ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 41 (2015), No. 5, pp. 1213–1233

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 41 (2015), No. 5, pp. 1213–1233
Online ISSN: 1735-8515

APPROXIMATION OF AN ADDITIVE MAPPING IN VARIOUS NORMED SPACES

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(Communicated by Madjid Eshaghi Gordji)

Abstract. In this paper, using the fixed point and direct methods, we prove the generalized Hyers-Ulam-Rassias stability of the following Cauchy-Jensen additive functional equation:

$$f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y+z}{2}\right)=f(x)+f(z) \tag{0.1}$$

in various normed spaces.
Keywords: Hyers-Ulam-Rassias stability, non-Archimedean normed spaces, random normed spaces.
MSC(2010): 39B22, 39B52, 39B82, 46S10, 47S10, 46S40.

1. Introduction

A classical question in the theory of functional equations is the following: *When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?*. If the problem accepts a solution, we say that the equation is *stable*. The first stability problem concerning group homomorphisms was raised by Ulam [45] in 1940. In the next year, Hyers [23] gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces. In 1978, Rassias [38] proved a generalization of Hyers's theorem for additive mappings.

Theorem 1.1. *(Th.M. Rassias): Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y)-f(x)-f(y)\| \le \epsilon(\|x\|^p+\|y\|^p)$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $0 \le p < 1$. Then the limit $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and $L : E \to E'$ is the

Article electronically published on October 17, 2015.
Received: 25 January 2014, Accepted: 22 July 2014.
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unique linear mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p$$

for all $x \in E$*. Also, if for each* $x \in E$ *the function* $f(tx)$ *is continuous in* $t \in \mathbb{R}$*, then* L *is linear.*

This new concept is known as generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. Furthermore, in 1994, a generalization of Rassias's theorem was obtained by Găvruta [21] by replacing the bound $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$.
In 1983, a generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [44] for mapping $f : X \to Y$, where X is a normed space and Y is a Banach space. In 1984, Cholewa [11] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group and, in 2002, Czerwik [12] proved the generalized Hyers-Ulam stability of the quadratic functional equation. The reader is referred to ([1]- [42]) and references therein for detailed information on stability of functional equations.

In 1897, Hensel [22] has introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications (see [13, 27, 30, 31, 36]).

Katsaras [26] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view (see [20, 29]). In particular, Bag and Samanta [5], following Cheng and Mordeson [10], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Karmosil and Michalek type [28]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [6].

Definition 1.2. By a *non-Archimedean field* we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot| : \mathbb{K} \to [0, \infty)$ such that, for all $r, s \in \mathbb{K}$, the following conditions hold:

(a) $|r| = 0$ if and only if $r = 0$;
(b) $|rs| = |r||s|$;
(c) $|r + s| \leq \max\{|r|, |s|\}$.

Clearly, by (b), $|1| = |-1| = 1$ and so, by induction, it follows from (c) that $|n| \leq 1$ for all $n \geq 1$.

Definition 1.3. Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$.

(1) A function $\|\cdot\| : X \to \mathbb{R}$ is a *non-Archimedean norm* (valuation) if it satisfies the following conditions:

(a) $\|x\| = 0$ if and only if $x = 0$ for all $x \in X$;

(b) $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}$ and $x \in X$;

(c) the strong triangle inequality (ultra-metric) holds, that is,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}$$

for all $x, y \in X$.

(2) The space $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*.

Note that $||x_n - x_m|| \leq max\{||x_{j+1} - x_j|| : m \leq j \leq n-1\}$ for all $m, n \in \mathbb{N}$ with $n > m$.

Definition 1.4. Let $(X, \|\cdot\|)$ be a non-Archimedean normed space.

(a) A sequence $\{x_n\}$ is a *Cauchy sequence* in X if $\{x_{n+1} - x_n\}$ converges to zero in X.

(b) The non-Archimedean normed space $(X, \|\cdot\|)$ is said to be *complete* if every Cauchy sequence in X is convergent.

The most important examples of non-Archimedean spaces are p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: for all $x, y > 0$, there exists a positive integer n such that $x < ny$.

Example 1.5. Fix a prime number p. For any nonzero rational number x, there exists a unique positive integer n_x such that $x = \frac{a}{b}p^{n_x}$, where a and b are positive integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$, which is called the *p-adic number field*. In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum_{k \geq n_x}^{\infty} a_k p^k$, where $|a_k| \leq p - 1$. The addition and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $|\sum_{k \geq n_x}^{\infty} a_k p^k|_p = p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}_p$ and $\mathbb{Q}_p$ is a locally compact filed.

In random stability, we adopt the usual terminology, notions and conventions of the theory of random normed spaces as in [43].

Throughout this paper, let $\triangle^+$ denote the set of all probability distribution functions $F : \mathbb{R} \cup [-\infty, +\infty] \to [0, 1]$ such that F is left-continuous and nondecreasing on $\mathbb{R}$ and $F(0) = 0, F(+\infty) = 1$. It is clear that the set $D^+ = \{F \in \triangle^+ : l^- F(-\infty) = 1\}$, where $l^- F(x) = \lim_{t \to x^-} F(t)$, is a subset of $\triangle^+$. The set $\triangle^+$ is partially ordered by the usual point-wise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. For any $a \geq 0$, the element $H_a(t)$ of D^+ is defined by

$$H_a(t) = \begin{cases} 0, & \text{if} \quad t \leq a, \\ 1, & \text{if} \quad t > a. \end{cases}$$

We can easily show that the maximal element in $\triangle^+$ is the distribution function $H_0(t)$.

Definition 1.6. A function $T : [0,1]^2 \to [0,1]$ is a *continuous triangular norm* (briefly, a t-norm) if T satisfies the following conditions:
(a) T is commutative and associative;
(b) T is continuous;
(c) $T(x,1) = x$ for all $x \in [0,1]$;
(d) $T(x,y) \leq T(z,w)$ whenever $x \leq z$ and $y \leq w$ for all $x,y,z,w \in [0,1]$.

Three typical examples of continuous t-norms are as follows: $T(x,y) = xy,\ T(x,y) = \max\{x+y-1,0\},\ T(x,y) = \min(x,y)$.

Recall that, if T is a t-norm and $\{x_n\}$ is a sequence in $[0,1]$, then $T_{i=1}^n x_i$ is defined recursively by $T_{i=1}^1 x_1 = x_1$ and $T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n)$ for all $n \geq 2$. $T_{i=n}^\infty x_i$ is defined by $T_{i=1}^\infty x_{n+i}$.

Definition 1.7. A *random normed space* (briefly, RN-space) is a triple (X,μ,T), where X is a vector space, T is a continuous t-norm and $\mu : X \to D^+$ is a mapping such that the following conditions hold:
(a) $\mu_x(t) = H_0(t)$ for all $t > 0$ if and only if $x = 0$;
(b) $\mu_{\alpha x}(t) = \mu_x\left(\frac{t}{|\alpha|}\right)$ for all $\alpha \in \mathbb{R}$ with $\alpha \neq 0$, $x \in X$ and $t \geq 0$;
(c) $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$ for all $x,y \in X$ and $t,s \geq 0$.

Every normed space $(X, \|\cdot\|)$ defines a random normed space (X,μ,T_M), where $\mu_u(t) = \frac{t}{t+\|u\|}$ for all $t > 0$ and T_M is the minimum t-norm. This space X is called the *induced random normed space.*
If the t-norm T is such that $\sup_{0<a<1} T(a,a) = 1$, then every RN-space (X,μ,T) is a metrizable linear topological space with the topology τ (called the μ-*topology* or the (ϵ,δ)-*topology*, where $\epsilon > 0$ and $\lambda \in (0,1)$) induced by the base $\{U(\epsilon,\lambda)\}$ of neighborhoods of θ, where

$$U(\epsilon,\lambda) = \{x \in X : \mu_x(\epsilon) > 1-\lambda\}.$$

Definition 1.8. Let (X,μ,T) be an RN-space.
(a) A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ (write $x_n \to x$ as $n \to \infty$) if

$$\lim_{n\to\infty} \mu_{x_n - x}(t) = 1$$

for all $t > 0$.
(b) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* in X if

$$\lim_{n\to\infty} \mu_{x_n - x_m}(t) = 1$$

for all $t > 0$.
(c) The RN-space (X,μ,T) is said to be *complete* if every Cauchy sequence in X is convergent.

Theorem 1.9. *If (X,μ,T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then* $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$.

Definition 1.10. Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies the following conditions:

(a) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;
(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(c) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.11. *Let (X,d) be a complete generalized metric space and* $J : X \to X$ *be a strictly contractive mapping with Lipschitz constant* $L < 1$. *Then, for all* $x \in X$, *either* $d(J^n x, J^{n+1} x) = \infty$ *for all nonnegative integers* n *or there exists a positive integer* n_0 *such that*

(a) $d(J^n x, J^{n+1} x) < \infty$ *for all* $n \geq n_0$;
(b) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;
(c) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;
(d) $d(y, y^*) \leq \frac{d(y, Jy)}{1-L}$ *for all* $y \in Y$.

2. Non-Archimedean Stability of Functional Equation (0.1)

In this section, we deal with the stability problem for the Cauchy-Jensen additive functional equation (0.1) in non-Archimedean normed spaces.

2.1. A Fixed Point Approach.

Theorem 2.1. *Let* X *be a non-Archimedean normed space and that* Y *be a complete non-Archimedean space. Let* $\varphi : X^3 \to [0, \infty)$ *be a function such that there exists an* $\alpha < 1$ *with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{\alpha \varphi(x, y, z)}{|2|} \tag{2.1}$$

for all $x, y, z \in X$. *Let* $f : X \to Y$ *be a mapping with* $f(0) = 0$ *satisfying*

$$\left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y+z}{2}\right) - f(x) - f(z) \right\|_Y \leq \varphi(x, y, z) \tag{2.2}$$

for all $x, y, z \in X$. *Then there exists a unique additive mapping* $L : X \to Y$ *such that*

$$\|f(x) - L(x)\|_Y \leq \frac{\alpha \varphi(x, 2x, x)}{|2| - |2|\alpha} \tag{2.3}$$

for all $x \in X$.

Proof. Putting $y = 2x$ and $z = x$ in (2.2), we get

$$\|f(2x) - 2f(x)\|_Y \leq \varphi(x, 2x, x) \tag{2.4}$$

for all $x \in X$. So

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\|_Y \leq \varphi\left(\frac{x}{2}, x, \frac{x}{2}\right) \leq \frac{\alpha \varphi(x, 2x, x)}{|2|} \tag{2.5}$$

for all $x \in X$. Consider the set $S := \{h : X \to Y\}$ and introduce the generalized metric on S:

$$d(g,h) = \inf\Big\{\mu \in (0,+\infty) : \|g(x) - h(x)\|_Y \le \mu\varphi(x,2x,x),\ \forall x \in X\Big\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [33]). Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$. Let $g,h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\|g(x) - h(x)\|_Y \le \epsilon\varphi(x,2x,x)$$

for all $x \in X$. Hence

$$\begin{aligned}\|Jg(x) - Jh(x)\|_Y &= \left\|2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right)\right\|_Y = |2|\left\|g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right)\right\|_Y \\ &\le |2|\epsilon\varphi\left(\frac{x}{2},x,\frac{x}{2}\right) \le \alpha\cdot\epsilon\varphi(x,2x,x)\end{aligned}$$

for all $x \in X$. So $d(g,h) = \varepsilon$ implies that $d(Jg,Jh) \le \alpha\varepsilon$. This means that $d(Jg,Jh) \le \alpha d(g,h)$ for all $g,h \in S$. It follows from (2.5) that $d(f,Jf) \le \frac{\alpha}{|2|}$. By Theorem 1.11, there exists a mapping $L : X \to Y$ satisfying the following:

(1) L is a fixed point of J, i.e.,

$$\frac{L(x)}{2} = L\left(\frac{x}{2}\right) \tag{2.6}$$

for all $x \in X$. The mapping L is a unique fixed point of J in the set $M = \{g \in S : d(h,g) < \infty\}$. This implies that L is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0,\infty)$ satisfying $\|f(x) - L(x)\|_Y \le \mu\varphi(x,2x,x)$ for all $x \in X$;

(2) $d(J^n f, L) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = L(x) \tag{2.7}$$

for all $x \in X$;

(3) $d(f,L) \le \frac{1}{1-\alpha}d(f,Jf)$, which implies the inequality $d(f,L) \le \frac{\alpha}{|2|-|2|\alpha}$. This implies that the inequalities (2.3) holds.

It follows from (2.1) and (2.2) that

$$\begin{aligned}&\left\|L\left(\frac{x+y+z}{2}\right) + L\left(\frac{x-y+z}{2}\right) - L(x) - L(z)\right\|_Y \\ &= \lim_{n\to\infty} |2|^n \left\|f\left(\frac{x+y+z}{2^{n+1}}\right) + f\left(\frac{x-y+z}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right\|_Y \\ &\le \lim_{n\to\infty} |2|^n \varphi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right) \le \lim_{n\to\infty} |2|^n.\frac{\alpha^n\varphi(x,y,z)}{|2|^n} = 0\end{aligned}$$

for all $x, y, z \in X$. So

$$L\left(\frac{x+y+z}{2}\right) + L\left(\frac{x-y+z}{2}\right) = L(x) + L(z)$$

for all $x, y, z \in X$. □

Corollary 2.2. *Let θ be a positive real number and r is a real number with $0 < r < 1$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying*

$$\left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y+z}{2}\right) - f(x) - f(z) \right\|_Y \leq \theta \left(\|x\|^r + \|y\|^r + \|z\|^r\right)$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq \frac{|2|\theta(2 + |2|^r)\|x\|^r}{|2|^{r+1} - |2|^2}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x, y, z) = \theta \left(\|x\|^r + \|y\|^r + \|z\|^r\right)$$

for all $x, y, z \in X$. Then we can choose $\alpha = |2|^{1-r}$ and we get the desired result. □

Theorem 2.3. *Let X be a non-Archimedean normed space and that Y be a complete non-Archimedean space. Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x, y, z) \leq |2|\alpha\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying (2.2). Then there exists a unique additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq \frac{\varphi(x, 2x, x)}{|2| - |2|\alpha} \tag{2.8}$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1. Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{g(2x)}{2}$$

for all $x \in X$. Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\|g(x) - h(x)\|_Y \leq \epsilon\varphi(x, 2x, x)$$

for all $x \in X$. Hence

$$\begin{aligned}\|Jg(x) - Jh(x)\|_Y &= \left\|\frac{g(2x)}{2} - \frac{h(2x)}{2}\right\|_Y = \frac{\|g(2x) - h(2x)\|_Y}{|2|} \\ &\leq \frac{\epsilon\varphi\left(2x, 4x, 2x\right)}{|2|} \leq \frac{|2|\alpha \cdot \epsilon\varphi(x, 2x, x)}{|2|}\end{aligned}$$

for all $x \in X$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq \alpha\varepsilon$. This means that $d(Jg, Jh) \leq \alpha d(g, h)$ for all $g, h \in S$. It follows from (2.4) that $d(f, Jf) \leq \frac{1}{|2|}$. By Theorem 1.11, there exists a mapping $L : X \to Y$ satisfying the following:

(1) L is a fixed point of J, i.e.,

$$L(2x) = 2L\left(x\right) \tag{2.9}$$

for all $x \in X$. The mapping L is a unique fixed point of J in the set $M = \{g \in S : d(h, g) < \infty\}$. This implies that L is a unique mapping satisfying (2.9) such that there exists a $\mu \in (0, \infty)$ satisfying $\|f(x) - L(x)\|_Y \leq \mu\varphi(x, 2x, x)$ for all $x \in X$;

(2) $d(J^n f, L) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = L(x) \tag{2.10}$$

for all $x \in X$;

(3) $d(f, L) \leq \frac{1}{1-\alpha} d(f, Jf)$, which implies the inequality $d(f, L) \leq \frac{1}{|2|-|2|\alpha}$. This implies that the inequalities (2.8) holds. The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Let θ be a positive real number and let r be a real number with $r > 1$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying*

$$\left\|f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y+z}{2}\right) - f(x) - f(z)\right\|_Y \leq \theta\left(\|x\|^r + \|y\|^r + \|z\|^r\right)$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq \frac{\theta(2 + |2|^r)\|x\|^r}{|2| - |2|^r}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.3 by taking

$$\varphi(x, y, z) = \theta\left(\|x\|^r + \|y\|^r + \|z\|^r\right)$$

for all $x, y, z \in X$. Then we can choose $\alpha = |2|^{r-1}$ and we get the desired result. □

2.2. **A Direct Method.** In this section, using a direct method, we prove the generalized Hyers-Ulam-Rassias stability of the Cauchy-Jensen additive functional equation (0.1) in non-Archimedean space.

Theorem 2.5. *Let G be an additive semigroup and that X is a non-Archimedean Banach space. Assume that $\zeta : G^3 \to [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty} |2|^n \zeta\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0 \tag{2.11}$$

for all $x, y, z \in G$. Suppose that, for any $x \in G$, the limit

$$£(x) = \lim_{n\to\infty} \max_{0\le k<n} |2|^k \zeta\left(\frac{x}{2^{k+1}}, \frac{x}{2^k}, \frac{x}{2^{k+1}}\right) \tag{2.12}$$

exists. Let $f : G \to X$ be a mapping with $f(0) = 0$ satisfying

$$\left\|f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y+z}{2}\right) - f(x) - f(z)\right\|_X \le \zeta(x, y, z), (\text{for all } x, y, z \in G) \tag{2.13}$$

Then the limit $A(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for all $x \in G$ and defines an additive mapping $A : G \to X$ such that

$$\|f(x) - A(x)\| \le £(x). \tag{2.14}$$

Moreover, if

$$\lim_{j\to\infty} \lim_{n\to\infty} \max_{j\le k<n+j} |2|^k \zeta\left(\frac{x}{2^{k+1}}, \frac{x}{2^k}, \frac{x}{2^{k+1}}\right) = 0$$

then A is the unique additive mapping satisfying (2.14).

Proof. Putting $y = 2x$ and $z = x$ in (2.13), we get

$$\|f(2x) - 2f(x)\|_Y \le \zeta(x, 2x, x) \tag{2.15}$$

for all $x \in G$. Replacing x by $\frac{x}{2^{n+1}}$ in (2.15), we obtain

$$\left\|2^{n+1} f\left(\frac{x}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right)\right\| \le |2|^n \zeta\left(\frac{x}{2^{n+1}}, \frac{x}{2^n}, \frac{x}{2^{n+1}}\right). \tag{2.16}$$

Thus, it follows from (2.11) and (2.16) that the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}_{n\ge1}$ is a Cauchy sequence. Since X is complete, it follows that $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}_{n\ge1}$ is convergent. Set

$$A(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right). \tag{2.17}$$

By induction on n, one can show that

$$\left\|2^n f\left(\frac{x}{2^n}\right) - f(x)\right\| \le \max\left\{|2|^k \zeta\left(\frac{x}{2^{k+1}}, \frac{x}{2^k}, \frac{x}{2^{k+1}}\right); 0 \le k < n\right\} \tag{2.18}$$

for all $n \geq 1$ and $x \in G$. By taking $n \to \infty$ in (2.18) and using (2.12), one obtains (2.14). By (2.11), (2.13) and (2.17), we get

$$\begin{aligned}
&\left\|A\left(\frac{x+y+z}{2}\right)+A\left(\frac{x-y+z}{2}\right)-A(x)-A(z)\right\| \\
&=\lim_{n\to\infty}|2|^n\left\|f\left(\frac{x+y+z}{2^{n+1}}\right)+f\left(\frac{x-y+z}{2^{n+1}}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{z}{2^n}\right)\right\| \\
&\leq \lim_{n\to\infty}|2|^n\zeta\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right)=0
\end{aligned}$$

for all $x, y, z \in X$. So

$$A\left(\frac{x+y+z}{2}\right)+A\left(\frac{x-y+z}{2}\right)=A(x)+A(z) \tag{2.19}$$

for all $x, y, z \in G$. Letting $y = 0$ in (2.19), we get

$$2A\left(\frac{x+z}{2}\right)=A(x)+A(z) \tag{2.20}$$

for all $x, z \in G$. Since

$$A(0)=\lim_{n\to+\infty}2^nf\left(\frac{0}{2^n}\right)=\lim_{n\to+\infty}2^nf(0)=0,$$

by letting $y = 2x$ and $z = x$ in (2.19), we get

$$A(2x)=2A(x)$$

for all $x \in G$. Replacing x by $2x$ and z by $2z$ in (2.20), we get

$$A(x+z)=A(x)+A(z)$$

for all $x, z \in G$. Hence $A : G \to X$ is additive.
To prove the uniqueness property of A, let L be another mapping satisfying (2.14). Then we have

$$\begin{aligned}
&\left\|A(x)-L(x)\right\|_X \\
&=\lim_{n\to\infty}|2|^n\left\|A\left(\frac{x}{2^n}\right)-L\left(\frac{x}{2^n}\right)\right\|_X \\
&\leq \lim_{k\to\infty}|2|^n\max\left\{\left\|A\left(\frac{x}{2^n}\right)-f\left(\frac{x}{2^n}\right)\right\|_X,\left\|f\left(\frac{x}{2^n}\right)-L\left(\frac{x}{2^n}\right)\right\|_X\right\} \\
&\leq \lim_{j\to\infty}\lim_{n\to\infty}\max_{j\leq k<n+j}|2|^k\zeta\left(\frac{x}{2^{k+1}},\frac{x}{2^k},\frac{x}{2^{k+1}}\right)=0
\end{aligned}$$

for all $x \in G$. Therefore, $A = L$. This completes the proof. □

Corollary 2.6. *Let $\xi : [0, \infty) \to [0, \infty)$ be a function satisfying*

$$\xi\left(\frac{t}{|2|}\right)\leq\xi\left(\frac{1}{|2|}\right)\xi(t),\quad \xi\left(\frac{1}{|2|}\right)<\frac{1}{|2|}$$

for all $t \geq 0$. Assume that $\kappa > 0$ and $f : G \to X$ is a mapping with $f(0) = 0$ such that

$$\left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y+z}{2}\right) - f(x) - f(z) \right\|_Y \leq \kappa\left(\xi(|x|) + \xi(|y|) + \xi(|z|)\right)$$

for all $x, y, z \in G$. Then there exists a unique additive mapping $A : G \to X$ such that

$$\|f(x) - A(x)\| \leq \frac{(2+|2|)\xi(|x|)}{|2|}$$

Proof. If we define $\zeta : G^3 \to [0, \infty)$ by $\zeta(x, y, z) := \kappa\left(\xi(|x|) + \xi(|y|) + \xi(|z|)\right)$, then we have

$$\lim_{n\to\infty} |2|^n \, \zeta\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0$$

for all $x, y, z \in G$. On the other hand, it follows that

$$\pounds(x) = \zeta\left(\frac{x}{2}, x, \frac{x}{2}\right) = \frac{(2+|2|)\xi(|x|)}{|2|}$$

exists for all $x \in G$. Also, we have

$$\lim_{j\to\infty} \lim_{n\to\infty} \max_{j\leq k<n+j} |2|^k \, \zeta\left(\frac{x}{2^{k+1}}, \frac{x}{2^k}, \frac{x}{2^{k+1}}\right) = \lim_{j\to\infty} |2|^j \, \zeta\left(\frac{x}{2^{j+1}}, \frac{x}{2^j}, \frac{x}{2^{j+1}}\right) = 0.$$

Thus, applying Theorem 2.5, we have the conclusion. This completes the proof. □

Theorem 2.7. *Let G be an additive semigroup and that X is a non-Archimedean Banach space. Assume that $\zeta : G^3 \to [0, +\infty)$ is a function such that*

$$\lim_{n\to\infty} \frac{\zeta\left(2^n x, 2^n y, 2^n z\right)}{|2|^n} = 0 \tag{2.21}$$

for all $x, y, z \in G$. Suppose that, for any $x \in G$, the limit

$$\pounds(x) = \lim_{n\to\infty} \max_{0\leq k<n} \frac{\zeta\left(2^k x, 2^{k+1} x, 2^k x\right)}{|2|^k} \tag{2.22}$$

exists and $f : G \to X$ be a mapping with $f(0) = 0$ and satisfying (2.13). Then the limit $A(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in G$ and

$$\|f(x) - A(x)\| \leq \frac{\pounds(x)}{|2|}. \tag{2.23}$$

for all $x \in G$. Moreover, if

$$\lim_{j\to\infty} \lim_{n\to\infty} \max_{j\leq k<n+j} \frac{\zeta\left(2^k x, 2^{k+1} x, 2^k x\right)}{|2|^k} = 0,$$

then A is the unique mapping satisfying (2.23).

Proof. It follows from (2.15), we get

$$\left\|f(x)-\frac{f(2x)}{2}\right\|_X \leq \frac{\zeta(x,2x,x)}{|2|} \tag{2.24}$$

for all $x \in G$. Replacing x by $2^n x$ in (2.24), we obtain

$$\left\|\frac{f(2^n x)}{2^n}-\frac{f(2^{n+1}x)}{2^{n+1}}\right\|_X \leq \frac{\zeta\left(2^n x, 2^{n+1}x, 2^n x\right)}{|2|^{n+1}}. \tag{2.25}$$

Thus it follows from (2.21) and (2.25) that the sequence $\left\{\frac{f(2^n x)}{2^n}\right\}_{n\geq 1}$ is convergent. Set

$$A(x) := \lim_{n\to\infty}\frac{f(2^n x)}{2^n}.$$

On the other hand, it follows from (2.25) that P

$$\begin{aligned}\left\|\frac{f(2^p x)}{2^p}-\frac{f(2^q x)}{2^q}\right\| = \left\|\sum_{k=p}^{q-1}\frac{f(2^{k+1}x)}{2^{k+1}}-\frac{f(2^k x)}{2^k}\right\| &\leq \max_{p\leq k<q}\left\{\left\|\frac{f(2^{k+1}x)}{2^{k+1}}-\frac{f(2^k x)}{2^k}\right\|\right\}\\ &\leq \frac{1}{|2|}\max_{p\leq k<q}\frac{\zeta\left(2^k x, 2^{k+1}x, 2^k x\right)}{|2|^k}\end{aligned}$$

for all $x \in G$ and $p, q \geq 0$ with $q > p \geq 0$. Letting $p = 0$, taking $q \to \infty$ in the last inequality and using (2.22), we obtain (2.23).

The rest of the proof is similar to the proof of Theorem 2.5. □

Corollary 2.8. *Let $\xi : [0,\infty) \to [0,\infty)$ be a function satisfying*

$$\xi\left(|2|\,t\right) \leq \xi\left(|2|\right)\xi(t), \quad \xi\left(|2|\right) < |2|$$

for all $t \geq 0$. Assume that $\kappa > 0$ and $f : G \to X$ is a mapping with $f(0) = 0$ satisfying

$$\left\|f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y+z}{2}\right)-f(x)-f(z)\right\| \leq \kappa\left(\xi(|x|)\cdot\xi(|y|)\cdot\xi(|z|)\right)$$

for all $x, y, z \in G$. Then there exists a unique additive mapping $A : G \to X$ such that

$$\|f(x)-A(x)\| \leq \kappa\xi(|x|)^3.$$

Proof. If we define $\zeta : G^3 \to [0,\infty)$ by

$$\zeta(x,y,z) := \kappa\left(\xi(|x|)\cdot\xi(|y|)\cdot\xi(|z|)\right)$$

and apply Theorem 2.7, then we get the conclusion. □

3. Random Stability of the Functional Equation (0.1)

In this section, using the fixed point and direct methods, we prove the generalized Hyers-Ulam-Rassias stability of the functional equation (0.1) in the random normed spaces.

3.1. **Direct Method.**

Theorem 3.1. *Let X be a real linear space, $(Z,\mu',\min)$ be an RN-space and $\varphi : X^3 \to Z$ be a function such that there exists $0<\alpha<\frac{1}{2}$ such that*

$$\mu'_{\varphi(\frac{x}{2},\frac{y}{2},\frac{z}{2})}(t) \geq \mu'_{\varphi(x,y,z)}\left(\frac{t}{\alpha}\right) \tag{3.1}$$

for all $x,y,z\in X$ and $t>0$ and $\lim_{n\to\infty}\mu'_{\varphi(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n})}\left(\frac{t}{2^n}\right)=1$ for all $x,y,z\in X$ and $t>0$. Let $(Y,\mu,\min)$ be a complete RN-space. If $f:X\to Y$ is a mapping with $f(0)=0$ such that

$$\mu_{f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y+z}{2}\right)-f(x)-f(z)}(t) \geq \mu'_{\varphi(x,y,z)}(t) \tag{3.2}$$

for all $x,y,z\in X$ and $t>0$. Then the limit $A(x)=\lim_{n\to\infty}2^n f\left(\frac{x}{2^n}\right)$

exists for all $x\in X$ and defines a unique additive mapping $A:X\to Y$ such that and

$$\mu_{f(x)-A(x)}(t) \geq \mu'_{\varphi(x,2x,x)}\left(\frac{(1-2\alpha)t}{\alpha}\right). \tag{3.3}$$

for all $x\in X$ and $t>0$.

Proof. Putting $y=2x$ and $z=x$ in (3.2), we see that

$$\mu_{f(2x)-2f(x)}(t) \geq \mu'_{\varphi(x,2x,x)}(t). \tag{3.4}$$

Replacing x by $\frac{x}{2}$ in (3.4), we obtain

$$\mu_{2f\left(\frac{x}{2}\right)-f(x)}(t) \geq \mu'_{\varphi\left(\frac{x}{2},x,\frac{x}{2}\right)}(t) \geq \mu'_{\varphi(x,2x,x)}\left(\frac{t}{\alpha}\right) \tag{3.5}$$

for all $x\in X$. Replacing x by $\frac{x}{2^n}$ in (3.5) and using (3.1), we obtain

$$\mu_{2^{n+1}f\left(\frac{x}{2^{n+1}}\right)-2^n f\left(\frac{x}{2^n}\right)}(t) \geq \mu'_{\varphi\left(\frac{x}{2^{n+1}},\frac{x}{2^n},\frac{x}{2^{n+1}}\right)}\left(\frac{t}{2^n}\right) \geq \mu'_{\varphi(x,2x,x)}\left(\frac{t}{2^n\alpha^{n+1}}\right)$$

and so

$$\begin{aligned}
\mu_{2^n f\left(\frac{x}{2^n}\right)-f(x)}\left(\sum_{k=0}^{n-1}2^k\alpha^{k+1}t\right) &= \mu_{\sum_{k=0}^{n-1}2^{k+1}f\left(\frac{x}{2^{k+1}}\right)-2^k f\left(\frac{x}{2^k}\right)}\left(\sum_{k=0}^{n-1}2^k\alpha^{k+1}t\right)\\
&\geq T_{k=0}^{n-1}\left(\mu_{2^{k+1}f\left(\frac{x}{2^{k+1}}\right)-2^k f\left(\frac{x}{2^k}\right)}(2^k\alpha^{k+1}t)\right)\\
&\geq T_{k=0}^{n-1}\left(\mu'_{\varphi(x,2x,x)}(t)\right)\\
&= \mu'_{\varphi(x,2x,x)}(t).
\end{aligned}$$

This implies that

$$\mu_{2^n f\left(\frac{x}{2^n}\right)-f(x)}(t) \geq \mu'_{\varphi(x,2x,x)}\left(\frac{t}{\sum_{k=0}^{n-1}2^k\alpha^{k+1}}\right). \tag{3.6}$$

Replacing x by $\frac{x}{2^p}$ in (3.6), we obtain

$$\mu_{2^{n+p}f\left(\frac{x}{2^{n+p}}\right)-2^p f\left(\frac{x}{2^p}\right)}(t) \geq \mu'_{\varphi(x,2x,x)}\left(\frac{t}{\sum_{k=p}^{n+p-1} 2^k \alpha^{k+1}}\right). \tag{3.7}$$

Since $\lim_{p,n\to\infty} \mu'_{\varphi(x,2x,x)}\left(\frac{t}{\sum_{k=p}^{n+p-1} 2^k \alpha^{k+1}}\right) = 1$, it follows that $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}_{n=1}^{\infty}$ is a Cauchy sequence in a complete RN-space $(Y, \mu, \min)$ and so there exists a point $A(x) \in Y$ such that $\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$. Fix $x \in X$ and put $p = 0$ in (3.7) and so, for any $\epsilon > 0$,

$$\begin{aligned} \mu_{A(x)-f(x)}(t+\epsilon) &\geq T\left(\mu_{A(x)-2^n f\left(\frac{x}{2^n}\right)}(\epsilon), \mu_{2^n f\left(\frac{x}{2^n}\right)-f(x)}(t)\right) \\ &\geq T\left(\mu_{A(x)-2^n f\left(\frac{x}{2^n}\right)}(\epsilon), \mu'_{\varphi(x,2x,x)}\left(\frac{t}{\sum_{k=0}^{n-1} 2^k \alpha^{k+1}}\right)\right). \end{aligned} \tag{3.8}$$

Taking $n \to \infty$ in (3.8), we get

$$\mu_{A(x)-f(x)}(t+\epsilon) \geq \mu'_{\varphi(x,2x,x)}\left(\frac{(1-2\alpha)t}{\alpha}\right). \tag{3.9}$$

Since ϵ is arbitrary, by taking $\epsilon \to 0$ in (3.9), we get

$$\mu_{A(x)-f(x)}(t) \geq \mu'_{\varphi(x,2x,x)}\left(\frac{(1-2\alpha)t}{\alpha}\right).$$

Replacing x, y and z by $\frac{x}{2^n}$, $\frac{y}{2^n}$ and $\frac{z}{2^n}$ in (3.2), respectively, we get

$$\mu_{2^n f\left(\frac{x+y+z}{2^{n+1}}\right)+2^n f\left(\frac{x-y+z}{2^{n+1}}\right)-2^n f\left(\frac{x}{2^n}\right)-2^n f\left(\frac{z}{2^n}\right)}(t) \geq \mu'_{\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right)}\left(\frac{t}{2^n}\right)$$

for all $x, y, z \in X$ and $t > 0$. Since $\lim_{n\to\infty} \mu'_{\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right)}\left(\frac{t}{2^n}\right) = 1$, we conclude that A satisfies (0.1). On the other hand

$$2A\left(\frac{x}{2}\right) - A(x) = \lim_{n\to\infty} 2^{n+1} f\left(\frac{x}{2^{n+1}}\right) - \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = 0.$$

This implies that $A : X \to Y$ is an additive mapping. To prove the uniqueness of the additive mapping A, assume that there exists another additive mapping $L : X \to Y$ which satisfies (3.3). Then we have

$$\begin{aligned} \mu_{A(x)-L(x)}(t) &= \lim_{n\to\infty} \mu_{2^n A\left(\frac{x}{2^n}\right)-2^n L\left(\frac{x}{2^n}\right)}(t) \\ &\geq \lim_{n\to\infty} \min\left\{\mu_{2^n A\left(\frac{x}{2^n}\right)-2^n f\left(\frac{x}{2^n}\right)}\left(\frac{t}{2}\right), \mu_{2^n f\left(\frac{x}{2^n}\right)-2^n L\left(\frac{x}{2^n}\right)}\left(\frac{t}{2}\right)\right\} \\ &\geq \lim_{n\to\infty} \mu'_{\varphi\left(\frac{x}{2^n}, \frac{2x}{2^n}, \frac{x}{2^n}\right)}\left(\frac{(1-2\alpha)t}{2^n}\right) \geq \lim_{n\to\infty} \mu'_{\varphi(x,2x,x)}\left(\frac{(1-2\alpha)t}{2^n \alpha^n}\right). \end{aligned}$$

Since $\lim_{n\to\infty} \mu'_{\varphi(x,2x,x)}\left(\frac{(1-2\alpha)t}{2^n\alpha^n}\right) = 1$. Therefore, it follows that $\mu_{A(x)-L(x)}(t) = 1$ for all $t > 0$ and so $A(x) = L(x)$. This completes the proof. □

Corollary 3.2. *Let X be a real normed linear space, $(Z,\mu',\min)$ be an RN-space and $(Y,\mu,\min)$ be a complete RN-space. Let r be a positive real number with $r>1$, $z_0\in Z$ and $f:X\to Y$ be a mapping with $f(0)=0$ satisfying*

$$\mu_{f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y+z}{2}\right)-f(x)-f(z)}(t)\geq \mu'_{(\|x\|^r+\|y\|^r+\|z\|^r)z_0}(t) \tag{3.10}$$

for all $x,y\in X$ and $t>0$. Then the limit $A(x)=\lim_{n\to\infty}2^n f\left(\frac{x}{2^n}\right)$ exists for all $x\in X$ and defines a unique additive mapping $A:X\to Y$ such that and

$$\mu_{f(x)-A(x)}(t)\geq \mu'_{\|x\|^p z_0}\left(\frac{(2^r-2)t}{2^r+2}\right)$$

for all $x\in X$ and $t>0$.

Proof. Let $\alpha=2^{-r}$ and $\varphi:X^3\to Z$ be a mapping defined by $\varphi(x,y,z)=(\|x\|^r+\|y\|^r+\|z\|^r)z_0$. Then, from Theorem 3.1, the conclusion follows. □

Theorem 3.3. *Let X be a real linear space, $(Z,\mu',\min)$ be an RN-space and $\varphi:X^3\to Z$ be a function such that there exists $0<\alpha<2$ such that $\mu'_{\varphi(2x,2y,2z)}(t)\geq\mu'_{\alpha\varphi(x,y,z)}(t)$ for all $x\in X$ and $t>0$ and*

$$\lim_{n\to\infty}\mu'_{\varphi(2^n x,2^n y,2^n z)}(2^n x)=1$$

for all $x,y,z\in X$ and $t>0$. Let $(Y,\mu,\min)$ be a complete RN-space. If $f:X\to Y$ be a mapping with $f(0)=0$ satisfying (3.2). Then the limit $A(x)=\lim_{n\to\infty}\frac{f(2^n x)}{2^n}$ exists for all $x\in X$ and defines a unique additive mapping $A:X\to Y$ such that and

$$\mu_{f(x)-A(x)}(t)\geq\mu'_{\varphi(x,2x,x)}((2-\alpha)t). \tag{3.11}$$

for all $x\in X$ and $t>0$.

Proof. It follows from (3.4) that

$$\mu_{\frac{f(2x)}{2}-f(x)}(t)\geq\mu'_{\varphi(x,2x,x)}(2t). \tag{3.12}$$

Replacing x by $2^n x$ in (3.12), we obtain that

$$\mu_{\frac{f(2^{n+1}x)}{2^{n+1}}-\frac{f(2^n x)}{2^n}}(t)\geq\mu'_{\varphi(2^n x,2^{n+1}x,2^n x)}(2^{n+1}t)\geq\mu_{\varphi(x,2x,x)}\left(\frac{2^{n+1}t}{\alpha^n}\right).$$

The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.4. *Let X be a real normed linear space, $(Z,\mu',\min)$ be an RN-space and $(Y,\mu,\min)$ be a complete RN-space. Let r be a positive real number with $0<r<1$, $z_0\in Z$ and $f:X\to Y$ be a mapping with $f(0)=0$ satisfying (3.10). Then the limit $A(x)=\lim_{n\to\infty}\frac{f(2^n x)}{2^n}$ exists for all $x\in X$ and defines a unique additive mapping $A:X\to Y$ such that and*

$$\mu_{f(x)-A(x)}(t)\geq\mu'_{\|x\|^p z_0}\left(\frac{(2-2^r)t}{2^r+2}\right)$$

for all $x\in X$ and $t>0$.

Proof. Let $\alpha = 2^r$ and $\varphi : X^3 \to Z$ be a mapping defined by $\varphi(x,y,z) = (\|x\|^r + \|y\|^r + \|z\|^r)z_0$. Then, from Theorem 3.3, the conclusion follows. □

3.2. **Fixed Point Method.**

Theorem 3.5. *Let X be a linear space, (Y,μ,T_M) be a complete RN-space and Φ be a mapping from X^3 to D^+ ($\Phi(x,y,z)$ is denoted by $\Phi_{x,y.z}$) such that there exists $0<\alpha<\frac{1}{2}$ such that*

$$\Phi_{2x,2y,2z}(t) \leq \Phi_{x,y,z}(\alpha t) \tag{3.13}$$

for all $x,y,z \in X$ and $t>0$. Let $f : X \to Y$ be a mapping with $f(0)=0$ satisfying

$$\mu_{f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y+z}{2}\right)-f(x)-f(z)}(t) \geq \Phi_{x,y,z}(t) \tag{3.14}$$

for all $x,y,z \in X$ and $t>0$. Then, for all $x \in X$, $A(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists and $A : X \to Y$ is a unique additive mapping such that

$$\mu_{f(x)-A(x)}(t) \geq \Phi_{x,2x,x}\left(\frac{(1-2\alpha)t}{\alpha}\right) \tag{3.15}$$

for all $x \in X$ and $t>0$.

Proof. Putting $y=2x$ and $z=x$ in (3.14), we have

$$\mu_{2f(\frac{x}{2})-f(x)}(t) \geq \Phi_{\frac{x}{2},x,\frac{x}{2}}(t) \geq \Phi_{x,2x,x}\left(\frac{t}{\alpha}\right) \tag{3.16}$$

for all $x \in X$ and $t>0$. Consider the set $S := \{g : X \to Y\}$ and the generalized metric d in S defined by

$$d(f,g) = \inf_{u\in(0,\infty)} \left\{\mu_{g(x)-h(x)}(ut) \geq \Phi_{x,2x,x}(t),\ \forall x \in X,\ t>0\right\}, \tag{3.17}$$

where $\inf \emptyset = +\infty$. It is easy to show that (S,d) is complete (see [33, Lemma 2.1]). Now, we consider a linear mapping $J : (S,d) \to (S,d)$ such that

$$Jh(x) := 2h\left(\frac{x}{2}\right) \tag{3.18}$$

for all $x \in X$. First, we prove that J is a strictly contractive mapping with the Lipschitz constant 2α. In fact, let $g,h \in S$ be such that $d(g,h)<\epsilon$. Then we have $\mu_{g(x)-h(x)}(\epsilon t) \geq \Phi_{x,2x,x}(t)$ for all $x \in X$ and $t>0$ and so

$$\begin{aligned}\mu_{Jg(x)-Jh(x)}(2\alpha\epsilon t) = \mu_{2g(\frac{x}{2})-2h(\frac{x}{2})}(2\alpha\epsilon t) &= \mu_{g(\frac{x}{2})-h(\frac{x}{2})}(\alpha\epsilon t)\\ &\geq \Phi_{\frac{x}{2},x,\frac{x}{2}}(\alpha t)\\ &\geq \Phi_{x,2x,x}(t)\end{aligned}$$

for all $x \in X$ and $t>0$. Thus $d(g,h)<\epsilon$ implies that $d(Jg,Jh)<2\alpha\epsilon$. This means that $d(Jg,Jh) \leq 2\alpha d(g,h)$ for all $g,h \in S$. It follows from (3.16) that

$$d(f,Jf) \leq \alpha.$$

By Theorem 1.11, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, that is,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{3.19}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set $\Omega = \{h \in S : d(g,h) < \infty\}$. This implies that A is a unique mapping satisfying (3.19) such that there exists $u \in (0,\infty)$ satisfying $\mu_{f(x)-A(x)}(ut) \geq \Phi_{x,2x,x}(t)$ for all $x \in X$ and $t > 0$.

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$.

(3) $d(f,A) \leq \frac{d(f,Jf)}{1-2\alpha}$ with $f \in \Omega$, which implies the inequality $d(f,A) \leq \frac{\alpha}{1-2\alpha}$ and so

$$\mu_{f(x)-A(x)}\left(\frac{\alpha t}{1-2\alpha}\right) \geq \Phi_{x,2x,x}(t)$$

for all $x \in X$ and $t > 0$. This implies that the inequality (3.15) holds. On the other hand

$$\mu_{2^n f\left(\frac{x+y+z}{2^{n+1}}\right)+2^n f\left(\frac{x-y+z}{2^{n+1}}\right)-2^n f\left(\frac{x}{2^n}\right)-2^n f\left(\frac{z}{2^n}\right)}(t) \geq \Phi_{\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}}\left(\frac{t}{2^n}\right)$$

for all $x,y,z \in X$, $t > 0$ and $n \geq 1$. By (3.13), we know that

$$\Phi_{\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}}\left(\frac{t}{2^n}\right) \geq \Phi_{x,y,z}\left(\frac{t}{(2\alpha)^n}\right).$$

Since $\lim_{n\to\infty} \Phi_{x,y,z}\left(\frac{t}{(2\alpha)^n}\right) = 1$ for all $x,y,z \in X$ and $t > 0$, we have

$$\mu_{A\left(\frac{x+y+z}{2}\right)+A\left(\frac{x-y+z}{2}\right)-A(x)-A(z)}(t) = 1$$

for all $x,y,z \in X$ and $t > 0$. Thus the mapping $A : X \to Y$ satisfying (0.1). Furthermore

$$\begin{aligned} A(2x) - 2A(x) &= \lim_{n\to\infty} 2^n f\left(\frac{x}{2^{n-1}}\right) - 2\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \\ &= 2\left[\lim_{n\to\infty} 2^{n-1} f\left(\frac{x}{2^{n-1}}\right) - \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)\right] \\ &= 0. \end{aligned}$$

This completes the proof. □

Corollary 3.6. *Let X be a real normed space, $\theta \geq 0$ and r be a real number with $r > 1$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying*

$$\mu_{f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y+z}{2}\right)-f(x)-f(z)}(t) \geq \frac{t}{t+\theta\left(\|x\|^r+\|y\|^r+\|z\|^r\right)} \tag{3.20}$$

for all $x, y, z \in X$ *and* $t > 0$. *Then* $A(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ *exists for all* $x \in X$ *and* $A : X \to Y$ *is a unique additive mapping such that*

$$\mu_{f(x)-A(x)}(t) \geq \frac{(2^r - 2)t}{(2^r - 2)t + (2^r + 2)\theta\|x\|^r}$$

for all $x \in X$ *and* $t > 0$.

Proof. The proof follows from Theorem 3.5 if we take

$$\Phi_{x,y,z}(t) = \frac{t}{t + \theta\big(\|x\|^r + \|y\|^r + \|z\|^r\big)}$$

for all $x, y, z \in X$ and $t > 0$. In fact, if we choose $\alpha = 2^{-r}$, then we get the desired result. □

Theorem 3.7. *Let* X *be a linear space,* (Y, μ, T_M) *be a complete RN-space and* Φ *be a mapping from* X^3 *to* D^+ *(*$\Phi(x, y, z)$ *is denoted by* $\Phi_{x,y,z}$*) such that for some* $0 < \alpha < 2$

$$\Phi_{\frac{x}{2},\frac{y}{2},\frac{z}{2}}(t) \leq \Phi_{x,y,z}(\alpha t)$$

for all $x, y, z \in X$ *and* $t > 0$. *Let* $f : X \to Y$ *be a mapping with* $f(0) = 0$ *satisfying (3.14). Then the limit* $A(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ *exists for all* $x \in X$ *and* $A : X \to Y$ *is a unique additive mapping such that*

$$\mu_{f(x)-A(x)}(t) \geq \Phi_{x,2x,x}((2 - \alpha)t) \tag{3.21}$$

for all $x \in X$ *and* $t > 0$.

Proof. Putting $y = 2x$ and $z = x$ in (3.14), we have

$$\mu_{\frac{f(2x)}{2}-f(x)}(t) \geq \Phi_{x,2x,x}(2t) \tag{3.22}$$

for all $x \in X$ and $t > 0$. Let (S, d) be the generalized metric space defined in the proof of Theorem 3.1. Now, we consider a linear mapping $J : (S, d) \to (S, d)$ such that

$$Jh(x) := \frac{1}{2}h(2x) \tag{3.23}$$

for all $x \in X$. It follows from (3.22) that $d(f, Jf) \leq \frac{1}{2}$. By Theorem 1.11, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, that is,

$$A(2x) = 2A(x) \tag{3.24}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set $\Omega = \{h \in S : d(g, h) < \infty\}$. This implies that A is a unique mapping satisfying (3.24) such that there exists $u \in (0, \infty)$ satisfying $\mu_{f(x)-A(x)}(ut) \geq \Phi_{x,2x,x}(t)$ for all $x \in X$ and $t > 0$.

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = A(x)$$

for all $x \in X$.
(3) $d(f,A) \le \frac{d(f,Jf)}{1-\frac{\alpha}{2}}$ with $f \in \Omega$, which implies the inequality

$$\mu_{f(x)-A(x)}\left(\frac{t}{2-\alpha}\right) \ge \Phi_{x,2x,x}(t)$$

for all $x \in X$ and $t > 0$. This implies that the inequality (3.21) holds. The rest of the proof is similar to the proof of Theorem 3.5. □

Corollary 3.8. *Let X be a real normed space, $\theta \ge 0$ and r be a real number with $0 < r < 1$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying (3.20). Then the limit $A(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in X$ and $A : X \to Y$ is a unique additive mapping such that*

$$\mu_{f(x)-A(x)}(t) \ge \frac{(2-2^r)t}{(2-2^r)t + (2^r+2)\theta\|x\|^r}$$

for all $x \in X$ and $t > 0$.

Proof. The proof follows from Theorem 3.7 if we take

$$\Phi_{x,y}(t) = \frac{t}{t + \theta(\|x\|^r + \|y\|^r + \|z\|^r)}$$

for all $x, y, z \in X$ and $t > 0$. In fact, if we choose $\alpha = 2^r$, then we get the desired result. □

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