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A HYBRID METHOD FOR SINGULARLY PERTURBED DELAY BOUNDARY VALUE PROBLEMS EXHIBITING A RIGHT BOUNDARY LAYER

F. Z. GENG* AND S. P. QIAN

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Abstract. The aim of this paper is to present a numerical method for singularly perturbed convection-diffusion problems with a delay. The method is a combination of the asymptotic expansion technique and the reproducing kernel method (RKM). First an asymptotic expansion for the solution of the given singularly perturbed delayed boundary value problem is constructed. Then the reduced regular delayed differential equation is solved analytically using the RKM. An error estimate and two numerical examples are provided to illustrate the effectiveness of the present method. The results of numerical examples show that the present method is accurate and efficient.
Keywords: Reproducing kernel method; singularly perturbed problems; delay boundary value problems.
MSC2010: Primary: 65L11; Secondary: 65L03.

1. Introduction

Singularly perturbed delay differential equations have attracted much attention. The numerical treatment of such problems present some major computational difficulties due to the presence of boundary and interior layers and the delayed terms. Therefore, it is important to develop suitable numerical methods to solve such problems.

Recently, there has been growing interest in numerical methods for solving singularly perturbed delay differential equations and a few special purpose methods have been developed by some authors [3, 4, 7, 15–17, 26–29]. Amiraliyev, Erdogan and Amiraliyeva [3,4,7] proposed exponentially fitted methods for singularly perturbed delay initial value problems. Kadalbajoo, Sharma and Gupta [15–17] presented some methods for solving singularly perturbed delay

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boundary value problems. Rai and Sharma [27–29] developed numerical methods for singularly perturbed delay differential turning point problems. Patidar and Sharma [26] developed a uniformly convergent non-standard finite difference methods for singularly perturbed differential-difference equations with delay and advance. Mohapatra and Natesan [24] proposed a numerical method for a class of singularly perturbed differential-difference equations with small delay and shift terms, based on the upwind finite difference operator on an adaptive grid. Subburayan and Ramanujam [31] presented an initial value technique for singularly perturbed convection-diffusion problems with a negative shift.

Reproducing kernel theory has been applied to many fields [1–17,19–32]. Recently, the RKM has been proposed for some differential and integral equations based on reproducing kernel theory [1, 2, 8–14, 19–23, 25, 30, 32]. However, it is very difficult to extend the application of the method to singularly perturbed delay differential equations.

In this paper, based on the asymptotic expansion technique and the RKM, an effective numerical method shall be presented for solving a class of singularly perturbed delay boundary value problems.

Motivated by the work of [18, 31], we consider the following singularly perturbed problems:

(1.1)
$$\begin{cases} -\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) + c(x)u(x-1) = f(x), \ \ x \in \Omega^* = (0,1) \cup (1,2), \\ u(x) = \Phi(x), x \in [-1,0], \ u(2) = l, \end{cases}$$

where $0 < \varepsilon \ll 1$, $a(x) \geq \alpha_1 > \alpha > 0$, $b(x) \geq \beta_0 > 0, \gamma_0 \leq c(x) < 0$, $2\alpha+5\beta_0+5\gamma_0 \geq \eta > 0, \eta \ \ \alpha(\alpha_1-\alpha) > -2\gamma_0$, $a(x), b(x)$, $\Phi(x)$ and $f(x)$ are assumed to be sufficiently smooth on $\overline{\Omega} = [0, 2]$, and such that (1.1) has a unique solution.

From [18], (1.1) exhibits a strong boundary layer at $x = 2$.

The rest of the paper is organized as follows. In the next section, an asymptotic expansion for the solution of the problem (1.1) is constructed. The RKM for reduced regular delayed differential equation is introduced in Section 3. The method for solving terminal value problem (2.2) is presented in Section 4. Error analysis is derived in Section 5. The numerical examples are given in Section 6. Section 7 ends this paper with a brief conclusion.

2. **An asymptotic expansion**

In this section, an asymptotic expansion approximation to the solution of problem (1.1) is constructed.

Let $u_0(x)$ be the solution of the reduced problem of given by

$$(2.1) \quad \begin{cases} a(x)u_0'(x) + b(x)u_0(x) + c(x)u_0(x-1) = f(x), \ \ x \in (0,2], \\ u_0(x) = \Phi(x), \ x \in [-1,0]. \end{cases}$$

Denote by $v_r(x) = e^{-\int_x^2 \frac{a(s)}{\varepsilon}ds}$ the solution (1.1) of the terminal value problem (TVP)

$$\begin{cases} \varepsilon v'(x) - a(x)v(x) = 0, \;\; x \in [0,2), \\ v(2) = 1. \end{cases} \tag{2.2}$$

In [12], an asymptotic expansion approximation was given by

$$u_{as}(x) = \begin{cases} u_0(x) + k_1 x, \;\; x \in [0,1], \\ u_0(x) + k_2 v_r(x) + k_3, \;\; x \in [1,2], \end{cases} \tag{2.3}$$

where

$$k_2 = \frac{l - u_0(2)}{1 + (\frac{a(1)}{\varepsilon} - 1)v_r(1)}, \;\; k_1 = k_2 v_r(1)\frac{a(1)}{\varepsilon}, \;\; k_3 = k_2 v_r(1)(\frac{a(1)}{\varepsilon} - 1).$$

Theorem 2.1. *Let $u(x)$ be the solution of problem* (1.1). *The asymptotic expansion approximation $u_{as}(x)$ satisfies*

$$|u(x) - u_{as}(x)| \le c\,\varepsilon, \;\; x \in [0,2],$$

where c is a positive constant.

3. RKM for reduced problem (2.1)

Let

$$Lu_0(x) = \begin{cases} a(x)u_0'(x) + b(x)u_0(x), \;\; x \in (0,1], \\ a(x)u_0'(x) + b(x)u_0(x) + c(x)u_0(x-1), \;\; x \in (1,2]. \end{cases}$$

Problem (2.1) is equivalent to

$$\begin{cases} Lu_0(x) = g(x), \\ u_0(0) = \Phi(0). \end{cases} \tag{3.1}$$

where

$$g(x) = \begin{cases} f(x) - c(x)\Phi(x-1), \;\; x \in (0,1], \\ f(x), \;\; x \in (1,2]. \end{cases}$$

Introducing a new unknown function

$$w(x) = u_0(x) - \Phi(x), \tag{3.2}$$

problem (3.1) with inhomogeneous boundary conditions can be equivalently reduced to the problem of finding a function $w(x)$ satisfying

$$\begin{cases} Lw(x) = F(x), \\ w(0) = 0. \end{cases} \tag{3.3}$$

where $F(x) = g(x) - L\Phi(x)$.

To solve (3.3), we first define a reproducing kernel space $W^3[0,2]$.

Definition 3.1. $W^3[0,2] = \{u(x) \mid u''(x)$ is absolutely continuous, $u^{(3)}(x) \in L^2[0,2], u(0) = 0\}$. The inner product and norm in $W^3[0,2]$ are given, respectively, by

$$(u(y), v(y))_3 = u(0)v(0) + u'(0)v'(0) + u''(0)v''(0) + \int_0^2 u^{(3)}v^{(3)}dy$$

and

$$\| u \|_3 = \sqrt{(u,u)_3},\ u, v \in W^3[0,2].$$

Theorem 3.2. $W^3[0,2]$ *is a reproducing kernel space and its reproducing kernel is*

$$k(x,y) = \begin{cases} k_1(x,y), & y \le x, \\ k_1(y,x), & y > x. \end{cases} \tag{3.4}$$

where $k_1(x,y) = \frac{1}{120}y\left(10x^2(y+3)y - 5x\left(y^3 - 24\right) + y^4\right)$.

For the proof, one may refer to [6].

Definition 3.3. $W^1[0,2] = \{u(x) \mid u(x)$ is an absolutely continuous real value function, $u'(x) \in L^2[0,2]\}$. The inner product and norm in $W^1[0,2]$ are given, respectively, by

$$(u(y), v(y))_1 = u(0)v(0) + \int_0^2 u'v'dy$$

and

$$\| u \|_1 = \sqrt{(u,u)_1},\ u, v \in W^1[0,2].$$

Theorem 3.4. $W^1[0,2]$ *is a reproducing kernel space and its reproducing kernel is*

$$\overline{k}(x,y) = \begin{cases} 1+y, & y \le x, \\ 1+x, & y > x, \end{cases} \tag{3.5}$$

For the proof, refer to [6].

In (3.3), obviously, $L : W^3[0,2] \to W^1[0,2]$ is a bounded linear operator. Put $\varphi_i(x) = \overline{k}(x, x_i)$ and $\psi_i(x) = L^*\varphi_i(x)$ where L^* is the adjoint operator of L . The orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ of $W^3[0,2]$ can be derived from the Gram-Schmidt orthogonalization process applied to $\{\psi_i(x)\}_{i=1}^{\infty}$,

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), (\beta_{ii} > 0, i = 1, 2, ...).$$

Theorem 3.5. *If* $\{x_i\}_{i=1}^{\infty}$ *is dense in* $[0,2]$*, then* $\{\psi_i(x)\}_{i=1}^{\infty}$ *is the complete function system of* $W^3[0,2]$.

Proof. Note here that

$$\begin{aligned} \psi_i(x) &= (L^*\varphi_i)(x) = ((L^*\varphi_i)(y), k(x,y)) \\ &= (\varphi_i(y), L_y k(x,y)) = L_y k(x,y)|_{y=x_i}. \end{aligned}$$

Hence, $\psi_i(x) \in W^3[0,2]$.
For each fixed $u(x) \in W^3[0,2]$, let $(u(x), \psi_i(x)) = 0, (i = 1, 2, ...)$, which means that

$$(u(x), (L^*\varphi_i)(x)) = (Lu(\cdot), \varphi_i(\cdot)) = (Lu)(x_i) = 0.$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense in $[0,2]$, $(Lu)(x) = 0$. It follows that $u \equiv 0$ from the existence of L^{-1}. □

Theorem 3.6. *If $\{x_i\}_{i=1}^{\infty}$ is dense in $[0,2]$, then the solution of (3.3) is*

$$w(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}F(x_k)\overline{\psi}_i(x). \tag{3.6}$$

Proof. From Theorem 3.4, it follows that $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ is a complete orthonormal basis in $W^3[0,2]$.
Note that $(w(x), \varphi_i(x)) = w(x_i)$ for each $w(x) \in W^1[0,2]$; hence we have

$$\begin{aligned} w(x) &= \sum_{i=1}^{\infty}(w(x), \overline{\psi}_i(x))\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}(w(x), L^*\varphi_k(x))\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}(Lw(x), \varphi_k(x))\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}(F(x), \varphi_k(x))\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}F(x_k)\overline{\psi}_i(x). \end{aligned} \tag{3.7}$$

So, the proof is complete. □

The approximate solution $w_N(x)$ can be obtained by taking finitely many terms in the series representation of $w(x)$ and

$$w_N(s) = \sum_{i=1}^{N}\sum_{k=1}^{i}\beta_{ik}F(x_k)\overline{\psi}_i(x). \tag{3.8}$$

Combining (3.8) and (3.2), leads to the approximate solution of (3.1)

$$u_{0,N}(x) = w_N(x) + \Phi(x) = \sum_{i=1}^{N}\sum_{k=1}^{i}\beta_{ik}F(x_k)\overline{\psi}_i(x) + \Phi(x). \tag{3.9}$$

Furthermore, the approximation to the asymptotic expansion $u_{as}(x)$ is immediately obtained

$$U_{as}^N(x) = \begin{cases} u_{0,N}(x) + k_1 x, \ x \in [0,1], \\ u_{0,N}(x) + k_2 v_r(x) + k_3, \ x \in [1,2], \end{cases} \tag{3.10}$$

4. Method for solving TVP(2.2)

In (3.10), a closed form expression is given for the boundary layer component. However, in many cases of variable coefficients, this closed form representation is not available. Also, it is difficult to obtain its numerical solution. Here we develop a piecewise RKM for finding the approximate solution of TVP(2.2).

Putting $s = 2 - x$ and $y(s) = v(x(s))$, TVP(2.2) becomes

$$\begin{cases} \varepsilon y'(s) + \overline{a}(s)y(s) = 0, \ \ s \in (0, 2], \\ y(0) = 1, \end{cases} \tag{4.1}$$

where $\overline{a}(s) = a(2 - s)$.

We will solve initial value problem (4.1) using the RKM in a piecewise fashion.

Divide $[0, 2]$ into M subintervals $[s_j, s_{j+1}], \ j = 0, 1, \ldots, M - 1$, with $s_0 = 0$ and $s_M = 2$. Denote by h_i the length of the ith subinterval, i.e. $h_i = |s_{i+1} - s_i|$. Then apply the RKM presented in the above section to (4.1) on every subinterval $[s_i, s_{i+1}]$.

Using the RKM(take N_1 equidistant nodes) to solve (4.1) on interval $[s_0, s_1]$, one obtains the approximate solution $y_{1,N_1}(s)$ of (4.1) on $[s_0, s_1]$. Then an initial value condition of (4.1) on $[s_1, s_2]$ is determined approximately by

$$y(s_1) = y_{1,N_1}(s_1).$$

So we can obtain the approximate solution $y_{2,N_1}(s)$ of (4.1) on the second interval $[s_1, s_2]$ by the RKM. In the same way, the approximate solutions $y_{i,N_1}(s)$ of (4.1) on the intervals $[s_{i-1}, s_i], \ i = 3, 4, \ldots, M$.

After obtaining the approximate solutions on all subintervals, these solutions are combined to obtain the approximate solution $y^M_{N_1}(s)$ of (4.1) on the entire interval $[0, 2]$. Obviously, $y^M_{N_1}(s)$ is continuous on $[0, 2]$. Naturally, the continuous approximate solution of (2.2) is obtained by $v^{M,N_1}_r(x) = y^M_{N_1}(2-x)$.

Therefore, the approximate solution to (1.1) can be further obtained by

$$U^{N,M,N_1}_{as}(x) = \begin{cases} u_{0,N}(x) + k_1 x, \ x \in [0, 1], \\ u_{0,N}(x) + k_2 v^{M,N_1}_r(x) + k_3, \ x \in [1, 2]. \end{cases} \tag{4.2}$$

5. Error analysis

From [3, 4, 7, 15–17, 24, 26–29, 31], we have the following Lemma.

Lemma 5.1. *If $w(x)$ is the solution of* (3.3)*, then there exists a positive C such that*

$$\|w(x)\|_\infty = \max_{x\in[0,2]} |w(x)| \leq C \parallel Lw(x) \parallel_\infty = C \parallel F(x) \parallel_\infty .$$

Lemma 5.2. *If $0 = x_1 < x_2 < \cdots < x_N = 2$, and if $a(x), b(x), \Phi(x)$ and $f(x)$ are sufficiently smooth, then the approximate solution $w_N(x)$ of (3.3) satisfies*

$$\| Lw_N(x) - F(x) \|_\infty = \max_{x\in[0,2]} |Lw_N - F| \leq d_1 h^2,$$

where d_1 is a positive constant and $h = \max_{1\leq i\leq N-1} |x_{i+1} - x_i|$.

Proof. For the details of the proof, one may refer to [8]. □

Theorem 5.3. *The approximate solution $w_N(s)$ of (3.3) satisfies*

$$\|w_N(x) - w(x)\|_\infty \leq d_2 \; h^2,$$

where d_2 is a positive constant.

Proof. Note that $w_N(x) - w(x)$ is the solution of

$$\begin{cases} LV = Lw_N - F, & x \in (0, 2], \\ V(0) = 0. \end{cases}$$

From Lemma 5.1, it follows that

$$\|w_N(x) - w(x)\|_\infty \leq C \; \| \; Lw_N(x) - Lw(x) \; \|_\infty \; .$$

By using Lemma 5.2, we get

$$\|w_N(x) - w(x)\|_\infty \leq C \; \| \; Lw_N(x) - Lw(x) \; \|_\infty \leq C \; d_1 \; h^2 = d_2 \; h^2,$$

where d_2 is a positive constant. □

Theorem 5.4. *The approximate solution $u_{0,N}(s)$ of (3.1) satisfies*

$$\|u_{0,N}(x) - u_0(x)\|_\infty \leq d_2 \; h^2.$$

Proof. In view of

$$u_{0,N}(x) = w_N(x) + \Phi(x), \;\; u_0(x) = w(x) + \Phi(x),$$

by Lemma 5.1, one sees that

$$\|u_{0,N}(x) - u_0(x)\|_\infty \leq d_2 \; h^2.$$

□

From Lemma 5.1 and 5.2, the following theorem can be obtained.

Theorem 5.5. *The errors between the approximate solution $U_{as}^N(x)$ and the exact solution $u(x)$ of (1.1) satisfies*

$$\|U_{as}^N(x) - u(x)\|_\infty \leq d_3 \; h^2 + c \; \varepsilon,$$

where d_3 is a positive constant, c and ε are shown in Theorem 2.1.

Proof. From (2.3) and (3.10), there exists a positive constant α such that

$$\|U_{as}^N(x) - u_{as}(x)\|_\infty \le \alpha \, \|u_{0,N}(x) - u_0(x)\|_\infty.$$

An application of Lemma 5.2 then yields that

$$\|U_{as}^N(x) - u_{as}(x)\|_\infty \le d_3 \; h^2, \tag{5.1}$$

where d_3 is a positive constant. From Theorem 2.1, we have

$$\|u_{as}(x) - u(x)\|_\infty \le c\,\varepsilon. \tag{5.2}$$

Note that

$$\begin{aligned} \|U_{as}^N(x) - u(x)\|_\infty \quad &= \|U_{as}^N(x) - u_{as}(x) + u_{as}(x) - u(x)\|_\infty \\ &\le \|U_{as}^N(x) - u_{as}(x)\|_\infty + \|u_{as}(x) - u(x)\|_\infty. \end{aligned} \tag{5.3}$$

Combining (5.1), (5.2) and (5.3), leads to

$$\|U_{as}^N(x) - u(x)\|_\infty \le d_3 \; h^2 + c \; \varepsilon.$$

□

6. **Numerical examples**

In this section, two examples are given to illustrate the numerical method discussed in this paper. All computations are performed by using Mathematica 7.0. Since the exact solution for the second problem is not available, the maximum absolute errors E_ε^N are evaluated by

$$E_\varepsilon^N = \max_{x\in[0,2]} |U_{as}^N(x) - U_{as}^{2N}(x)|.$$

Example 6.1. Consider the following singularly perturbed delay boundary value problem used in [31]

$$\begin{cases} -\varepsilon u''(x) + 3u'(x) - u(x-1) = 0, \;\; x \in \Omega^*, \\ u(x) = 1, \;\; x \in [-1, 0], u(2) = 2. \end{cases}$$

Its solution is

$$u(x) = \begin{cases} c_1 \left(e^{\frac{3x}{\varepsilon}} - 1\right) + \frac{x}{3} + 1, \;\; x \in [0, 1], \\ e^{\frac{3x-6}{\varepsilon}} \left(\frac{2}{3} c_1 e^{3/\varepsilon} + \frac{2c_1}{3} - c_2 - \frac{2\varepsilon}{27} + \frac{23}{18}\right) - \frac{1}{3} c_1 x e^{\frac{3x-3}{\varepsilon}} - \frac{c_1 x}{3} + c_2 + \frac{x\varepsilon}{27} \\ \quad + \frac{1}{18}(x-1)^2 + \frac{x}{3}, \;\; x \in [1, 2], \end{cases}$$

where

$$\begin{aligned} c_1 &= \frac{e^{-6/\varepsilon}\left(-\frac{\varepsilon^2}{27} + \frac{4\varepsilon}{9} - 3\right)}{\frac{2}{3}\left(e^{-3/\varepsilon} - e^{-6/\varepsilon}\right)\varepsilon - 4e^{-6/\varepsilon} + 3} \\ c_2 &= \frac{c_1 e^{3/\varepsilon}\left(-e^{-3/\varepsilon} + \frac{4}{\varepsilon} + 1\right)}{1 - e^{-3/\varepsilon}} + \frac{\frac{2}{27} e^{-3/\varepsilon}\varepsilon - \frac{\varepsilon}{27} - \frac{23 e^{-3/\varepsilon}}{18} + 1}{1 - e^{-3/\varepsilon}}. \end{aligned}$$

Take $x_i = \frac{2(i-1)}{N-1}$, $i = 1, 2, \ldots, N$ and $h_i = 0.1\varepsilon, i = 1, 2, \ldots, 60$, $h_i = \frac{2-6\varepsilon}{M-60}, i = 61, 62, \ldots, M$. The maximum absolute errors using the present method (PM) are compared with [31] in Table 1 for $\varepsilon = 2^{-6}$, 2^{-10}, 2^{-13}. From the comparison, we can see that the present method can produce more accurate numerical results. Taking $\varepsilon = 2^{-20}$, $N = 100, M = 256, N_1 = 30$, $h_i = 0.05\varepsilon, i = 1, 2, \ldots, 200$, $h_i = \frac{2-10\varepsilon}{56}, i = 201, 202, \ldots, 256$, the absolute errors $|U_{as}^N - u(x)|$ and $|U_{as}^{N,M,N_1} - u(x)|$ are shown in Figure 1.

Example 6.2. Consider the following singularly perturbed delay boundary value problem used in [31]

$$\begin{cases} -\varepsilon u''(x) + (x+10)u'(x) - u(x-1) = x, \ \ x \in \Omega^*, \\ u(x) = x, \ x \in [-1, 0], u(2) = 2. \end{cases}$$

The exact solution of this problem is not available. Taking $x_i = (i-1)h, \ h = \frac{2}{N-1}, \ \ i = 1, 2, \ldots, N$, the numerical results compared with [31] are given in Table 2 for $\varepsilon = 2^{-6}, \ 2^{-10}, \ 2^{-13}$. By such comparison, the present method can provide more accurate approximate solutions than [31]. Taking $\varepsilon = 2^{-20}, \ N = 100, 200$, the maximum absolute errors E_ε^N are shown in Figure 2. Taking $N_1 = 30$ and $M = 128, 256$, the absolute errors $|v_r^{M,N_1}(x) - v_r(x)|$ are shown in Figure 3.

Table 1. Comparison of the maximum absolute errors E_ε^N for Example 6.1

ε	$N = 64([12])$	$N = 128([12])$	$N = 128(\mathrm{PM}(3.10))$	$N = M = 128, N_1 = 30(\mathrm{PM}(4.2))$
2^{-6}	7.85×10^{-4}	2.73×10^{-4}	1.24×10^{-6}	7.80×10^{-5}
2^{-10}	7.85×10^{-4}	2.73×10^{-4}	1.24×10^{-6}	4.80×10^{-5}
2^{-13}	7.85×10^{-4}	2.73×10^{-4}	1.24×10^{-6}	8.40×10^{-6}

Table 2. Comparison of the maximum absolute errors E_ε^N for Example 6.2

ε	$N = 64([12])$	$N = 64(\mathrm{PM})$	$N = 128([12])$	$N = 128(\mathrm{PM})$
2^{-6}	2.64×10^{-3}	1.51×10^{-4}	8.39×10^{-4}	6.70×10^{-5}
2^{-10}	2.64×10^{-3}	1.51×10^{-4}	8.39×10^{-4}	6.70×10^{-5}
2^{-13}	2.64×10^{-3}	1.51×10^{-4}	8.39×10^{-4}	6.70×10^{-5}

7. Conclusion

This paper extends the applications of the reproducing kernel method to singularly perturbed delay boundary value problems. An effective method is developed for solving singularly perturbed delay boundary value problems exhibiting a strong right boundary layer. The present method combines the advantages of the asymptotic expansion technique and the reproducing kernel method. The numerical results compared with the existing method show that the present method is a valid technique for treating singularly perturbed delay boundary value problems exhibiting a right boundary layer.

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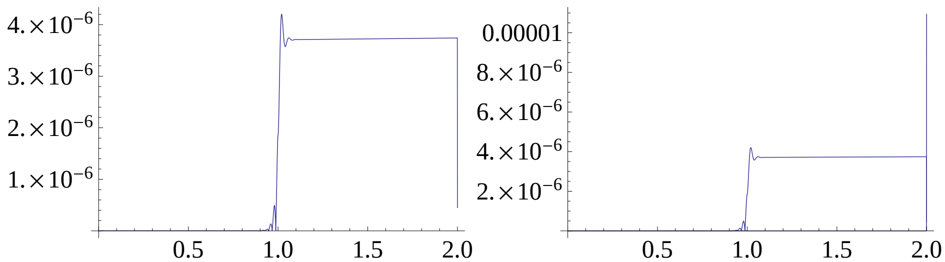

FIGURE 1. Absolute errors between the approximate solution and exact solution of Example 6.1 for $\varepsilon = 2^{-20}$(the left:$|U_{as}^{N} - u(x)|$; the right: $|U_{as}^{N,M,N_1} - u(x)|$).

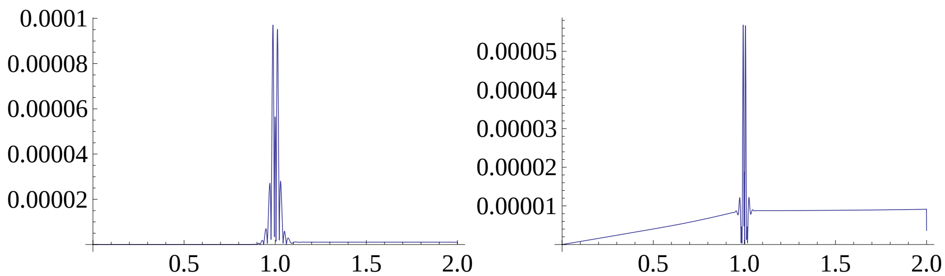

FIGURE 2. The maximum absolute errors E_{ε}^{N} for $\varepsilon = 2^{-20}$(the left:$N = 100$; the right: $N = 200$).

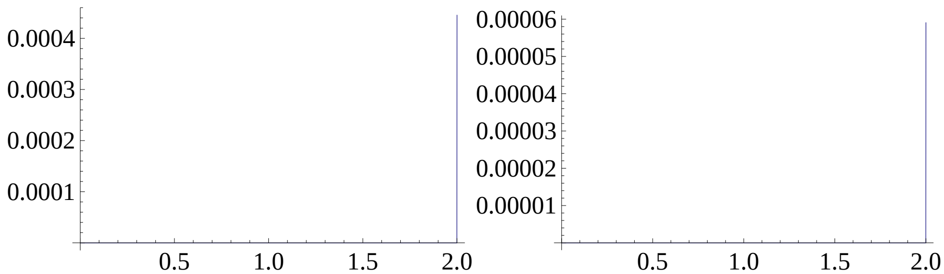

FIGURE 3. The absolute errors $|v_r^{M,N_1}(x) - v_r(x)|$ for $\varepsilon = 2^{-20}$(the left:$M = 128$; the right: $M = 256$).

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