \geq 1}$ is convergent. Set

$$A(x) := \lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right).$$

By induction on $n$, one can show that

$$\left\| 2^n f \left( \frac{x}{2^n} \right) - f(x) \right\| \leq \max \left\{ |2|^k \zeta \left( \frac{x}{2^k+1}, \frac{x}{2^k}, \frac{x}{2^k+1} \right); 0 \leq k < n \right\}$$

for all $x \in G$. This completes the proof.
for all $n \geq 1$ and $x \in G$. By taking $n \to \infty$ in (2.18) and using (2.12), one obtains (2.14). By (2.11), (2.13) and (2.17), we get

$$
\left\| A \left( \frac{x+y+z}{2} \right) + A \left( \frac{x-y+z}{2} \right) - A(x) - A(z) \right\|
= \lim_{n \to \infty} |2^n|^2 \frac{2^n f \left( \frac{x+y+z}{2^n} \right) + f \left( \frac{x-y+z}{2^n} \right) - f \left( \frac{x}{2^n} \right) - f \left( \frac{z}{2^n} \right)}
\leq \lim_{n \to \infty} |2|^n \zeta \left( \frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n} \right) = 0
$$

for all $x, y, z \in X$. So

$$
(2.19) \quad A \left( \frac{x+y+z}{2} \right) + A \left( \frac{x-y+z}{2} \right) = A(x) + A(z)
$$

for all $x, y, z \in G$. Letting $y = 0$ in (2.19), we get

$$
(2.20) \quad 2A \left( \frac{x+z}{2} \right) = A(x) + A(z)
$$

for all $x, z \in G$. Since

$$
A(0) = \lim_{n \to +\infty} 2^n f \left( \frac{0}{2^n} \right) = \lim_{n \to +\infty} 2^n f(0) = 0,
$$

by letting $y = 2x$ and $z = x$ in (2.19), we get

$$
A(2x) = 2A(x)
$$

for all $x \in G$. Replacing $x$ by $2x$ and $z$ by $2z$ in (2.20), we get

$$
A(x+z) = A(x) + A(z)
$$

for all $x, z \in G$. Hence $A : G \to X$ is additive.

To prove the uniqueness property of $A$, let $L$ be another mapping satisfying (2.14). Then we have

$$
\begin{align*}
\| A(x) - L(x) \|_X &= \lim_{n \to \infty} |2^n|^n \| A \left( \frac{x}{2^n} \right) - L \left( \frac{x}{2^n} \right) \|_X \\
&\leq \lim_{k \to \infty} \max_{j \leq k < n+1} 2^k \zeta \left( \frac{x}{2^k}, \frac{x}{2^{k+1}}, \frac{x}{2^k} \right) = 0
\end{align*}
$$

for all $x \in G$. Therefore, $A = L$. This completes the proof.

\[ \square \]

**Corollary 2.6.** Let $\xi : [0, \infty) \to [0, \infty)$ be a function satisfying

$$
\xi \left( \frac{t}{2} \right) \leq \xi \left( \frac{1}{2} \right) \xi(t), \quad \xi \left( \frac{1}{2} \right) \leq \frac{1}{2}
$$

for all $x \in G$. The approximation of an additive mapping in various normed spaces.
for all \( t \geq 0 \). Assume that \( \kappa > 0 \) and \( f : G \to X \) is a mapping with \( f(0) = 0 \) such that
\[
\left\| f \left( \frac{x + y + z}{2} \right) + f \left( \frac{x - y + z}{2} \right) - f(x) - f(z) \right\|_Y \leq \kappa (\xi(|x|) + \xi(|y|) + \xi(|z|))
\]
for all \( x, y, z \in G \). Then there exists a unique additive mapping \( A : G \to X \) such that
\[
\left\| f(x) - A(x) \right\| \leq \frac{(2 + |2|)\xi(|x|)}{|2|}
\]
for all \( x, y, z \in G \). Assume that \( > 0 \) and \( f : G \to X \) is a mapping with \( f(0) = 0 \) such that

Proof. If we define \( \xi : G^3 \to [0, \infty) \) by \( \xi(x, y, z) := \kappa (\xi(|x|) + \xi(|y|) + \xi(|z|)) \), then we have
\[
\lim_{n \to \infty} |2|^n \xi \left( \frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n} \right) = 0
\]
for all \( x, y, z \in G \). On the other hand, it follows that
\[
\mathcal{L}(x) = \xi \left( \frac{x}{2}, \frac{x}{2} \right) = \frac{(2 + |2|)\xi(|x|)}{|2|}
\]
exists for all \( x \in G \). Also, we have
\[
\lim_{n \to \infty} \lim_{n \to \infty} \max_{0 \leq k < n} |2|^k \xi \left( \frac{x}{2^k}, \frac{y}{2^k}, \frac{z}{2^k} \right) = \lim_{n \to \infty} \max_{0 \leq k < n} \xi \left( \frac{x}{2^k}, \frac{y}{2^k}, \frac{z}{2^k} \right) = 0.
\]
Thus, applying Theorem 2.5, we have the conclusion. This completes the proof.

Theorem 2.7. Let \( G \) be an additive semigroup and that \( X \) is a non-Archimedean Banach space. Assume that \( \xi : G^3 \to [0, +\infty) \) is a function such that
\[
\lim_{n \to \infty} \frac{\xi \left( 2^n x, 2^n y, 2^n z \right)}{|2|^n} = 0
\]
for all \( x, y, z \in G \). Suppose that, for any \( x \in G \), the limit
\[
\mathcal{L}(x) = \lim_{n \to \infty} \max_{0 \leq k < n} \xi \left( \frac{x}{2^k}, \frac{x}{2^{k+1}}, \frac{x}{2^{k+1}} \right) = \lim_{n \to \infty} \frac{\xi \left( 2^n x \right)}{|2|^n}
\]
exists and \( f : G \to X \) be a mapping with \( f(0) = 0 \) and satisfying (2.13). Then the limit \( A(x) := \lim_{n \to \infty} \frac{f(2^n x)}{|2|^n} \) exists for all \( x \in G \) and
\[
\left\| f(x) - A(x) \right\| \leq \frac{\mathcal{L}(x)}{|2|}
\]
for all \( x \in G \). Moreover, if
\[
\lim_{j \to \infty} \lim_{n \to \infty} \max_{0 \leq k < n+j} \frac{\xi \left( 2^k x, 2^{k+1} x, 2^k x \right)}{|2|^k} = 0,
\]
then \( A \) is the unique mapping satisfying (2.23).
Proof. It follows from (2.15), we get
\[
(2.24) \quad \left\| f(x) - \frac{f(2x)}{2} \right\|_X \leq \frac{\zeta(x, 2x, x)}{|2|}
\]
for all \( x \in G \). Replacing \( x \) by \( 2^n x \) in (2.24), we obtain
\[
(2.25) \quad \left\| \frac{f(2^n x)}{2^n} - \frac{f(2^{n+1} x)}{2^{n+1}} \right\|_X \leq \frac{\zeta \left( \frac{2^n x, 2^{n+1} x, 2^n x}{|2|^{n+1}} \right)}{2^n}.
\]
Thus it follows from (2.21) and (2.25) that the sequence \( \left\{ \frac{f(2^n x)}{2^n} \right\}_{n \geq 1} \) is convergent. Set
\[
A(x) := \lim_{n \to \infty} f \left( \frac{2^n x}{2^n} \right).
\]
On the other hand, it follows from (2.25) that
\[
\left\| \frac{f(2^p x)}{2^p} - \frac{f(2^q x)}{2^q} \right\| = \left\| \sum_{k=p}^{q-1} \frac{f(2^k+1 x)}{2^{k+1}} - \frac{f(2^k x)}{2^k} \right\| \leq \max_{p \leq k < q} \left\{ \left\| \frac{f(2^{k+1} x)}{2^{k+1}} - \frac{f(2^k x)}{2^k} \right\| \right\} \leq \frac{1}{|2|} \max_{p \leq k < q} \zeta \left( \frac{2^k x, 2^{k+1} x, 2^k x}{|2|^k} \right)
\]
for all \( x \in G \) and \( p, q \geq 0 \) with \( q > p \geq 0 \). Letting \( p = 0 \), taking \( q \to \infty \) in the last inequality and using (2.22), we obtain (2.23).

The rest of the proof is similar to the proof of Theorem 2.5. \( \square \)

Corollary 2.8. Let \( \xi : [0, \infty) \to [0, \infty) \) be a function satisfying
\[
\xi \left( \left| 2^t \right| \right) \leq \xi \left( \left| 2^t \right| \right) \xi(t), \quad \xi \left( \left| 2^t \right| \right) < \left| 2^t \right|
\]
for all \( t \geq 0 \). Assume that \( \kappa > 0 \) and \( f : G \to X \) is a mapping with \( f(0) = 0 \) satisfying
\[
\left\| f \left( \frac{x + y + z}{2} \right) + f \left( \frac{x - y - z}{2} \right) - f(x) - f(z) \right\| \leq \kappa \left( \xi \left( \left| x \right| \right) \cdot \xi \left( \left| y \right| \right) \cdot \xi \left( \left| z \right| \right) \right)
\]
for all \( x, y, z \in G \). Then there exists a unique additive mapping \( A : G \to X \) such that
\[
\| f(x) - A(x) \| \leq \kappa \xi \left( \left| x \right| \right)^3.
\]

Proof. If we define \( \zeta : G^3 \to [0, \infty) \) by
\[
\zeta(x, y, z) := \kappa \left( \xi \left( \left| x \right| \right) \cdot \xi \left( \left| y \right| \right) \cdot \xi \left( \left| z \right| \right) \right)
\]
and apply Theorem 2.7, then we get the conclusion. \( \square \)

3. Random Stability of the Functional Equation (0.1)

In this section, using the fixed point and direct methods, we prove the generalized Hyers-Ulam-Rassias stability of the functional equation (0.1) in the random normed spaces.
3.1. Direct Method.

**Theorem 3.1.** Let \( X \) be a real linear space, \( (Z, \mu', \min) \) be an RN-space and \( \varphi : X^3 \to Z \) be a function such that there exists \( 0 < \alpha < \frac{1}{2} \) such that

\[
(3.1) \quad \mu'_{\varphi}(\hat{x}, \hat{y}, \hat{z})(t) \geq \mu'_{\varphi(x,y,z)} \left( \frac{t}{\alpha} \right)
\]

for all \( x, y, z \in X \) and \( t > 0 \) and \( \lim_{n \to \infty} \mu'_{\varphi}(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}) \left( \frac{t}{2^n} \right) = 1 \) for all \( x, y, z \in X \) and \( t > 0 \). Let \( (Y, \mu, \min) \) be a complete RN-space. If \( f : X \to Y \) is a mapping with \( f(0) = 0 \) such that

\[
(3.2) \quad \mu_{f(\hat{x})}(t) \geq \mu'_{\varphi(x,y,z)}(t)
\]

for all \( x, y, z \in X \) and \( t > 0 \). Then the limit \( A(x) = \lim_{n \to \infty} 2^n f(\frac{x}{2^n}) \) exists for all \( x \in X \) and defines a unique additive mapping \( \hat{A} : X \to Y \) such that

\[
(3.3) \quad \mu_{f(x)-A(x)}(t) \geq \mu'_{\varphi(x,2x,x)} \left( \frac{(1-2\alpha)t}{\alpha} \right).
\]

for all \( x \in X \) and \( t > 0 \).

**Proof.** Putting \( y = 2x \) and \( z = x \) in (3.2), we see that

\[
(3.4) \quad \mu_{f(2x)-2f(x)}(t) \geq \mu'_{\varphi(x,2x,x)}(t).
\]

Replacing \( x \) by \( \hat{z} \) in (3.4), we obtain

\[
(3.5) \quad \mu_{f(x)}(\hat{z}) \geq \mu'_{\varphi(x, \hat{z}, x)}(t) \geq \mu'_{\varphi(x,2x,x)} \left( \frac{t}{2n} \right)
\]

for all \( x \in X \). Replacing \( x \) by \( \frac{x}{2^n} \) in (3.5) and using (3.1), we obtain

\[
\mu_{2^n f(\frac{x}{2^n})}(t) \geq \mu'_{\varphi(\frac{x}{2^n}, \frac{x}{2^n}, \frac{x}{2^n})} \left( \frac{t}{2^n} \right) \geq \mu'_{\varphi(x,2x,x)} \left( \frac{t}{2^n \alpha^{k+1}} \right)
\]

and so

\[
\mu_{2^n f(\frac{x}{2^n})}(t) \geq \mu_{\sum_{k=0}^{n-1} 2^k \alpha^{k+1} t} \sum_{k=0}^{n-1} 2^k \alpha^{k+1} t) \geq T_{n-1} \left( \mu'_{\varphi(x,2x,x)}(t) \right)
\]

This implies that

\[
(3.6) \quad \mu_{2^n f(\frac{x}{2^n})}(t) \geq \mu'_{\varphi(x,2x,x)} \left( \frac{t}{\sum_{k=0}^{n-1} 2^k \alpha^{k+1}} \right).
\]
Replacing \( x \) by \( \frac{3x}{2\pi} \) in (3.6), we obtain

\[
(3.7) \quad \mu_{2^n}(\phi(x,2x,x)) - 2^n \phi(x,2x,x)(t) \geq \mu_{\phi}(x,2x,x) \left( \frac{t}{\sum_{k=0}^{\infty} 2^k \alpha^{k+1}} \right).
\]

Since \( \lim_{n \to \infty} \mu_{\phi}(x,2x,x) \left( \frac{t}{\sum_{k=0}^{\infty} 2^k \alpha^{k+1}} \right) = 1 \), it follows that \( \{ 2^n \phi(x,2x,x) \}_{n=1}^{\infty} \) is a Cauchy sequence in a complete RN-space \((Y, \mu, \min)\) and so there exists a point \( A(x) \in Y \) such that \( \lim_{n \to \infty} 2^n \phi(x,2x,x) = A(x) \). Fix \( x \in X \) and put \( p = 0 \) in (3.7) and so, for any \( \epsilon > 0 \),

\[
(3.8) \quad \mu_{A(x)-f(x)}(t + \epsilon) \geq T \left( \mu_{A(x)-2^n f(x,2x,x)}(\epsilon), \mu_{\phi}(x,2x,x) \left( \frac{t}{\sum_{k=0}^{\infty} 2^k \alpha^{k+1}} \right) \right).
\]

Taking \( n \to \infty \) in (3.8), we get

\[
\mu_{A(x)-f(x)}(t + \epsilon) \geq \mu_{\phi}(x,2x,x) \left( \frac{1 - 2\alpha t}{\alpha} \right) \cdot (1 - 2\alpha t).
\]

Since \( \epsilon \) is arbitrary, by taking \( \epsilon \to 0 \) in (3.9), we get

\[
\mu_{A(x)-f(x)}(t) \geq \mu_{\phi}(x,2x,x) \left( \frac{1 - 2\alpha t}{\alpha} \right) \cdot (1 - 2\alpha t).
\]

Replacing \( x, y \) and \( z \) by \( \frac{3x}{2\pi}, \frac{3y}{2\pi} \) and \( \frac{3z}{2\pi} \) in (3.2), respectively, we get

\[
\mu_{2^n f(x,2x,x,2x,x,2x,x)}(t + \epsilon) \geq \mu_{\phi}(x,2x,x,2x,x,2x,x) \left( \frac{t}{2^n} \right) \cdot (1 - 2\alpha t).
\]

for all \( x, y, z \in X \) and \( t > 0 \). Since \( \lim_{n \to \infty} \mu_{\phi}(x,2x,x,2x,x,2x,x) \left( \frac{t}{2^n} \right) = 1 \), we conclude that \( A \) satisfies (0.1). On the other hand

\[
2A \left( \frac{x}{2} \right) - A(x) = \lim_{n \to \infty} 2^n \phi(x,2x,x,2x,x,2x,x) = 0.
\]

This implies that \( A : X \to Y \) is an additive mapping. To prove the uniqueness of the additive mapping \( A \), assume that there exists another additive mapping \( L : X \to Y \) which satisfies (3.3). Then we have

\[
\mu_{A(x)-L(x)}(t) = \lim_{n \to \infty} \mu_{2^n A(x,2x,x,2x,x,2x,x,2x,x)}(t) \geq \mu_{2^n A(x,2x,x,2x,x,2x,x,2x,x)}(t) \cdot \mu_{\phi}(x,2x,x,2x,x,2x,x,2x,x) \left( \frac{t}{2^n} \right) = 1.
\]

Therefore, it follows that \( \mu_{A(x)-L(x)}(t) = 1 \) for all \( t > 0 \) and so \( A(x) = L(x) \). This completes the proof. \( \square \)
Corollary 3.2. Let $X$ be a real normed linear space, $(Z, \mu', \min)$ be an RN-space and $(Y, \mu, \min)$ be a complete RN-space. Let $r$ be a positive real number with $r > 1$, $z_0 \in Z$ and $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying
\begin{equation}
\mu_{f(x; y)}(t) = \mu_{\phi(x; y; z)}(t) \geq \mu_{\phi(x; y; z)}(t)
\end{equation}
for all $x, y \in X$ and $t > 0$. Then the limit $A(x) = \lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right)$ exists for all $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that and
\[ \mu_f(x; y)(t) \geq \mu'_{\phi(x; y; z)}(t) \]
for all $x \in X$ and $t > 0$.

Proof. Let $\alpha = 2^{-r} \pi$ and $\varphi : X^3 \to Z$ be a mapping defined by $\varphi(x, y, z) = (\|x\|^r + \|y\|^r + \|z\|^r)z_0$. Then, from Theorem 3.1, the conclusion follows. \qed

Theorem 3.3. Let $X$ be a real linear space, $(Z, \mu', \min)$ be an RN-space and $\varphi : X^3 \to Z$ be a function such that there exists $0 < \alpha < 2$ such that $\mu'_{\varphi(x, y; z)}(t) \geq \mu'_{\varphi(x, y; z)}(t)$ for all $x \in X$ and $t > 0$ and
\[ \lim_{n \to \infty} \mu_{\varphi(2^n x, 2^n y, 2^n z)}(2^n x) = 1 \]
for all $x, y, z \in X$ and $t > 0$. Let $(Y, \mu, \min)$ be a complete RN-space. If $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying (3.2). Then the limit $A(x) = \lim_{n \to \infty} f(2^n x) \alpha^n$ exists for all $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that and
\[ \mu_f(x)(t) \geq \mu'_{\varphi(x, y; z)}((2 - \alpha)t). \]
for all $x \in X$ and $t > 0$.

Proof. It follows from (3.4) that
\begin{equation}
\mu_{\phi(x; y)}(t) \geq \mu'_{\phi(x; 2x, x)}(2t).
\end{equation}
Replacing $x$ by $2^n x$ in (3.12), we obtain that
\[ \mu_f(x; y; z)(t) \geq \mu_{\phi(x; 2^n x, 2^{n+1} x, 2^n x)}(2^{n+1} t) \geq \mu_{\phi(x, 2x, x)} \left( \frac{2^{n+1} t}{\alpha^n} \right). \]
The rest of the proof is similar to the proof of Theorem 3.1. \qed

Corollary 3.4. Let $X$ be a real normed linear space, $(Z, \mu', \min)$ be an RN-space and $(Y, \mu, \min)$ be a complete RN-space. Let $r$ be a positive real number with $0 < r < 1$, $z_0 \in Z$ and $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying (3.10). Then the limit $A(x) = \lim_{n \to \infty} f(2^n x) \alpha^n$ exists for all $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that and
\[ \mu_f(x; y)(t) \geq \mu'_{\phi(x; y; z)}(t) \]
for all $x \in X$ and $t > 0$. 

Proof. Let \( \alpha = 2^r \) and \( \varphi : X^3 \to Z \) be a mapping defined by \( \varphi(x, y, z) = (\|x\|^{r} + \|y\|^{r} + \|z\|^{r})z_0 \). Then, from Theorem 3.3, the conclusion follows. \( \square \)

3.2. Fixed Point Method.

**Theorem 3.5.** Let \( X \) be a linear space, \( (Y, \mu, T_M) \) be a complete RN-space and \( \Phi \) be a mapping from \( X^3 \) to \( D^+ \) \( (\Phi(x, y, z) \) is denoted by \( \Phi_{x,y,z} \) \) such that there exists \( 0 < \alpha < \frac{1}{2} \) such that

\[
\Phi_{2x,2y,2z}(t) \leq \Phi_{x,y,z}(\alpha t)
\]

for all \( x, y, z \in X \) and \( t > 0 \). Let \( f : X \to Y \) be a mapping with \( f(0) = 0 \) satisfying

\[
\mu_{f(\frac{\alpha + 1}{2} + f(\frac{\alpha + 2}{2} - x + f(z))(t) \geq \Phi_{x,y,z}(t)
\]

for all \( x, y, z \in X \) and \( t > 0 \). Then, for all \( x \in X \), \( A(x) := \lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right) \) exists and \( A : X \to Y \) is a unique additive mapping such that

\[
\mu_{f(x) - A(x)}(t) \geq \Phi_{x,2x.x} \left( \frac{1 - 2\alpha t}{\alpha} \right)
\]

for all \( x \in X \) and \( t > 0 \).

**Proof.** Putting \( y = 2x \) and \( z = x \) in (3.14), we have

\[
\mu_{2f(\frac{x}{2}) - f(x)}(t) \geq \Phi_{2x,2x,2x}(t) \geq \Phi_{x,2x,x} \left( \frac{1}{\alpha} \right)
\]

for all \( x \in X \) and \( t > 0 \). Consider the set \( S := \{ g : X \to Y \} \) and the generalized metric \( d \) in \( S \) defined by

\[
d(f, g) = \inf_{u \in (0, \infty)} \left\{ \mu_{g(x) - h(x)}(ut) \geq \Phi_{x,2x,x}(t), \forall x \in X, t > 0 \right\},
\]

where \( \inf \emptyset = +\infty \). It is easy to show that \( (S, d) \) is complete (see [33, Lemma 2.1]). Now, we consider a linear mapping \( J : (S, d) \to (S, d) \) such that

\[
Jh(x) := 2h \left( \frac{x}{2} \right)
\]

for all \( x \in X \). First, we prove that \( J \) is a strictly contractive mapping with the Lipschitz constant \( 2\alpha \). In fact, let \( g, h \in S \) be such that \( d(g, h) < \epsilon \). Then we have \( \mu_{g(x) - h(x)}(\epsilon t) \geq \Phi_{x,2x,x}(t) \) for all \( x \in X \) and \( t > 0 \) and so

\[
\mu_{Jg(x) - Jh(x)}(2\alpha t) = \mu_{2g(\frac{x}{2}) - 2h(\frac{x}{2})}(2\alpha t) = \mu_{g(\frac{x}{2}) - h(\frac{x}{2})}(2\alpha t) \geq \Phi_{\frac{x}{2},\frac{x}{2},\frac{x}{2}}(\alpha t) \geq \Phi_{x,2x,x}(t)
\]

for all \( x \in X \) and \( t > 0 \). Thus \( d(g, h) < \epsilon \) implies that \( d(Jg, Jh) < 2\alpha \epsilon \). This means that \( d(Jg, Jh) \leq 2\alpha d(g, h) \) for all \( g, h \in S \). It follows from (3.16) that

\[
d(f, Jf) \leq \alpha.
\]
By Theorem 1.11, there exists a mapping $A : X \to Y$ satisfying the following:

(1) $A$ is a fixed point of $J$, that is,

$$A\left(\frac{x}{2}\right) = \frac{1}{2} A(x)$$

(3.19)

for all $x \in X$. The mapping $A$ is a unique fixed point of $J$ in the set $\Omega = \{ h \in S : d(g, h) < \infty \}$. This implies that $A$ is a unique mapping satisfying (3.19) such that there exists $u \in (0, \infty)$ satisfying $\mu_{f(x) - A(x)}(ut) \geq \Phi_{x,y,z}(t)$ for all $x \in X$ and $t > 0$.

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right) = A(x)$$

for all $x \in X$.

(3) $d(f, A) \leq \frac{d(f,Jf)}{1-2\alpha}$ with $f \in \Omega$, which implies the inequality $d(f, A) \leq \frac{\alpha t}{1-2\alpha}$ and so

$$\mu_{f(x) - A(x)} \left( \frac{\alpha t}{1-2\alpha} \right) \geq \Phi_{x,y,z}(t)$$

for all $x \in X$ and $t > 0$. This implies that the inequality (3.15) holds. On the other hand

$$\mu_{2^n f \left( \frac{x+y+z}{2^n} \right) + 2^n f \left( \frac{x+y+z}{2^n} \right) - 2^n f \left( \frac{x+y}{2^n} \right)}(t) \geq \Phi_{x,y,z} \left( \frac{t}{(2\alpha)^n} \right)$$

for all $x, y, z \in X, t > 0$ and $n \geq 1$. By (3.13), we know that

$$\Phi_{\frac{x+y}{2^n}, \frac{x+y+z}{2^n}} \left( \frac{t}{2^n} \right) \geq \Phi_{x,y,z} \left( \frac{t}{(2\alpha)^n} \right).$$

Since $\lim_{n \to \infty} \Phi_{x,y,z} \left( \frac{t}{(2\alpha)^n} \right) = 1$ for all $x, y, z \in X$ and $t > 0$, we have

$$\mu_{A \left( \frac{x+y+z}{2^n} \right) + A \left( \frac{x+y}{2^n} \right) - A(x) - A(z)}(t) = 1$$

for all $x, y, z \in X$ and $t > 0$. Thus the mapping $A : X \to Y$ satisfying (0.1). Furthermore

$$A(2x) - 2A(x) = \lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right) - 2 \lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right)$$

$$= 2 \left[ \lim_{n \to \infty} 2^{n-1} f \left( \frac{x}{2^{n-1}} \right) - \lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right) \right]$$

$$= 0.$$

This completes the proof. \hspace{1cm} \square

**Corollary 3.6.** Let $X$ be a real normed space, $\theta \geq 0$ and $r$ be a real number with $r > 1$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying

$$\mu_{J^n f \left( \frac{x+y+z}{2^n} \right) + f \left( \frac{x+y+z}{2^n} \right) - f(x) - f(z)}(t) \geq \frac{t}{t + \theta(\|x\|^r + \|y\|^r + \|z\|^r)}$$

(3.20)
for all $x, y, z \in X$ and $t > 0$. Then $A(x) = \lim_{n \to \infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for all $x \in X$ and $A : X \to Y$ is a unique additive mapping such that

$$\mu_{f(x)-A(x)}(t) \geq \frac{(2^r - 2)t}{(2^r - 2)t + (2^r + 2)\theta\|x\|^r}$$

for all $x \in X$ and $t > 0$.

**Proof.** The proof follows from Theorem 3.5 if we take

$$\Phi_{x,y,z}(t) = \frac{t}{t + \theta(\|x\|^r + \|y\|^r + \|z\|^r)}$$

for all $x, y, z \in X$ and $t > 0$. In fact, if we choose $\alpha = 2^{-r}$, then we get the desired result. □

**Theorem 3.7.** Let $X$ be a linear space, $(Y, \mu, T_M)$ be a complete RN-space and $\Phi$ be a mapping from $X^3$ to $D^+$ ($\Phi(x,y,z)$ is denoted by $\Phi_{x,y,z}$) such that for some $0 < \alpha < 2$

$$\Phi_{x,y,z}(t) \leq \Phi_{x,y,z}(\alpha t)$$

for all $x, y, z \in X$ and $t > 0$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying (3.14). Then the limit $A(x) := \lim_{n \to \infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in X$ and $A : X \to Y$ is a unique additive mapping such that

$$\mu_{f(x)-A(x)}(t) \geq \Phi_{x,2x,x}(2t)$$

for all $x \in X$ and $t > 0$.

**Proof.** Putting $y = 2x$ and $z = x$ in (3.14), we have

$$\mu_{f(2x)-f(x)}(t) \geq \Phi_{x,2x,x}(2t)$$

for all $x \in X$ and $t > 0$. Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 3.1. Now, we consider a linear mapping $J : (S, d) \to (S, d)$ such that

$$Jh(x) := \frac{1}{2} h(2x)$$

for all $x \in X$. It follows from (3.22) that $d(f, Jf) \leq \frac{1}{2}$. By Theorem 1.11, there exists a mapping $A : X \to Y$ satisfying the following:

1. $A$ is a fixed point of $J$, that is,

$$A(2x) = 2A(x)$$

for all $x \in X$. The mapping $A$ is a unique fixed point of $J$ in the set $\Omega = \{h \in S : d(g, h) < \infty\}$. This implies that $A$ is a unique mapping satisfying (3.24) such that there exists $u \in (0, \infty)$ satisfying $\mu_{f(x)-A(x)}(ut) \geq \Phi_{x,2x,x}(t)$ for all $x \in X$ and $t > 0$.

2. $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n \to \infty} \frac{f(2^n x)}{2^n} = A(x)$$
for all $x \in X$.

(3) $d(f, A) \leq \frac{d(f, Jf)}{1-\frac{r}{2}}$ with $f \in \Omega$, which implies the inequality

$$\mu_{f(x)-A(x)} \left( \frac{t}{2-\alpha} \right) \geq \Phi_{x,2x,x}(t)$$

for all $x \in X$ and $t > 0$. This implies that the inequality (3.21) holds. The rest of the proof is similar to the proof of Theorem 3.5.

\[ \Box \]

**Corollary 3.8.** Let $X$ be a real normed space, $\theta \geq 0$ and $r$ be a real number with $0 < r < 1$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying (3.20). Then the limit $A(x) = \lim_{n \to \infty} \frac{f(2^nx)}{2^n}$ exists for all $x \in X$ and $A : X \to Y$ is a unique additive mapping such that

$$\mu_{f(x)-A(x)}(t) \geq \frac{(2-2^r)t}{(2-2^r)t + (2^r + 2)\theta\|x\|^r}$$

for all $x \in X$ and $t > 0$.

**Proof.** The proof follows from Theorem 3.7 if we take

$$\Phi_{x,y}(t) = \frac{t}{t + \theta(\|x\|^r + \|y\|^r + \|z\|^r)}$$

for all $x, y, z \in X$ and $t > 0$. In fact, if we choose $\alpha = 2^r$, then we get the desired result.

\[ \Box \]

**References**


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